q-distributions in complex systems: a brief review

S. Picoli Jr., R. S. Mendes, L. C. Malacarne and R. P. B. Santos
Departamento de Física and National Institute of Science and Technology for Complex Systems, Universidade Estadual de Maringá, Avenida Colombo 5790, 87020-900 Maringá, PR, Brazil

(Received on 26 March, 2009)

The nonextensive statistical mechanics proposed by Tsallis is today an intense and growing research field. Probability distributions which emerges from the nonextensive formalism (q-distributions) have been applied to an impressive variety of problems. In particular, the role of q-distributions in the interdisciplinary field of complex systems has been expanding. Here, we make a brief review of q-exponential, q-Gaussian and q-Weibull distributions focusing some of their basic properties and recent applications. The richness of systems analyzed may indicate future directions in this field.

Keywords: q-exponential, q-Gaussian, q-Weibull, Nonextensive statistics

1. INTRODUCTION

Common characteristics of complex systems include long-range correlations, multifractality and non-Gaussian distributions with asymptotic power law behavior. Typically, such systems are not well described by approaches based on the usual statistical mechanics. In this scenario, a new formalism capable of providing a better description of complex systems is welcome. This is the case of the generalized (nonextensive) statistical mechanics proposed by Tsallis - nowadays, an intense and growing research field[1–4].

Concepts related with nonextensive statistical mechanics have found applications in a variety of disciplines including physics, chemistry, biology, mathematics, geography, economics, medicine, informatics, linguistics among others[5–7]. Probability distributions which emerge from the nonextensive formalism - also called q-distributions - have been applied to an impressive variety of problems in diverse research areas including the interdisciplinary field of complex systems.

In the present work we focus on q-exponential, q-Gaussian and q-Weibull distributions. We summarized some of their basic properties and provide useful references of recent applications. The richness of systems analyzed may indicate future directions in this research line.

2. q-EXPONENTIAL DISTRIBUTION

The q-exponential distribution is given by the probability density function (pdf)

$$p_{qe}(x) = p_0 \left[1-(1-q)\,\frac{x}{x_0}\right]^{1/(1-q)} \qquad (1)$$

for $1-(1-q)x/x_0 \geq 0$. If $p_0 = (2-q)/x_0$, eq. (1) is normalized.

In the limit $q \to 1$, eq. (1) recovers the usual exponential distribution in the same way in which the q-exponential function, defined as $e_q^{-x} \equiv [1-(1-q)x]^{1/(1-q)}$, recovers exponential function in the limit $q \to 1$ ($e_1^{-x} \equiv e^{-x}$). If $q < 1$, eq. (1) has a finite value for any finite real value of x since, by definition, $p_{qe}(x) = 0$ for $1-(1-q)x/a < 0$. If $q > 1$, eq. (1) exhibits power law asymptotic behavior,

$$p_{qe}(x) \sim x^{-1/(q-1)}. \qquad (2)$$

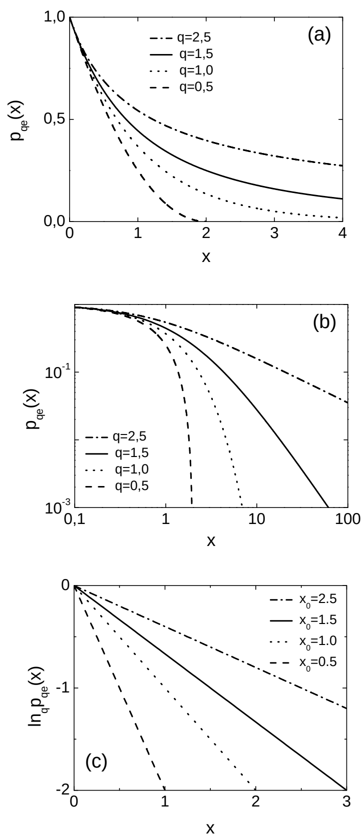

FIG. 1: **q-exponential distribution**. a) Plot of $p_{qe}(x)$ versus x, with $p_0 = x_0 = 1$ and typical values of q. b) Log-log plot of the curves in a). c) $\ln_q p_{qe}(x)$ versus x for $p_0 = 1$ and typical values of x_0.

Note also that $p_{qe}(x) \simeq 1+x$ for small x, independently of the q value. Figures 1a and 1b show $p_{qe}(x)$ versus x for typical values of q.

The q-exponential distribution, for $q > 1$, corresponds to the Zipf-Mandelbrot law[8] and a Burr-type distribution[9]. In this sense, the q-exponential is a generalization of these distributions for $q < 1$. Thus, by choosing suitable values for q, q-exponentials may be used to represent both short and long tailed distributions. This feature also holds for the other q-distributions.

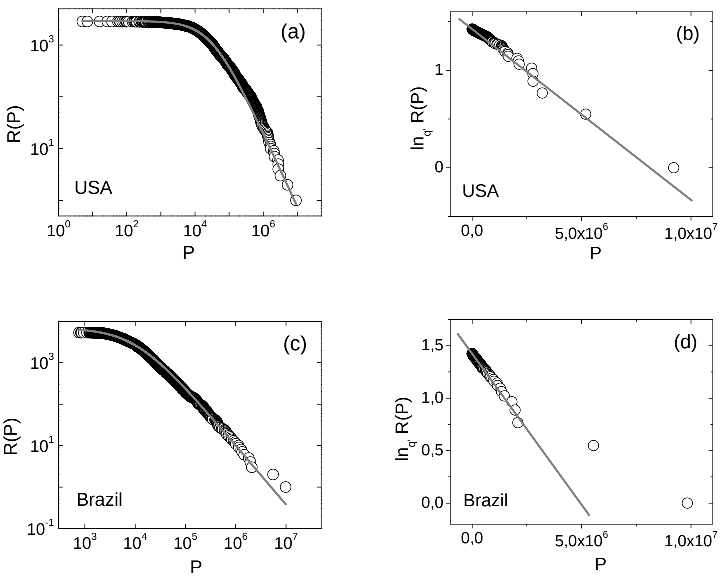

FIG. 2: **Population of cities.** a) Empirical cdf $R(P)$, where P is the population of USA cities. The solid line is a q-exponential, given by eq. (3), with $q' = 1.7$ ($q \simeq 1.4$), $x'_0 = 21,250$ and $c' = 2,919$. b) $\ln_{q'} R(P)$ versus P, with $q' = 1.7$, for the same data shown in (a). The solid line is a linear fit to the data. c) Empirical cdf $R(P)$, where P is the population of Brazilian cities. The solid line is a q-exponential, given by eq. (3), with $q' = 1.7$ ($q \simeq 1.4$), $x'_0 = 7,073$ and $c' = 6,968$. d) $\ln_{q'} R(P)$ versus P, with $q' = 1.7$, for the same data shown in (c). The solid line is a linear fit to the data.

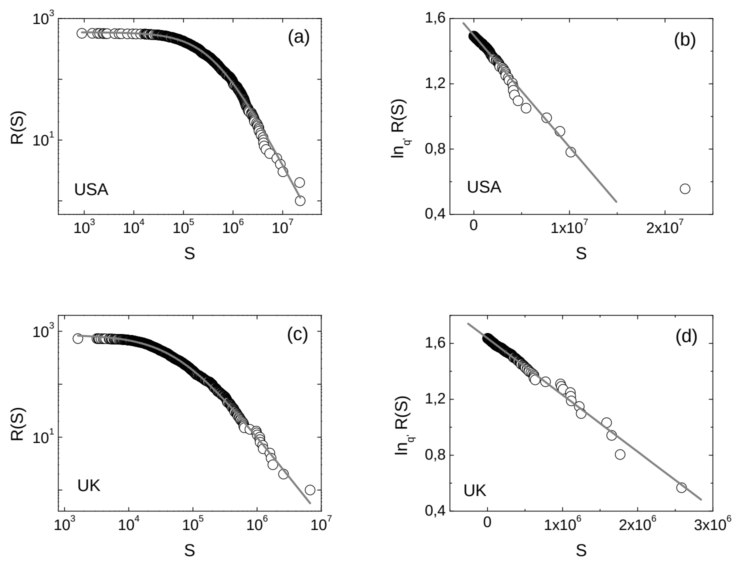

FIG. 3: **Circulation of magazines.** a) Empirical cdf $R(S)$, where S is the circulation of 570 USA magazines in 2004. The solid line is a q-exponential, given by eq. (3), with $q' = 1.65$ ($q \simeq 1.4$), $x'_0 = 255,204$ and $c' = 594$. b) $\ln_{q'} R(S)$ versus S, with $q' = 1.65$, for the same data shown in (a). The solid line is a linear fit to the data. c) Empirical cdf $R(S)$, where S is the circulation of 727 UK magazines in 2005. The solid line is a q-exponential, given by eq. (3), with $q' = 1.65$ ($q \simeq 1.4$), $x'_0 = 37,493$ and $c' = 860$. b) $\ln_{q'} R(S)$ versus S, with $q' = 1.65$, for the same data shown in (c). The solid line is a linear fit to the data.

The cumulative distribution function (cdf) associated to eq. (1) is given by

$$R_{qe}(x) = \int_x^\infty p_{qe}(y)\,dy = p'_0 \left[1-(1-q')\frac{x}{x'_0}\right]^{1/(1-q')}, \qquad (3)$$

defined for $q < 2$, with $q' = 1/(2-q)$, $x'_0 = x_0/(2-q)$ and $p'_0 = p_0 x_0/(2-q)$. Observe that $R_{qe}(x)$ and $p_{qe}(x)$ exhibit the same mathematical form.

It is possible to visualize q-exponential distributions as straight lines in graphs with appropriate scales. Applying the q-logarithm function, defined as $\ln_q x \equiv [x^{(1-q)} - 1]/(1-q)$, with $\ln_1 x \equiv \ln(x)$, in both sides of eq. (1), we have

$$\ln_q p_{qe}(x) = \ln_q p_0 - [1+(1-q)\ln_q p_0]\frac{x}{x_0}. \qquad (4)$$

A similar result holds for $R_{qe}(x)$. Figure 1c shows $\ln_q p_{qe}(x)$ versus x for typical values of x_0.

The q-exponential function given by eq. (1) has been employed in a growing number of theoretical and empirical works on a large variety of themes. Examples include scale-free networks[10–14], dynamical systems[15–27], algebraic structures[28–31] among other topics in statistical physcics[32–36].

As specific examples of q-exponential distributions in complex systems, let us consider results on population of cities[37] and circulation of magazines[38]. Figure 2 shows the cumulative distribution of the population of cities in the USA and Brazil. Figure 3 shows the cumulative distribution of circulation of magazines in the USA and UK. In both cases - population of cities and circulation of magazines - the empirical data are consistent with a q-exponential distribution, with $q \simeq 1.4$.

q-exponential distributions have also been applied in the empirical study of stock markets[39–42], DNA sequences[43], family names[44], human behavior[45–47], geomagnetic records[48, 49], train delays[50], reaction kinetics[51], air networks[52], hydrological phenomena[53], fossil register[54], basketball[55], earthquakes[56–58], world track records[59], voting processes[60], internet[61], individual success[62], citations of scientific papers[63, 64], football[65], linguistics[66, 67] and solar neutrinos[68, 69].

3. q-GAUSSIAN DISTRIBUTION

The q-Gaussian distribution is specified by the pdf

$$p_{qg}(x) = p_0 \left[1 - (1-q) \left(\frac{x}{x_0} \right)^2 \right]^{1/(1-q)}, \tag{5}$$

for $1-(1-q)(x/x_0)^2 \geq 0$ and $p_{qg}(x) = 0$ otherwise. It is normalized if $p_0 = (2/x_0)\sqrt{(q-1)/\pi}\,\Gamma[1/(q-1)]/\Gamma[1/(q-1)-1/2]$. In addition, eq. (5) presents unit variance if $x_0^2 = 5-3q$, with $q < 5/3$.

In the limit $q \to 1$, eq. (5) recovers the usual Gaussian distribution, so $q \neq 1$ indicates a departure from Gaussian statistics. For $q > 1$, the tails of q-Gaussian decrease as power laws,

$$p_{qg}(|x|) \sim |x|^{-2/(q-1)}. \tag{6}$$

Figures 4a and 4b show $p_{qg}(x)$ for typical values of q.

Applying the q-logarithm function in both sides of eq. (5), we have

$$\ln_q p_{qg}(x) = \ln_q p_0 - [1 + (1-q)\ln_q p_0] \left(\frac{x}{x_0} \right)^2. \tag{7}$$

Figure 4c shows $\ln_q p_{qg}(x)$ versus x^2 for typical values of x_0.

Recent works have been focused on the study of mathematical properties of q-Gaussian functions[70–78], including methods for generating random numbers which follow q-Gaussian distributions[79, 80]. q-Gaussians have been employed in the study of a wide range of themes including probabilistic models[81, 82], stellar plasmas[83], porous-medium equation[84], Bose-condensed gases[85–87], dynamical systems[88–90], polymeric networks[91], small-world networks[92], fingering processes[93], processes with stochastic volatility[94, 95] and nonlinear diffusion[96, 97].

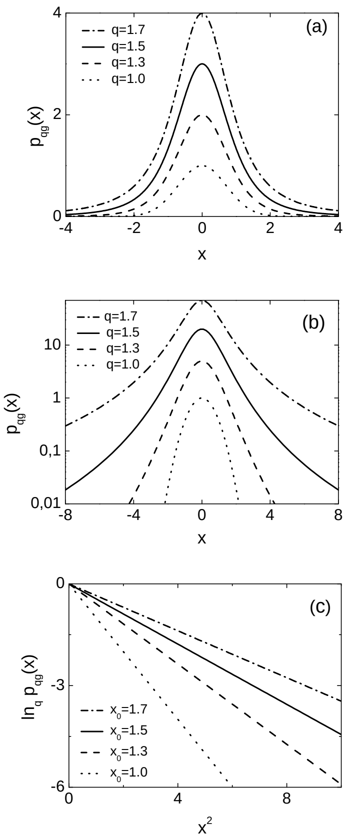

FIG. 4: **q-Gaussian distribution**. a) Plot of $p_{qg}(x)$ versus x, with $p_0 = x_0 = 1$, for typical values of q. Some curves were vertically shifted for a better visualization. b) The same curves shown in a), but for mono-log scale. Some curves were also shifted . c) $\ln_q p_{qg}(x)$ versus x^2 for $p_0 = 1$ and typical values of x_0.

In order to illustrate a recent application of q-Gaussian distributions in complex systems, we mention here results on the dynamics of earthquakes[98]. Figure 5 shows the distribution of energy differences between successive earthquakes at the San Andreas Fault. The empirical data is consistent with a q-Gaussian distribution, with $q = 1.75$.

Other recent applications of q-Gaussian distribution include stock markets[99–107], DNA molecules[108], the solar wind[109–111], galaxies[112], optical lattices[113], cellular aggregates[114] and the atmosphere[115].

4. q-WEIBULL DISTRIBUTION

The q-Weibull distribution is given by the pdf

$$p_{qw}(x) = p_0 \frac{r x^{r-1}}{x_0^r} \left[1 - (1-q) \left(\frac{x}{x_0} \right)^r \right]^{1/(1-q)}, \tag{8}$$

for $1-(1-q)(x/x_0)^r \geq 0$ and $p_{qw}(x) = 0$ otherwise. Eq. (8) is normalized if $p_0 = 2-q$.

In the limits $q \to 1$, $r \to 1$, and $q \to 1$, $r \to 1$, eq. (8) recovers Weibull, q-exponential and exponential distributions, re-

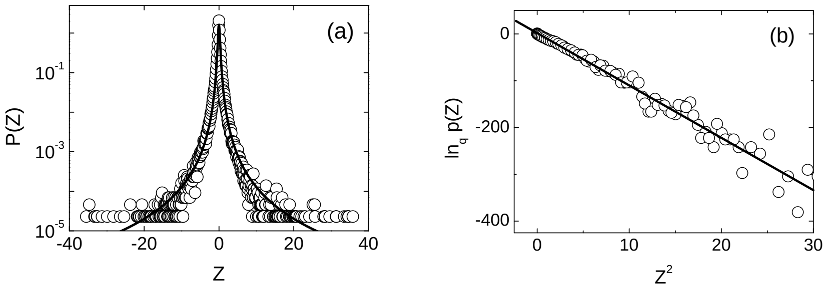

FIG. 5: **Earthquakes**. a) Empirical pdf $P(Z)$, where $Z = E(t+1) - E(t)$ is the energy difference between successive earthquakes at the San Andreas Fault in the period 1966-2006. The solid line is a q-Gaussian, given by eq. (5), with $q = 1.75$, $x_0 = 0.25$ and $p_0 = 1.63$. b) $\ln_q P(Z)$ versus Z^2, with $q = 1.75$, for small values of Z. The solid line is a linear fit to the data.

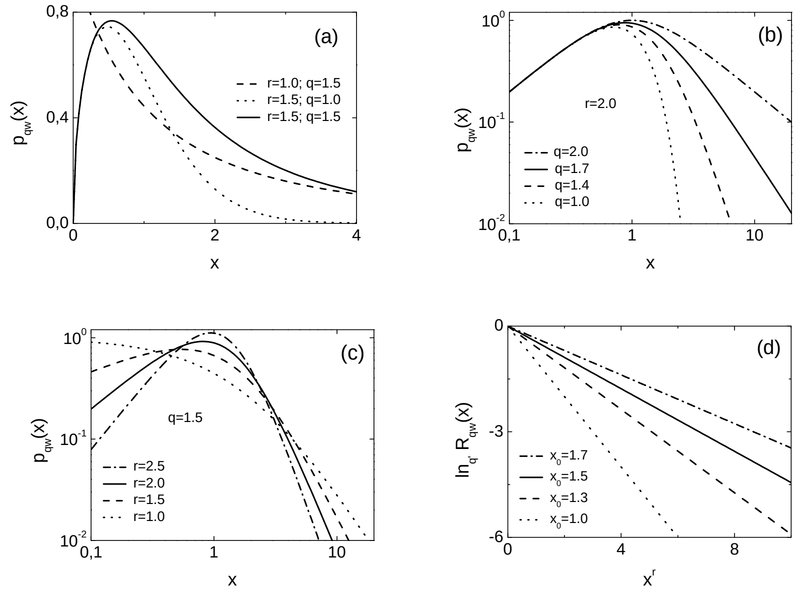

FIG. 6: ***q*-Weibull distribution**. a) Plot of $p_{qw}(x)$ versus x, with $p_0 = x_0 = 1$, and typical values of q and r. b) Log-log plot of $p_{qw}(x)$ versus x, with $p_0 = x_0 = 1$, $r = 2$ and typical values of q. c) a) Log-log plot of $p_{qw}(x)$ versus x, with $p_0 = x_0 = 1$, $q = 1.5$ and typical values of r. d) $\ln_{q'} R_{qw}(x)$ versus x^r for $p'_0 = 1$ and typical values of x_0.

spectively. If $q < 1$, $p_{qw}(x)$ has a finite limit since $p_{qw}(x) = 0$ for $1-(1-q)(x/x_0)^r < 0$. If $q > 1$, $p_{qw}(x)$ exhibits power law behavior both for small and large values of x. More specifically,

$$p_{qw}(x) \sim x^{-\xi}, \tag{9}$$

with $\xi = (1-r)$ for small x and $\xi = r[(2-q)/(q-1)] + 1$ for large x. Figures 6a, 6b and 6c show $p_{qw}(x)$ versus x for typical values of q and r.

The cdf associated to $p_{qw}(x)$ is given by

$$R_{qw}(x) = p'_0 \left[1 - (1-q')\left(\frac{x}{x'_0}\right)^r\right]^{1/(1-q')}, \tag{10}$$

with $q' = 1/(2-q)$, $(x'_0)^r = x_0^r/(2-q)$ and $p'_0 = p_0/(2-q)$. Applying the q-logarithm function in both sides of the cdf R_{qw}, we have

$$\ln_{q'} R_{qw}(x) = \ln_{q'} p'_0 - [1 + (1-q)\ln_{q'} p'_0]\left(\frac{x}{x_0}\right)^r. \tag{11}$$

Figure 6c shows $\ln_{q'} R_{qW}(x)$ versus x^r for typical values of x_0.

If $p_{qw}(x)$ is normalized ($p_0 = 2-q$), Eq. (11) reduces to $\ln_{q'} R_{qw}(x) = -(x/x_0)^r$. In this case,

$$\ln[-\ln_{q'}(R_{qw}(x))] = r\ln x - r\ln x_0. \tag{12}$$

As specific example of q-Weibull distribution in complex systems, we now consider results on citations in scientific journals[116]. Figure 7 shows the distribution of the impact factor of scientific journals in comparison with a q-Weibull curve. The empirical data is consistent with a q-Weibull distribution, with $q = 1.45$ and $r = 1.50$.

Other recent works have been related to q-Weibull distributions. For example, new classes of generalized asymmetric distributions have been introduced which include q-Weibull as a special case[117, 118]. q-Weibull has also been applied in the study of fractal kinetics[119], dieletric breakdown in oxides[120], relaxation in heterogeneous systems[121], ciclone victims and highway lengths[55] among others.

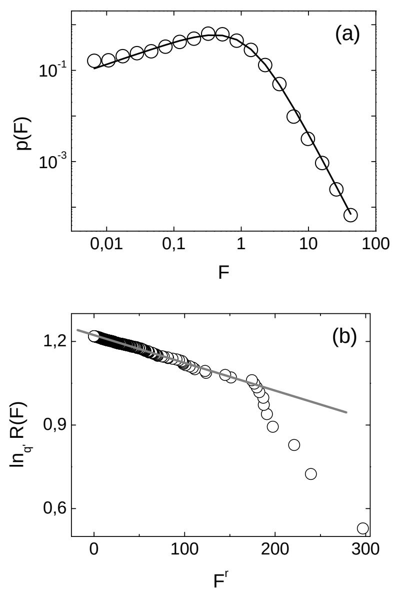

FIG. 7: **Citations in scientific journals.** a) Empirical pdf $p(F)$, where F is the 2004 impact factor for 5912 scientific journals. The solid line is a q-Weibull distribution, given by eq. (8), with $r = 1.5$, $q = 1.45$, $x_0 = 0.74$ and $p_0 = 0.58$. b) $\ln_{q'} R(F)$ versus F^r, with $q' = 1.82$ ($q = 1.45$) and $r = 1.5$. The solid line is a linear fit to the data.

5. BASIS FOR q-DISTRIBUTIONS

¿From the viewpoint of the principle of the maximum entropy, some q-distributions optimizes generalized entropies - more general entropic measures than the standard Boltzmann-Gibbs entropy. A striking example is the q-entropy proposed by Tsallis[1]

$$S_q = \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1}, \tag{13}$$

where W is the total number of microstates of the system, p_i are the occupation probabilities and q is a real parameter. The standard Boltzmann-Gibbs entropy is recovered in the limit $q \to 1$.

The maximization of S_q subject to specific constraints generates occupation probabilities following a q-exponential distribution. The q-exponential optimizes other generalized entropic measures such as the Renyi and normalized Tsallis entropies. However, only Tsallis entropy can provide an appropriate basis for the q-exponential distribution since it presents several properties essential for an entropy[122, 123]. Changing the constraints, the maximization of S_q also generates occupation probabilities following a q-Gaussian distribution.

Formally, q-distributions can arise when the exponential function of the original distribution is replaced by a q-exponential function. For example, this basic procedure applied in standard exponential, Gaussian and Weibull distributions leads to q-exponential, q-Gaussian and q-Weibull, respectively[55]. This viewpoint suggests the consideration of other q-distributions which could be obtained by simply replacing its exponential function by a q-exponential one.

q-distributions can also emerge from compound distributions[124]

$$p_q(x) = \int_0^\infty p(x,\lambda) f(\lambda) d\lambda, \tag{14}$$

where $f(\lambda)$ is a Gamma function. For example, if $p(x,\lambda)$ is a Weibull distribution, $p_q(x)$ is given by a q-Weibull distribution[120]. Naturally, other forms for $f(\lambda)$ may be considered to obtain alternative distributions. In a physical context, this scenario has been explored with success in superstatistics where nonequilibrium situations with local fluctuations of the environment are taken into account[125–127].

The generalized distributions considered here can also be obtained from the following ordinary differential equation:

$$\frac{dy}{dx} = \rho y^q. \tag{15}$$

In fact, if ρ is constant, the solution of eq. (15) is a q-exponential; if $\rho \propto x$, the solution is a q-Gaussian. If y is the cdf and $\rho \propto x^r$, we have a q-Weibull. By considering further terms in eq. (15), other q-distributions can be obtained[128]. q-distributions can also emerge in other contexts. For instance, q-Gaussian arises from the non-linear diffusion (porous media) equation[84] and from a generalization of the central limit theorem[3]. Another example is the q-lognormal distribution which emerges from generalized cascades[28].

6. CONCLUSION

The present work presents a brief overview of recent applications of some q-distributions largely used in the context of Tsallis statistics. It illustrates how q-exponential, q-Gaussian and q-Weibull distributions have been applied in the study of a wide variety of systems in several fields.

The success of q-distributions in describing diverse systems is in part due to its ability of exhibit heavy-tails and model power law phenomena - a typical characteristic of complex systems. The positive and exciting results obtained with q-distributions also indicate possible applications of Tsallis nonextensive statistical mechanics. Naturally, further work may be necessary to explore possible relations between the analyzed systems and the present theory.

[1] C. Tsallis, *Journal of Statistical Physics* **52**, 479 (1988).
[2] C. Tsallis, R. S. Mendes and A. R. Plastino, *Physica A-Statistical Mechanics and its Applications* **261**, 534 (1998).
[3] C. Tsallis, *Brazilian Journal of Physics* **39**, 337 (2009).
[4] C. Tsallis, *Introduction to Nonextensive Statistical Mechanics - Approaching a Complex World*, Springer, New York (2009).

[5] S. Abe and Y. Okamoto, eds., *Nonextensive Statistical Mechanics and Its Applications*, Springer, Berlin (2001).
[6] M. Gell-Mann and C. Tsallis, eds., *Nonextensive Entropy - Interdisciplinary Applications*, Oxford University Press, New Yor (2004).
[7] http://tsallis.cat.cbpf.br/biblio.htm
[8] B. B. Mandelbrot, *The Fractal Geometry of Nature*, Freeman, New York (1977).
[9] I. W. Burr, *Ann. Math. Stat.* **13**, 215 (1942).
[10] C. Tsallis, *European Physical Journal-Special Topics* **161**, 175 (2008).
[11] S. Thurner, F. Kyriakopoulos and C. Tsallis, *Physical Review E* **76**, 036111 (2007).
[12] D. R. White, N. Kejzar, C. Tsallis, D. Farmer and S. White, *Physical Review E* **73**, 016119 (2006).
[13] M. D. S. de Menezes, S. D. da Cunha, D. J. B. Soares and L. R. da Silva, *Progress of Theoretical Physics Supplement* **162**, 131 (2006).
[14] D. J. B. Soares, C. Tsallis, A. M. Mariz and L. R. da Silva, *Europhysics Letters* **70**, 70 (2005).
[15] J. P. Dal Molin, M. A. A. da Silva, I. R. da Silva and A. Caliri, *Brazilian Journal of Physics* **39**, 435 (2009).
[16] M. G. Campo, G. L. Ferri and G. B. Roston, *Brazilian Journal of Physics* **39**, 439 (2009).
[17] H. Hernandez-Saldana and A. Robledo, *Physica A-Statistical Mechanics and its Applications* **370**, 286 (2006).
[18] F. Baldovin, *Physica A-Statistical Mechanics and its Applications* **372**, 224 (2006).
[19] R. Jaganathan and S. Sinha, *Physics Letters A* **338**, 277 (2005).
[20] R. Ishizaki and M. Inoue, *Progress of Theoretical Physics* **114**, 943 (2005).
[21] A. Robledo, *Physics Letters A* **328**, 467 (2004).
[22] A. Robledo, *Physica A-Statistical Mechanics and its Applications* **342**, 104 (2004).
[23] A. Pluchino, V. Latora and A. Rapisarda, *Physica D-Nonlinear Phenomena* **193**, 315 (2004).
[24] Y. Y. Yamaguchi, J. Barre, F. Bouchet, T. Dauxois and S. Ruffo, *Physica A-Statistical Mechanics and its Applications* **337**, 36 (2004).
[25] R. S. Johal and U. Tirnakli, *Physica A-Statistical Mechanics and its Applications* **331**, 487 (2004).
[26] F. Baldovin and A. Robledo, *Physical Review E* **66**, 045104 (2002).
[27] F. Baldovin and A. Robledo, *Europhysics Letters* **60**, 518 (2002).
[28] S. M. D. Queiros, *Brazilian Journal of Physics* **39**, 448 (2009).
[29] P. G. S. Cardoso, E. P. Borges, T. C. P. Lobao and S. T. R. Pinho, *Journal of Mathematical Physics* **49**, 093509 (2008).
[30] D. Strzalka and F. Grabowski, *Modern Physics Letters B* **22**, 1525 (2008).
[31] E. P. Borges, *Physica A-Statistical Mechanics and its Applications* **340**, 95 (2004).
[32] G. D. Magoulas and A. Anastasiadis, *International Journal of Bifurcation and Chaos* **16**, 2081 (2006).
[33] R. Hanel and S. Thurner, *Physica A-Statistical Mechanics and its Applications* **365**, 162 (2006).
[34] R. S. Johal, A. Planes and E. Vives, *Physical Review E* **68**, 056113 (2003).
[35] Q. P. A. Wang, *Physics Letters A* **300**, 169 (2002).
[36] S. Abe and A. K. Rajagopal, *Europhysics Letters* **52**, 610 (2000).
[37] L. C. Malacarne, R. S. Mendes and E. K. Lenzi, *Physical Review E* **65**, 017106 (2002).
[38] S. Picoli, R. S. Mendes and L. C. Malacarne, *Europhysics Letters* **72**, 865 (2005).
[39] M. Politi and E. Scalas, *Physica A-Statistical Mechanics and its Applications* **387**, 2025 (2008).
[40] Z. Q. Jiang, W. Chen and W. X. Zhou, *Physica A-Statistical Mechanics and its Applications* **387**, 5818 (2008).
[41] T. Kaisoji, *Physica A-Statistical Mechanics and its Applications* **370**, 109 (2006).
[42] T. Kaisoji, *Physica A-Statistical Mechanics and its Applications* **343**, 662 (2004).
[43] T. Oikonomou, A. Provata and U. Tirnakli, *Physica A-Statistical Mechanics and its Applications* **387**, 2653 (2008).
[44] H. S. Yamada, *Physica A-Statistical Mechanics and its Applications* **387**, 1628 (2008).
[45] T. Takahashi, H. Oono, T. Inoue, S. Boku, Y. Kako, Y. Kitaichi, I. Kusumi, T. Masui, S. Nakagawa, K. Suzuki, T. Tanaka, T. Koyama and M. H. B. Radford, *Neuroendocrinology Letters* **29**, 351 (2008).
[46] T. Takahashi, H. Oono and M. H. B. Radford, *Physica A-Statistical Mechanics and its Applications* **387**, 2066 (2008).
[47] D. O. Cajueiro, *Physica A-Statistical Mechanics and its Applications* **364**, 385 (2006).
[48] L. F. Burlaga, A. F.-Vinas and C. Wang, *Journal of Geophysical Research* **112**, A07206 (2007).
[49] L. F. Burlaga and A. F.-Vinas, *Journal of Geophysical Research* **110**, A07110 (2005).
[50] K. Briggs and C. Beck, *Physica A-Statistical Mechanics and its Applications* **378**, 498 (2007).
[51] R. K. Niven, *Chemical Engineering Science* **61**, 3785 (2006).
[52] W. Li, Q. A. Wang, L. Nivanen, A. Le Mehaute, *European Physical Journal B* **48**, 95 (2005).
[53] C. J. Keilock, *Advances in Water Resources* **28**, 773 (2005).
[54] T. Shimada, S. Yukawa and N. Ito, *International Journal of Modern Physics C* **14**, 1267 (2003).
[55] S. Picoli, R. S. Mendes and L. C. Malacarne, *Physica A-Statistical Mechanics and its Applications* **324**, 678 (2003).
[56] T. Hasumi, *Physica A-Statistical Mechanics and its Applications* **388**, 477 (2009).
[57] A. H. Darooneh and C. Dadashinia, *Physica A-Statistical Mechanics and its Applications* **387**, 3647 (2008).
[58] S. Abe and N. Suzuki, *Journal of Geophysical Research-Solid Earth* **108**, 2113 (2003).
[59] J. Alvarez-Ramirez, M. Meraz and G. Gallegos, *Physica A-Statistical Mechanics and its Applications* **328**, 545 (2003).
[60] M. L. Lyra, U. M. S. Costa, R. N. Costa Filho and J. S. Andrade Jr., *Europhysics Letters* **62**, 131 (2003).
[61] S. Abe and N. Suzuki, *Physical Review E* **67**, 016106 (2003).
[62] E. P. Borges, *European Physical Journal B* **30**, 593 (2002).
[63] A. D. Anastasiadis, M. P. de Albuquerque and M. P. de Albuquerque, *Brazilian Journal of Physics* **39**, 511 (2009).
[64] C. Tsallis and M. P. Albuquerque, *European Physical Journal B* **13**, 777 (2000).
[65] L. C. Malacarne and R. S. Mendes, *Physica A-Statistical Mechanics and its Applications* **286**, 391 (2000).
[66] L. Egghe, *Journal of the American Society for Information Science* **50**, 233 (1999).
[67] S. Denisov, *Physics Letters A* **235**, 447 (1997).
[68] P. Quarati, A. Carbone, G. Gervino, G. Kaniadakis, A. Lavagno and E. Miraldi, *Nuclear Physics A* **621**, 345 (1997).
[69] G. Kaniadakis, A. Lavagno and P. Quarati, *Physics Letters B* **369**, 308 (1996).
[70] S. Umarov, C. Tsallis and S. Steinberg, *Milan Journal of Mathematics*, **76** 307 (2008).
[71] C. Vignat and A. Plastino, *Physics Letters A* **365**, 370 (2007).
[72] H. Suyari and M. Tsukada, *IEEE Transactions on Information Theory* **51**, 753 (2005).
[73] E. Ricard, *Communications in Mathematical Physics* **257**,

659 (2005).
[74] A. Nou, *Mathematische Annalen* **330**, 17 (2004).
[75] N. Saitoh and H. Yoshida, *Journal of Mathematical Physics* **41**, 5767 (2000).
[76] M. Marciniak, *Studia Mathematica* **129**, 113 (1998).
[77] M. Bozejko, B. Kummerer and R. Speicher, *Communications in Mathematical Physics* **185**, 129 (1997).
[78] H. vanLeeuwen and H. Maassen, *Journal of Physics A-Mathematical and General* **29**, 4741 (1996).
[79] W. J. Thistleton, J. A. Marsh, K. Nelson and C. Tsallis, *IEEE Transactions on Information Theory* **53**, 4805 (2007).
[80] C. Anteneodo, *Physica A-Statistical Mechanics and its Applications* **358**, 289 (2005).
[81] A. Rodrigues, V. Schwammle and C. Tsallis, *Journal of Statistical Mechanics-Theory and Experiment* P09006 (2008).
[82] J. A. Marsh, M. A. Fuentes, L. G. Moyano and C. Tsallis, *Physica A-Statistical Mechanics and its Applications* **372**, 183 (2006).
[83] F. Caruso, A. Pluchino, V. Latora, S. Vinciguerra and A. Rapisarda, *Physical Review E* **75**, 055101 (2007).
[84] V. Schwammle, F. D. Nobre and C. Tsallis, *European Physical Journal B* **66**, 537 (2008).
[85] A. L. Nicolin and R. Carretero-Gonzalez, *Physica A-Statistical Mechanics and its Applications* **387**, 6032 (2008).
[86] E. Erdemir and B. Tanatar, *Physica A-Statistical Mechanics and its Applications* **322**, 449 (2003).
[87] K. S. Fa, R. S. Mendes, P. R. B. Pedreira and E. K. Lenzi, *Physica A-Statistical Mechanics and its Applications* **295**, 242 (2001).
[88] U. Tirnakli, C. Beck and C. Tsallis, *Physical Review E* **75**, 040106 (2007).
[89] A. Pluchino, A. Rapisarda and C. Tsallis, *EPL* **80**, 26002 (2007).
[90] L. G. Moyano and C. Anteneodo, *Physical Review E* **74**, 021118 (2006).
[91] L. C. Malacarne, R. S. Mendes, E. K. Lenzi, S. Picoli and J. P. Dal Molin, *European Physical Journal E* **20**, 395 (2006).
[92] H. Hasegawa, *Physica A-Statistical Mechanics and its Applications* **365**, 383 (2006).
[93] P. Grosfils and J. P. Boon, *Europhysics Letters* **74**, 609 (2006).
[94] S. M. D. Queiros and C. Tsallis, *European Physical Journal B* **48**, 139 (2005).
[95] S. M. D. Queiros and C. Tsallis, *Europhysics Letters* **69**, 893 (2005).
[96] C. Tsallis and D. J. Bukman, *Physical Review E* **54**, R2197 (1996).
[97] A. R. Plastino and A. Plastino, *Physica A-Statistical Mechanics and its Applications* **222**, 347 (1995).
[98] F. Caruso, A. Pluchino, V. Latora, S. Vinciguerra and A. Rapisarda, *Physical Review E* **75**, 055101 (2007).
[99] N. Gradojevic and R. Gencay, *Economics Letters* **100**, 27 (2008).
[100] M. Kozaki and A. H. Sato, *Physica A-Statistical Mechanics and its Applications* **387**, 1225 (2008).
[101] T. S. Biro and R. Rosenfeld, *Physica A-Statistical Mechanics and its Applications* **387**, 1603 (2008).
[102] A. A. G. Cortines, R. Riera and C. Anteneodo, *European Physical Journal B* **60**, 385 (2007).
[103] S. Drozdz, M. Forczek, J. Kwapien, P. Oswiecimka and R. Rak, *Physica A-Statistical Mechanics and its Applications* **383**, 59 (2007).
[104] R. Rak, S. Drozdz and J. Kwapien, *Physica A-Statistical Mechanics and its Applications* **374**, 315 (2007).
[105] L. Zunino, B. M. Tabak, D. G. Perez, M. Caravaglia and O. A. Rosso, *European Physical Journal B* **60**, 111 (2007).
[106] F. M. Ramos, C. Rodrigues Neto, R. R. Rosa, L. D. Abreu S and M. J. A. Bolzam, *Nonlinear Analysis* **23**, 3521 (2001).
[107] F. Michael and M. D. Johnson, *Physica A-Statistical Mechanics and its Applications* **320**, 525 (2003).
[108] D. A. Moreira, E. L. Albuquerque, L. R. da Silva and D. S. Galvao, *Physica A-Statistical Mechanics and its Applications* **387**, 5477 (2008).
[109] L. F. Burlaga, A. F. Vinas, N. F. Ness and M. H. Acuna, *Astrophysical Journal* **644**, L83 (2006).
[110] M. P. Leubner and Z. Voros, *Astrophysical Journal* **618**, 547 (2005).
[111] M. P. Leubner and Z. Voros, *Nonlinear Processes in Geophysics* **12**, 171 (2005).
[112] A. Nakamichi and M. Morikawa, *Physica A-Statistical Mechanics and its Applications* **341**, 215 (2004).
[113] J. Jersblad, H. Ellmann, K. Stochkel, A. Kastberg, L. Sanchez-Palencia and R. Kaiser, *Physical Review A* **69**, 013410 (2004).
[114] A. Upadhyaya, J. P. Rieu, J. A. Glazier and Y. Sawada, *Physica A-Statistical Mechanics and its Applications* **293**, 549 (2001).
[115] E. Yee, P. R. Kosteniuk, G. M. Chandler, C. A. Biltoft and J. F. Bowers, *Boundary-Layer Meteorology* **66**, 127 (1993).
[116] S. Picoli, R. S. Mendes, L. C. Malacarne and E. K. Lenzi, *Europhysics Letters* **75**, 673 (2006).
[117] K. K. Jose and S. R. Naik, *Physica A-Statistical Mechanics and its Applications* **387**, 6943 (2008).
[118] A. M. Mathai, *Linear Algebra and Its Applications* **396**, 317 (2005).
[119] F. Brouers and O. Sotolongo-Costa, *Physica A-Statistical Mechanics and its Applications* **368**, 165 (2006).
[120] U. M. S. Costa, V. N. Freire, L. C. Malacarne, R. S. Mendes, S. Picoli, E. A. de Vasconcelos and E. F. da Silva, *Physica A-Statistical Mechanics and its Applications* **361**, 209 (2006).
[121] F. Brouers and O. Sotolongo-Costa, *Physica A-Statistical Mechanics and its Applications* **356**, 359 (2005).
[122] S. Abe, *Physica D-Nonlinear Phenomena* **193**, 84 (2004).
[123] S. Abe, *Physical Review E* **66**, 046134 (2002).
[124] N. L. Johnson and S. Kotz, *Wiley Series in Probability and Mathematical Statistics: Continuous Univariate Distributions - 1*, John Wiley and Sons, New York (1970).
[125] C. Beck, *Brazilian Journal of Physics* **39**, 357 (2009).
[126] C. Beck, *Physical Review Letters* **87**, 180601 (2001).
[127] G. Wilk and Z. Wlodarczyk, *Physical Review Letters* **84**, 2270 (2000).
[128] C. Tsallis, G. Bemski and R. S. Mendes, *Physics Letters A* **257**, 93 (1999).