Lie symmetries and related group-invariant solutions of a nonlinear Fokker-Planck equation based on the Sharma-Taneja-Mittal entropy

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In the framework of the statistical mechanics based on the Sharma-Taneja-Mittal entropy we derive a family of nonlinear Fokker-Planck equations characterized by the associated non-increasing Lyapunov functional. This class of equations describes kinetic processes in anomalous mediums where both super-diffusive and sub-diffusive mechanisms arise contemporarily and competitively. We classify the Lie symmetries and derive the corresponding group-invariant solutions for the physically meaningful Uhlenbeck-Ornstein process. For the purely diffusive process we show that any localized state asymptotically approaches a bell shape well fitted by a generalized Gaussian which is, in general, a quasi-self-similar solution for this class of purely diffusive equations.

Keywords: Nonlinear Fokker-Planck equation, Sharma-Taneja-Mittal entropy, Lie symmetries, Group invariant solutions

1. INTRODUCTION

One of the most important phenomenological equations of non equilibrium statistical mechanics is the Fokker-Planck equation. Its linear version is considered appropriate for the description of a wide variety of physical phenomena characterized by short-range interactions and/or short-time memories, typically associated with normal diffusion. This equation, intimately related with the Boltzmann-Gibbs entropy, rules the time evolution of the density probability distribution in the presence of an external force field toward the stationary state corresponding to the exponential Gibbs distribution [1].
Differently, nonlinear Fokker-Planck equations (NFPE) emerge in presence of anomalous diffusion, generally associated to non-Gaussian distributions like power-law or stretched exponential [2, 3], typically observed in complex systems with long-range interactions, memory effects persisting in time and, more in general, systems governed by a non ergodic dynamics dominated by fractal and hierarchical structures in the phase space.

Anomalous diffusion characterizes several phenomenologies like surface growth [4], transport of fluid in porous media [5], diffusion in plasmas [6], subrecoil laser cooling [7], diffusion of micelles dissolved in salted water [8], two dimensional rotating flow [9], anomalous diffusion at liquid surfaces [10] and others (see, for instance, [11] and reference therein).
It is now clear that these equations are strictly related to generalized entropies [12]. In this respect, recent attempts to extend the usual concepts of thermostatistics and kinetic theory to complex systems with non-Boltzmannian distributions have dealt wich the introduction of several entropic forms like the q-entropy [13], the κ-entropy [14] and the quantum-group entropy [15], among the others [16, 17].

We observe that several of the above entropies are special cases of a two-parameter family originally introduced by Shrama, Taneja and Mittal (STM-entropy) [18, 19] and recently reconsidered on a physical background in [20].

The purpose of the present work is to derive a NFPE in the picture of a generalized statistical mechanics based on the STM-entropy and to study its classical Lie symmetries and the related group invariant solutions (GIS) [21, 22].
This NFPE is characterized by the related non decreasing Lyapunov functional that, at equilibrium, coincides with the generalized free energy of the system. In fact, as known, Fokker-Planck equations describe dissipative phenomena where temperature is fixed instead of energy. Therefore, these equations are more properly associated with the canonical assemble and the appropriate thermodynamical potential is the free energy. In this respect, Lyapunov functionals related to some NFPE were derived and discussed in a relatively general context in [23–25].

Based on the monotonic behavior of the Lyapunov functional, one can show that any initial localized state, which evolves according to the corresponding NFPE, converges to a stationary solution that minimizes the Lyapunov functional. In addition, for the Uhlenbeck-Ornstein process, this stationary solution is nothing but a generalized Gaussian which maximizes the corresponding constrained entropic form [26, 27]. Remarkably, the NFPE we are introducing is characterized by a nonlinear diffusive term with two different power factors which make this equation more difficult to study as compared to the NFPE for poroses media (known as porous medium equation) which has only a single power factor dependency [25]. Another different NFPE with a double power diffusion coefficient has also been investigated in [12].

The paper is organized as follows. In section 2 we give a review of the STM-entropy while the associated NFPE is derived in section 3. In section 4 we classify the classical Lie symmetries and derive the corresponding GIS. In section 5 we briefly study the related purely diffusive equation, whilst section 6 contains a summary of our results.

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2. SHARMA-TENEJA-MITTAL ENTROPY

The STM entropy has been originally introduced in [18, 19] as a generalization of the following functional composition equation

$$\int f(x,y)\,dx\,dy = \int f(x)\,dx + \int f(y)\,dy\,, \tag{1}$$

in the most general composition equation

$$\int f(x,y)\,dx\,dy = \int y^a\,dy \int f(x)\,dx + \int x^b\,dx \int f(y)\,dy\,, \tag{2}$$

for $a \neq 0$ and $b \neq 0$, constants parameters.

Whereas the most general solution of Eq. (1) [28] is provided by the well-known Boltzmann-Gibbs (BG) entropy[1]

$$S^{\rm BG}[p] = -\int p(x)\,\ln\big(p(x)\big)\,dx\,, \tag{3}$$

the unique solution of Eq. (2), whenever $a \neq b$, introduces the STM entropy in the form

$$S^{\rm STM}[p] = -\int \frac{p(x)^a - p(x)^b}{a-b}\,dx\,, \tag{4}$$

for a density distribution function $p(x)$.

More recently, in [20], the same entropic form has been derived starting from the following ansatz

$$S[p] = -\int p(x)\,\Lambda\big(p(x)\big)\,dx\,, \tag{5}$$

where $\Lambda(x)$ plays the role of a generalized logarithm and for $\Lambda(x) = \ln(x)$, Eq. (5) reproduces the BG entropy (3).

By requiring that the equilibrium distribution, maximizing the entropic form (5) under suitable constraints, can be written by means of the generalized exponential $\mathcal{E}(x)$, the inverse function of $\Lambda(x)$: $\Lambda(\mathcal{E}(x)) = \mathcal{E}(\Lambda(x)) = x$, one derives the following functional-differential equation

$$\frac{d}{dx}\left[x\,\Lambda(x)\right] = \lambda\,\Lambda\left(\frac{x}{\alpha}\right)\,. \tag{6}$$

By solving this equation, with the boundary conditions $\Lambda(1) = 0$ and $(d/dx)\Lambda(x)\Big|_{x=1} = 1$, we obtain the two-parameter deformed logarithm

$$\Lambda(x) \equiv \ln_{\{\kappa,r\}}(x) = x^r\,\frac{x^\kappa - x^{-\kappa}}{2\,\kappa}\,, \tag{7}$$

which reduces to the standard logarithm in the $(\kappa, r) \to (0,0)$ limit. It satisfies the relation $\ln_{\{\kappa,r\}}(x) = \ln_{\{-\kappa,r\}}(x) = -\ln_{\{\kappa,-r\}}(1/x)$ which, for $r = 0$, reproduces the well known propriety of the standard logarithm: $\ln(x) = -\ln(1/x)$.

[1] In this paper, without sake of generality, we use unities with the Boltzmann constant $k = 1$.

The two constants α and λ, appearing in Eq. (6), are given by

$$\alpha = \left(\frac{1+r-\kappa}{1+r+\kappa}\right)^{1/2\kappa}\,, \tag{8}$$

$$\lambda = \frac{(1+r-\kappa)^{(r+\kappa)/2\kappa}}{(1+r+\kappa)^{(r-\kappa)/2\kappa}}\,, \tag{9}$$

and are related each to other in $\ln_{\{\kappa,r\}}(1/\alpha) = 1/\lambda$.

Using the solution (7) in the ansatz (5) we obtain the following two-parameter deformed entropy

$$\begin{aligned} S_{\kappa,r}[p] &= -\int p(x)\,\ln_{\{\kappa,r\}}(p(x))\,dx\,, \\ &= -\int \frac{p(x)^{1+r+\kappa} - p(x)^{1+r-\kappa}}{2\kappa}\,dx\,, \end{aligned} \tag{10}$$

which has the same form as Eq. (4) with $a = 1+r+\kappa$ and $b = 1+r-\kappa$. From Eq. (10), it is clear that the STM-entropy mimics the BG entropy, which is recovered in the $(\kappa, r) \to (0,0)$ limit, through the replacement of the standard logarithm with its generalized version (7).

In the following we require, as natural conditions, that the generalized exponential $\mathcal{E}(x) \equiv \exp_{\{\kappa,r\}}(x)$ be a strictly increasing and convex function. Moreover, we require that the equilibrium distribution, obtained by maximizing the constrained entropic form (10), has finite momenta of the nth-order, with $n \geq 0$ ($n = 0$ is the normalization). This implies the convergence of the following integrals

$$\int\limits_0^\infty x^n \exp_{\{\kappa,r\}}(-x)\,dx\,, \quad \text{and} \quad \int\limits_0^\infty x^n \exp_{\{\kappa,-r\}}(-x)\,dx\,. \tag{11}$$

In this way, the deformation parameters κ and r are restricted to the two dimensional region $I\!R^2 \supset \mathcal{R} = \{-|\kappa| \leq r \leq |\kappa|$, if $0 \leq |\kappa| < 1/[2\,(n+1)]$ and $|\kappa| - 1/(n+1) \leq r \leq 1/(n+1) - |\kappa|$, if $1/[2\,(n+1)] \leq |\kappa| < 1/(n+1)\}$ depicted in figure 1 (remark that the convergence of the nth-momentum implies the convergence of any mth-momentum with $m < n$).

For any $(\kappa, r) \in \mathcal{R}$, $\ln_{\{\kappa,r\}}(x)$ is a continuous, monotonic, increasing, concave and normalizable function for $x \in I\!R^+$, with $\ln_{\{\kappa,r\}}(I\!R^+) \subseteq I\!R$. Consequently, the STM-entropy results to be positive definite, continuous, symmetric, expandible, decisive, maximal, concave and Lesche stable [20].

The equilibrium distribution related to the entropy (10), constrained by the zeroth-momentum $\sigma^0 = \int p(x)\,dx \equiv 1$ and the nth-momentum $\sigma^n = \int x^n\,p(x)\,dx$, can be obtained through the following variational problem

$$\frac{\delta}{\delta p(x)}\left[S_{\kappa,r}[p] - \gamma\int p(x)\,dx - \beta\int x^n\,p(x)\,dx\right] = 0\,, \tag{12}$$

and assumes the expression

$$p(x) = \alpha\,\exp_{\{\kappa,r\}}\left(-\frac{1}{\lambda}\,(\gamma + \beta x^n)\right)\,, \tag{13}$$

where γ and β are the Lagrange multipliers associated to the constraints σ^0 and σ^n, respectively.

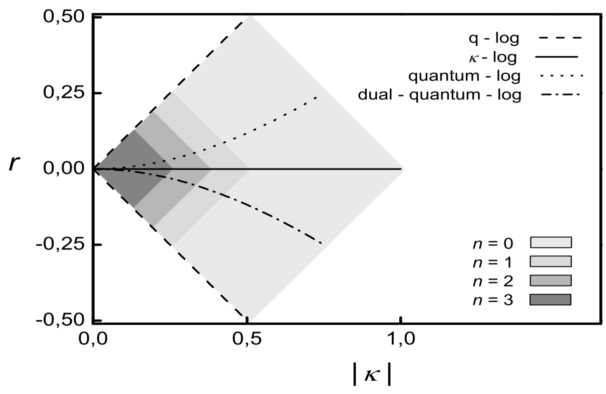

FIG. 1: Parameter space (κ, r) for the logarithm (7). The shaded zones corresponds to the regions of convergence of the integrals (11), for $n = 0, 1, 2, 3$. The four lines: dashed, solid, dotted and, dash-dotted represent the loci of points of the logarithms belonging to the q-entropy, κ-entropy, quantum-group entropy and the dual quantum-group entropy, respectively, as discussed in the text.

We conclude by recalling that several generalized entropies, which have been currently used in the study of anomalous statistical systems, belong to the STM family. In figure 1, we depicted the loci of points representing some of these one-parameter entropies: the q-entropy

$$S_{2-q}[p] = -\int \frac{p(x)^{2-q} - p(x)}{q-1}\, dx\,, \tag{14}$$

for $r = \pm|\kappa|$, with $q = 1 \mp 2|\kappa|$ [13], the κ-entropy

$$S_\kappa[p] = -\int \frac{p(x)^{1+\kappa} - p(x)^{1-\kappa}}{2\kappa}\, dx\,, \tag{15}$$

for $r = 0$ [14], the quantum group-entropy

$$S_{q_A}[p] = -\int \frac{p(x)^{q_A^{-1}} - p(x)^{q_A}}{q_A - q_A^{-1}}\, dx\,, \tag{16}$$

for $r = \sqrt{1+\kappa^2} - 1 > 0$, with $q_A = \sqrt{1+\kappa^2} + |\kappa|$ [15] and its dual form

$$S^*_{q_A}[p] = -\int \frac{p(x)^{2-q_A^{-1}} - p(x)^{2-q_A}}{q_A - q_A^{-1}}\, dx\,, \tag{17}$$

for $r = 1 - \sqrt{1+\kappa^2} < 0$, with $q_A = \sqrt{1+\kappa^2} - |\kappa|$ [29].

The thermostatistics theory based on the STM-entropy fulfills the Legendre structure [29]. The main proprieties of a statistical system described by this entropy, in the microcanonical formalism, has been investigated in [30]. Finally, in [31], the entropy $S_{\kappa,r}[p]$ has been derived from a generalized version of the Shannon-Khinchin axioms and the corresponding uniqueness theorem.

3. NONLINEAR FOKKER-PLANCK EQUATION

Following standard methods [11, 24, 32] a NFPE related to the STM-entropy (STM-NFPE) can be obtained starting from the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial v} = 0\,, \tag{18}$$

for a normalized density distribution $\rho \equiv \rho(v,t)$ describing a conservative particle system in the velocity space, where the nonlinear current $j \equiv j(v,t)$, given by

$$j = -\rho \frac{\partial}{\partial v}\left(\frac{\delta}{\delta\rho}\mathcal{L}_{\kappa,r}[\rho]\right)\,, \tag{19}$$

is related to the density field through the thermodynamic force $\partial(\delta\mathcal{L}_{\kappa,r}[\rho]/\delta\rho)/\partial v$. We introduce the functional $\mathcal{L}_{\kappa,r}[\rho]$ according to

$$\mathcal{L}_{\kappa,r}[\rho] \equiv U[\rho] - D\, S_{\kappa,r}[\rho]\,, \tag{20}$$

where $U[\rho]$ is the mean energy of the system

$$U[\rho] = \int \frac{1}{2} v^2 \rho(v,t)\, dv\,, \tag{21}$$

and D is a constant diffusion coefficient.

Within these settings, the nonlinear current becomes

$$j = -v\rho - D\lambda\rho \frac{\partial}{\partial v} \log_{\{\kappa,r\}}\left(\frac{\rho}{\alpha}\right)\,, \tag{22}$$

so that the STM-NFPE can be explicitly written in

$$\frac{\partial \rho}{\partial t} - \frac{\partial}{\partial v}\Big(v\rho\Big) - D\frac{\partial^2}{\partial v^2}\left[\rho^{1+r}\Big(\frac{r+\kappa}{2\kappa}\rho^\kappa - \frac{r-\kappa}{2\kappa}\rho^{-\kappa}\Big)\right] = 0\,, \tag{23}$$

which will be the subject of our investigations.

It is worthy to note that this equation embodies two well-known special cases: the linear Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} - \frac{\partial}{\partial v}\Big(v\rho\Big) - D\frac{\partial^2 \rho}{\partial v^2} = 0\,, \tag{24}$$

when $r = \kappa = 0$ and the q-NFPE

$$\frac{\partial \rho}{\partial t} - \frac{\partial}{\partial v}\Big(v\rho\Big) - D\frac{\partial^2}{\partial v^2}\rho^{2-q} = 0\,, \tag{25}$$

when $r = \pm|\kappa| = (1-q)/2$.

Remark that the particle current (22) is the sum of two contributes: $j = j^{\rm drift} + j^{\rm diff}$. A linear drift term $j^{\rm drift} \propto v\rho$, which describes the standard Uhlenbeck-Ornstein process and a nonlinear diffusive term

$$j^{\rm diff} \propto \frac{\partial}{\partial v}\Big[(r+\kappa)\,\rho^{1+r+\kappa} - (r-\kappa)\,\rho^{1+r-\kappa}\Big]\,, \tag{26}$$

given by the sum of two different power terms of ρ (whenever $r \neq \pm|\kappa|$). This kind of nonlinearity is substantially different from the one appearing in Eq. (25) formed by just a single power term of ρ.

As known, in the $j^{\rm drift} \to 0$ limit (purely diffusive case), Eq. (25) describes super-diffusive or sub-diffusive processes depending on the value of $q < 1$ or $q > 1$, respectively.

Let us now consider a linear combination of two q-NFPE (25) with two power indexes q_1 and q_2, respectively, according to

$$\begin{aligned}&(r+\kappa)\left[\frac{\partial\rho}{\partial t}-\frac{\partial}{\partial v}\left(v\,\rho+D\frac{\partial}{\partial v}\,\rho^{2-q_1}\right)\right]\\ -\quad&(r-\kappa)\left[\frac{\partial\rho}{\partial t}-\frac{\partial}{\partial v}\left(v\,\rho+D\frac{\partial}{\partial v}\,\rho^{2-q_2}\right)\right]=0\,.\end{aligned}\tag{27}$$

Clearly, this equation coincides with Eq. (23) when $q_1=1-r-\kappa$ and $q_2=1-r+\kappa$. On the other hand, from figure 1, we can see that the index $q_1=1-r-\kappa<1$ and the index $q_2=1-r+\kappa>1$. This means that STM-NFPE can be obtained as a linear combination of two different q-NFPE: the one describing a super-diffusive process and the other describing a sub-diffusive process. In other words, the STM-NFPE describes kinetic processes occurring in anomalous media where both super-diffusive and sub-diffusive mechanisms arise contemporarily and competitively.

The stationary state reached by the system described by the STM-NFPE can be obtained by imposing the condition of the current-free $j=0$. In this way, from Eq. (22), we obtain

$$\frac{1}{2}\,v^2+D\,\lambda\ln_{\{\kappa,r\}}\left(\frac{\rho^{\rm st}}{\alpha}\right)=-D\gamma\,,\tag{28}$$

where $D\gamma$ is the integration constant and

$$\rho^{\rm st}=\alpha\exp_{\{\kappa,r\}}\left(-\frac{1}{\lambda}\left(\gamma+\frac{v^2}{2D}\right)\right)\,,\tag{29}$$

which is a generalized Gaussian.

This solution is nothing but the optimizing maximal entropy distribution obtainable from the variational problem (12), with $n=2$, if we identify the Lagrange multiplier $\beta=1/D$.

Finally, let us inspect on the physical meaning of the functional $\mathcal{L}_{\kappa,r}[\rho]$ used to derive the STM-NFPE. Firstly, we observe that at equilibrium, with $\beta=1/D$, the functional (20) coincides with the generalized free energy of the system, i.e. $F_{\kappa,r}[\rho^{\rm eq}]\equiv\mathcal{L}_{\kappa,r}[\rho^{\rm eq}]$ [30].
Differently, out of equilibrium, it can be shown that Eq. (20) is a not increasing function which reaches its minimum when the system reaches the stationary state $\rho^{\rm st}(v)$. In fact, from Eqs. (18) and (19), we have

$$\begin{aligned}\frac{d}{dt}\,\mathcal{L}_{\kappa,r}[\rho]&=\int\frac{\partial\rho}{\partial t}\,\frac{\delta}{\delta\rho}\Big(\mathcal{L}_{\kappa,r}[\rho]\Big)\,dv\\&=-\int\frac{\partial j}{\partial v}\,\frac{\delta}{\delta\rho}\Big(\mathcal{L}_{\kappa,r}[\rho]\Big)\,dv\\&=\int j\,\frac{\partial}{\partial v}\left[\frac{\delta}{\delta\rho}\Big(\mathcal{L}_{\kappa,r}[\rho]\Big)\right]\,dv\\&=-\int\frac{j^2}{\rho}\,dv\le 0\,,\end{aligned}$$

where equality holds at equilibrium, when j vanishes.

According to [33], we can identify the quantity $\mathcal{L}_{\kappa,r}[\rho]$ with the Lyapunov functional of the system under inspection, which uniquely characterizes the Fokker-Planck equation (23).

4. LIE SYMMETRIES AND RELATED GROUP INVARIANT SOLUTIONS

A way to obtain special solutions of a PDE is based on the determination of its Lie symmetries. Here we explore the classical Lie symmetries of the STM-NFPE where the generators $\chi(v,\,t,\,\rho)$ are functions of the independent variables v and t and of the dependent variable ρ. More general symmetries, with generators depending also on the derivative of the field ρ, can also exist, although we do not explore this situation here.

4.1. Lie symmetries

The determination of the Lie symmetries can be accomplished by following well-known techniques described in standard textbooks [21, 22, 34] which we remind for the details. For the sake of clarity, it is useful to consider the three cases represented by the Eqs. (23), (24) and (25) separately.

a) Linear FP equation.

Firstly, we consider the linear Fokker-Planck equation (24) which has been widely studied in the past [35]. Its maximal symmetry group is composed by the following seven operators

$$\begin{aligned}&\chi_1=\frac{\partial}{\partial t}\,,\\&\chi_2=e^{-t}\,\frac{\partial}{\partial v}\,,\\&\chi_3=e^{t}\left(\frac{\partial}{\partial v}-\frac{1}{D}\,v\,\rho\,\frac{\partial}{\partial\rho}\right)\,,\\&\chi_4=e^{-2t}\left(v\,\frac{\partial}{\partial v}-\frac{\partial}{\partial t}-\rho\,\frac{\partial}{\partial\rho}\right)\,,\\&\chi_5=e^{2t}\left(v\,\frac{\partial}{\partial v}+\frac{\partial}{\partial t}-\frac{1}{D}\,v^2\,\rho\,\frac{\partial}{\partial\rho}\right)\,,\\&\chi_6=\rho\,\frac{\partial}{\partial\rho}\,,\\&\chi_7=\eta\,\frac{\partial}{\partial\rho}\,,\end{aligned}\tag{30}$$

where η is another solution of Eq. (24). The firsts two operators generate time and velocity translations, χ_3 generates dilations whilst χ_4 and χ_5 generate more complicated transformations involving both dilations and translations in velocity and time. Finally, the last two generators reflect the linearity of Eq. (24) in the sense that if ρ and η are solutions, then the same is also true for $k\,\rho+\eta$, with k a constant.

b) q-NFPE.

The maximal symmetry group of equation (25) is formed by the following four operators

$$
\begin{aligned}
\chi_1 &= \frac{\partial}{\partial t}\,, \\
\chi_2 &= e^{-t}\,\frac{\partial}{\partial v}\,, \\
\tilde{\chi}_3 &= \frac{1-q}{2}\,v\,\frac{\partial}{\partial v}+\rho\,\frac{\partial}{\partial \rho}\,. \\
\tilde{\chi}_4 &= e^{(q-3)\,t}\left(v\,\frac{\partial}{\partial v}-\frac{\partial}{\partial t}-\rho\,\frac{\partial}{\partial \rho}\right)\,.
\end{aligned}
\tag{31}
$$

The first two generators are identical to those of the linear case and produce time and velocity translations whilst the last two generators have a q-dependence and produce dilations.

c) STM-NFPE.

The maximal symmetry group of Eq. (23) reduces to the following two operators

$$
\begin{aligned}
\chi_1 &= \frac{\partial}{\partial t}\,, \\
\chi_2 &= e^{-t}\,\frac{\partial}{\partial v}\,,
\end{aligned}
\tag{32}
$$

which generate time and velocity translations. Any other symmetry is destroyed by the particular expression of the nonlinearity of the diffusive term.

4.2. Group-invariant solutions

Having classifiied the classical Lie symmetries, we derive now several physically meaningful solutions characterized by their invariance under some of the above symmetries transformations.

Let us begin by considering the GIS related to the generators χ_1 and χ_2 which are common to the whole family of STM-NFPE. Trivially, the invariant solution under the action of the generator χ_1 corresponds to the stationary state (29) considered in the previous section. Also trivial is the invariant solution related to the generator χ_2, which produces a time-dependent space translation. It is merely a constant. More significative GIS can be obtained starting from a linear combination of χ_1 and χ_2, with generator

$$
\chi=\frac{1}{u}\,\frac{\partial}{\partial t}+e^{-t}\,\frac{\partial}{\partial v}\,,
\tag{33}
$$

where u is a constant. The invariant corresponding to this symmetry is

$$
\xi=v+\frac{1}{u}\,e^{-t}\,,
\tag{34}
$$

which is the coordinate of a time-depending moving frame. Rewriting Eq. (23) in the ξ variable we obtain the following ordinary differential equation

$$
\begin{aligned}
&D\,\frac{d^2}{d\xi^2}\left(\frac{r+\kappa}{2\,\kappa}\eta(\xi)^{1+r+\kappa}-\frac{r-\kappa}{2\,\kappa}\eta(\xi)^{1+r-\kappa}\right) \\
&+\xi\,\frac{d}{d\xi}\eta(\xi)+\eta(\xi)=0\,,
\end{aligned}
\tag{35}
$$

where $\eta(\xi)\equiv\rho(v,t)$. Its solution, given by

$$
\eta(\xi)=\alpha\,\exp_{\{\kappa,r\}}\left(-\frac{1}{\lambda}\left(\gamma+\frac{\xi^2}{2D}\right)\right)\,,
\tag{36}
$$

is a generalized Gaussian that translates in the v-space, by preserving its shape in time. In fact, we can easily verify that the variance $(\Delta v)^2=\langle v^2\rangle-\langle v\rangle^2$ is conserved in time.

Moreover, this solution describes an isoentropic process with $dS_{\kappa,r}[\rho]/dt=0$. Notwithstanding, the Lyapunov function decreases in time since the system dissipates energy during the evolution toward the equilibrium

$$
\frac{d}{dt}\mathcal{L}_{\kappa,r}[\rho]=-\frac{e^{-t}}{u^2}\leq 0\,,
\tag{37}
$$

where equality holds at $t\to\infty$.

Let us now consider the two symmetries generated by the operators $\tilde{\chi}_3$ and $\tilde{\chi}_4$ which are typical of the q-NFPE.
Quite interesting, the first of these transformation introduces the following scaling

$$
\rho(v,t)\to e^{\varepsilon}\,\rho\big(e^{\frac{q-1}{2}\,\varepsilon}\,v,t\big)\,,
\tag{38}
$$

which, in the $q\to 1$ limit, reduces to the scaling $\rho\to c\,\rho$, holding for the linear equations.
Unfortunately, the related GIS

$$
\rho(v,t)\propto v^{\frac{2}{1-q}}\,f(t)\,,
\tag{39}
$$

with $f(t)$ obtainable from Eq. (25) with the ansatz (39), is physically meaningless, being divergent when $v\to 0$ ($q>1$) or $v\to\infty$ ($q<1$).

Finally, the operator $\tilde{\chi}_4$ introduces more complicate scaling involving all the variables ρ, v and t. The corresponding GIS is again divergent in both $v\to\infty$ and $t\to\infty$.

More in general, starting from suitable linear combinations of the generators (31) we can obtain some physically significative solutions.
Among the many, let us consider the following case

$$
\begin{aligned}
\chi &= e^{(q-3)\,t}\,v\,\frac{\partial}{\partial v}+\left(1-e^{(q-3)\,t}\right)\frac{\partial}{\partial t}-e^{(q-3)\,t}\,\rho\,\frac{\partial}{\partial \rho} \\
&\equiv \chi_1+\tilde{\chi}_3\,.
\end{aligned}
\tag{40}
$$

Introducing the global invariants

$$
\xi=v\,\left(1-e^{(q-3)\,t}\right)^{\frac{1}{q-3}}\,,
\tag{41}
$$

$$
\eta(\xi)=\rho(v,t)\,\left(1-e^{(q-3)\,t}\right)^{\frac{1}{3-q}}\,,
\tag{42}
$$

the q-NFPE becomes

$$
D\,\frac{d^2}{d\xi^2}\eta(\xi)^{2-q}+\xi\,\frac{d}{d\xi}\eta(\xi)+\eta(\xi)=0\,,
\tag{43}
$$

whose solution can be written in

$$
\eta(\xi)=\frac{1}{Z_q}\,\exp_q\left(-\frac{1}{2}\,\beta_q\,\xi^2\right)\,,
\tag{44}
$$

with

$$\beta_q = \frac{1}{2-q}\,\frac{Z_q^{1-q}}{D} \ , \tag{45}$$

Z_q the normalization constant and $\exp_q(x) = [1-(1-q)\,x]^{\frac{1}{1-q}}$ the q-exponential.

Returning to the original variables ρ and v, we obtain the well known q-Gaussian self-similar solution in the form

$$\rho(x,t) = \frac{1}{Z_q(t)}\,\exp_q\left(-\frac{1}{2}\,\beta_q(t)\,v^2\right) \ , \tag{46}$$

where

$$Z_q(t) = Z_q\left(1-\frac{e^{(q-3)\,t}}{m}\right)^{\frac{1}{3-q}} \ , \tag{47}$$

$$\beta_q(t) = \beta_q\left(\frac{Z_q}{Z_q(t)}\right)^2 \ . \tag{48}$$

Finally, for a discussion of the remaining GIS belonging to the linear Fokker-Planck equation, generated by the operators χ_3, χ_4 and χ_5 of Eq. (30), we remaind to the existent literature [34, 36].

5. NONLINEAR DIFFUSIVE EQUATION

From the previous analysis it follows that, in general, the self-similar function

$$\rho^{\rm G}(v,t) = \exp_{\{\kappa,r\}}\left(a(t)-b(t)\,v^2\right) \ , \tag{49}$$

is not an exact solution of Eq. (23), although it still plays a role in the study of purely diffusive processes related to this equation.

In order to clarify this role let us recall that, according to the following time-dependent transformation

$$\eta(v,t) = \frac{1}{a(t)}\,\rho\big(\zeta(v,t),\,\tau(t)\big) \ , \tag{50}$$

with

$$\zeta(v,t) = \sqrt{D}\,\frac{v}{a(t)} \ ,$$
$$\tau(t) = \ln a(t) \ , \tag{51}$$
$$a(t) = \big[1+(3-q)\,t\big]^{\frac{1}{3-q}} \ ,$$

the q-FPE is mapped in the following purely diffusive equation

$$\frac{\partial\eta}{\partial t} = \frac{\partial^2}{\partial v^2}\eta^{2-q} \ , \tag{52}$$

which is known as porous medium equation [37, 38].

Correspondingly, the self-similar q-Gaussian solution (46) is changed into[2]

$$\eta(v,t) = \frac{1}{t^{\frac{1}{3-q}}}\left(1-\frac{1-q}{2\,(2-q)\,(3-q)}\,\frac{v^2}{t^{\frac{2}{3-q}}}\right)^{\frac{1}{1-q}} \ , \tag{53}$$

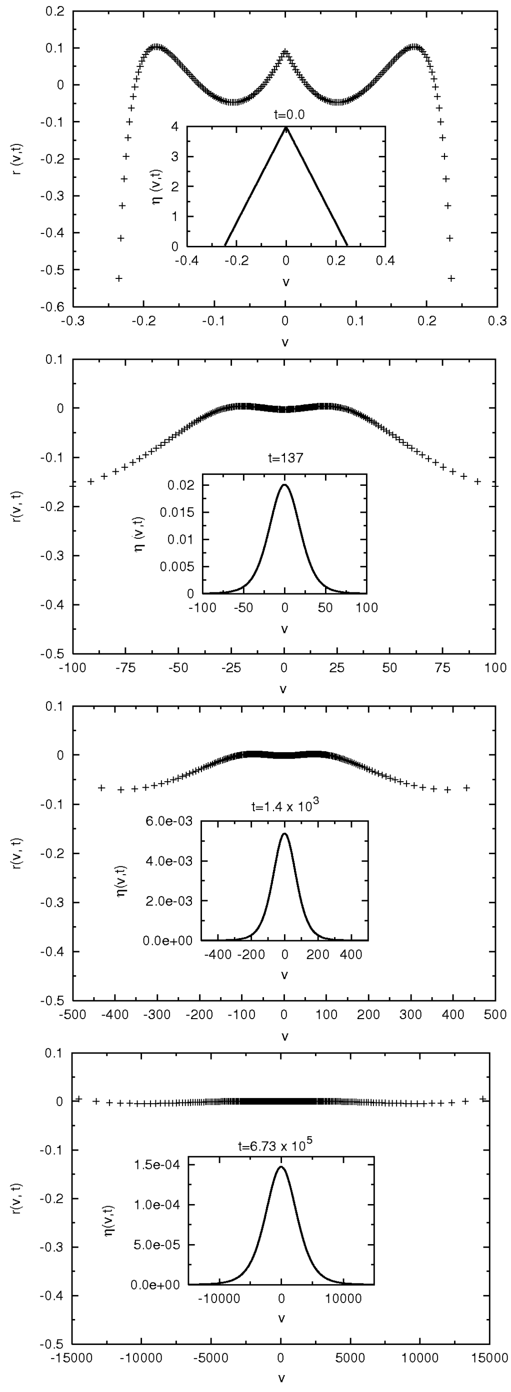

FIG. 2: Large time values behavior of the numerical solutions for $\kappa = 0.3$, $r = 0$, with an initial triangular probability distribution. The logarithm of the ratio (58) is plotted in each figure. The best fitted parameters are: $a = 1.324$, $b = 40.54$ at $t = 0.0$; $a = -4.862$, $b = 2.777\times10^{-3}$ at $t = 137$; $a = -7.640$, $b = 2.871\times10^{-4}$ at $t = 1.4\times10^3$ and $a = -23.39$, $b = 6.205\times10^{-7}$ at $t = 6.73\times10^5$, respectively. Inset figures show $\eta(v,t)$ at each time t.

that is the Barenblatt' self-similar solution [39].

Based on these results, let us introduce a time-dependent transformation for the whole family of the STM-NFPE, de-

[2] We posed $Z_q = (3-q)^{1/(q-3)}$.

fined by

$$\eta(v,t) = \rho\big(\zeta(v,t), \tau(t)\big) , \tag{54}$$

with $\zeta(v,t)$ and $\tau(t)$ as in Eq. (51), but now

$$a(t) = (1+2t)^{\frac{1}{2}} . \tag{55}$$

Remark that the transformation (54) maps (κ, r)-Gaussian into (κ, r)-Gaussian.

By applying such transformation on Eq. (23), we obtain the following evolution equation

$$\frac{\partial\eta}{\partial t} = \frac{\partial^2}{\partial v^2}\left[\eta^{1+r}\left(\frac{r+\kappa}{2\kappa}\eta^{\kappa} - \frac{r-\kappa}{2\kappa}\eta^{-\kappa}\right)\right] + \frac{\eta}{2t+1} , \tag{56}$$

that is a diffusive equation with an extra term representing a source for the field particle.

In the large time limit, this term becomes negligible with respect to the others, so that equation (56) can be well approximate by the purely (κ, r)-diffusive equation

$$\frac{\partial}{\partial t}\eta = \frac{\partial^2}{\partial v^2}\left[\eta^{1+r}\left(\frac{r+\kappa}{2\kappa}\eta^{\kappa} - \frac{r-\kappa}{2\kappa}\eta^{-\kappa}\right)\right] . \tag{57}$$

We recall now that any localized solution of STM-NFPE (23) asymptotically approaches the stationary state given by the generalized Gaussian (29) which is, therefore, transformed by means of equation (54) in another one that well approximates the solution of Eq. (57). This means that, any localized state is driven to a stationary state that is asymptotically well approximated by a (κ, r)-Gaussian function.

In order to confirm this asymptotic behavior, we numerically study the approach to the equilibrium of a given Cauchy problem for the diffusive equation (57).

In fact, the asymptotic behavior of the solutions of Eq. (57) can be evidenced by studying the time evolution of the function

$$r(v,t) = \ln\left(\frac{\rho(v,t)}{\rho^{G}(v,t)}\right) , \tag{58}$$

which is the logarithmic ratio between the numerical solution $\rho(v,t)$ of equation (57) and the generalized Gaussian function (49), with $a(t)$ and $b(t)$ the best fitted parameters of the numerical solution, at each time t.

In the inset of figure 2, we plotted the time evolution of $\eta(v,t)$ with an initial triangle shape, for the case $\kappa = 0.3$, $r = 0$. It is clear from this picture that $r(v,t) \to 0$ as $t \to \infty$, i.e. the function (58) gradually decreases to zero as time evolves, which gives a strong evidence that the numerical solution is asymptotically approaching the generalized Gaussian function.

We have run several simulations with different initial shapes to confirm this asymptotic behavior.

6. SUMMARY

In this work we have studied the classical Lie symmetries and the related group invariant solutions of a nonlinear Fokker-Planck equation based on the Sharma-Taneja-Mittal entropy. The analysis showed that the generalized Gaussian function, obtained by replacing the standard exponential with its generalized version, is recurrent in the expression of several GIS. In fact, it models the stationary state (29) as well as the traveling wave (36) and, limiting to the q-case, also the self-similar solution (46). In general, the (κ, r)-Gaussian is not a scale invariant solution of the STM-NFPE although it plays a role in the study of the evolution of localized initial states driven by the purely diffusive equation (57) related to the STM-NFPE. By performing several numerical simulations, with different initial shapes, we have found strong evidence that localized initial states are well approximated, for large time values, by the (κ, r)-Gaussian function instead of the standard Gaussian.

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