

Some properties of deformed q -numbers

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Nonextensive statistical mechanics has been a source of investigation in mathematical structures such as deformed algebraic structures. In this work, we present some consequences of q -operations on the construction of q -numbers for all numerical sets. Based on such a construction, we present a new product that distributes over the q -sum. Finally, we present different patterns of q -Pascal's triangles, based on q -sum, whose elements are q -numbers.

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1. INTRODUCTION

The q -operations [1, 2] that emerge from nonextensive statistical mechanics [3] seem to provide a natural background for its mathematical formulation. The definitions of q -sum and q -product, on the realm of real numbers,

$$x \oplus_q y := x + y + (1 - q)xy, \quad (1)$$

$$x \otimes_q y := [x^{1-q} + y^{1-q} - 1]_+^{\frac{1}{1-q}}, \quad x > 0, y > 0, \quad (2)$$

where $[p]_+ = \max\{p, 0\}$, allow some expressions of nonextensive statistical mechanics to be written with the same formal simplicity of the extensive ($q = 1$) formalism. For instance, the q -logarithm [4] of a product, and the q -exponential of a sum are written as

$$\ln_q xy = \ln_q x \oplus_q \ln_q y,$$

$$e_q^{x+y} = e_q^x \otimes_q e_q^y,$$

with

$$\ln_q x := \frac{x^{1-q} - 1}{1 - q}, \quad x > 0 \quad (3)$$

and

$$e_q^x := [1 + (1 - q)x]_+^{\frac{1}{1-q}}. \quad (4)$$

The q -sum and the q -product are associative, commutative, present neutral element (0 for q -sum and 1 for q -product) and opposite and inverse elements under restrictions. A reasonable question is whether those operations provide a structure

of commutative ring or even field. Since the q -product does not distribute over the q -sum, they do not define those algebraic structures.

There are instances of other structures that are distributive, though do not present other properties. For instance, the tropical algebra [5] — for which the T -sum of two extended real numbers ($\mathbb{R} \cup \{-\infty\}$) is the minimum between them and the T -product is the usual sum — does not have reciprocal elements in relation to the T -sum.

On the other hand, the relevant structure of a near-ring [6] is an example of a non-distributive ring; however, in this case, distributivity is required in at least one side. It has been known from long ago, as pointed out by Green [7], that practical examples of (both sides) of non-distributive algebraic structures are not so easy to find out. So the q -algebraic structure is a good example of both-side non-distributive structure.

Recently, we have generalized the q -algebraic structure into a biparametrized (q, q') -algebraic structure (and, more generally, into an n parameter algebraic structure) [8], in such a way that the two-parameter operators (q, q') -sum, (q, q') -product, and their inverses, present the same properties of the monoperameterized q -algebraic structure.

A remarkable feature of these algebraic structures is that the distributivity property does not hold. Though this “non-property” is very interesting, there are some proposals in the literature [1, 9] (that will be shown later) which change somehow the q -algebraic structure in order to recover distributivity. In all of those proposals [1, 2, 9], the operations are deformed but the numbers are not. In this work, we deform the numbers to obtain the q -numbers x_q for all numerical sets based on q -sum in such way that

$$x_q \oplus_q y_q = (x + y)_q. \quad (5)$$

Since the q -product is, in a sense that we will discuss later, intrinsically non-distributive, in order to obtain the distributive structure in a very natural way, we keep the q -sum and propose a new product such that

$$x_q \diamond_q y_q = (xy)_q. \quad (6)$$

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We also set up the a -numbers and k -numbers based on other deformed sums presented in [1, 9]. We call the attention to the interesting connection between the q -natural number and the Heine number [10]. Other mathematical objects, whose elements are q -numbers, may be generated by deformed operations; we exemplify some q -Pascal's triangles, derived by q -sum, that correspond to different patterns.

The paper is organized as follows: Sec. 2 introduces the q -numerical sets; Sec. 3 proposes a different product \diamond_q ; other mathematical objects as q -Pascal's triangles are addressed in Sec. 4. Finally, in Sec. 5 we draw our concluding remarks.

2. THE q -NUMERICAL SETS

The main idea is to use the classical construction of the numerical sets [11] for which elements are the respective deformed numbers. We use the notation $\mathbb{N}_q, \mathbb{Z}_q, \mathbb{Q}_q, \mathbb{R}_q$ for q -natural, q -integer, q -rational and q -real numerical sets respectively.

Consider an induction over an arbitrary generator g (that we assume different from 0 and $-1/(1-q)$ to avoid trivial structures) q -summed n times:

$$\begin{aligned} g & \\ g \oplus_q g &= 2g + (1-q)g^2 \\ g \oplus_q g \oplus_q g &= 3g + 3(1-q)g^2 + (1-q)^2g^3 \\ &\vdots \\ \underbrace{g \oplus_q \cdots \oplus_q g}_{n \text{ times}} &= \frac{[1 + (1-q)g]^n - 1}{1-q}. \end{aligned} \quad (7)$$

For simplicity of the expressions in this note, we shall choose $g = 1$, and obtain the deformed q -natural number summed n times:

$$\begin{aligned} n_q &= \underbrace{1 \oplus_q + \cdots + \oplus_q 1}_{n \text{ times}} \\ &= \frac{(2-q)^n - 1}{1-q} \\ &= \sum_{k=1}^n \binom{n}{k} (1-q)^{k-1}, \end{aligned} \quad (8)$$

where $\binom{n}{k}$ stands for the binomial coefficients. The q -neutral element is the same as the usual one, $0_q = 0$ ($n_q \oplus_q 0_q = n_q$), and also $1_q = 1$. Of course $n_q \rightarrow n$ as $q \rightarrow 1$.

The dependence on the parameter q provides a plethora of interesting different structures. For instance, with $q = 2$, we have a structure given by $\{1\}$; with $q = 3$, we have a structure isomorphic to the finite field with two elements $\{0, 1\}$. However, if $q < 2$, we have infinite structures whose elements are all real numbers (if complex numbers are allowed, there is more freedom on the parameter q).

It is not difficult to verify that the set $\mathbb{N}_q = \{n_q, n \in \mathbb{N}\}$ with the map $\sigma : \mathbb{N}_q \rightarrow \mathbb{N}_q, n_q \mapsto n_q \oplus 1_q$ is a model for the Peano axioms. Let us show, for example, some elements of the set \mathbb{N}_q for $q = 0$:

$$\mathbb{N}_0 = \{0, 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, \dots\}.$$

From the set of the deformed q -natural number, we may construct, as in the classical way, by means of the (difference) equivalence relation on $\mathbb{N}_q \times \mathbb{N}_q$, the set of deformed q -integer numbers \mathbb{Z}_q . We also draw some elements of this set for $q = 0$:

$$\mathbb{Z}_0 = \left\{ \dots, -\frac{127}{128}, -\frac{63}{64}, -\frac{31}{32}, -\frac{15}{16}, -\frac{7}{8}, -\frac{3}{4}, -\frac{1}{2}, 0, 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, \dots \right\}.$$

It is interesting to note that, in any case, the q -integers are strictly greater than $-1/(1-q)$.

The q -integer numbers were also studied by R. Cardo and A. Corvolan [12] based on the \odot_q operation introduced in [2]: $n_q = n \odot 1$, which is defined in the same way as the notion we introduced in (7) by induction.

Analogously, we have also constructed, as in the classical case, the deformed q -rational numbers \mathbb{Q}_q , by an (ratio) equivalence relation on $\mathbb{Z}_q \times \mathbb{Z}_q^*$, and the q -real numbers, \mathbb{R}_q , by Cauchy sequences. It would also be possible to construct the q -real numbers using the Dedekind cuts.

We have proved that, following the classical construction of the q -real numbers, they are given by

$$x_q = \frac{(2-q)^x - 1}{1-q}. \quad (9)$$

The asymptotical behavior of x_q ($x \rightarrow \infty$) is given by:

$$\lim_{x \rightarrow \infty} x_q = \begin{cases} \infty, & q < 1 \\ x, & q = 1 \\ \frac{1}{q-1}, & 1 < q \leq 2. \end{cases} \quad (10)$$

For $q > 2$, x_q may assume complex values.

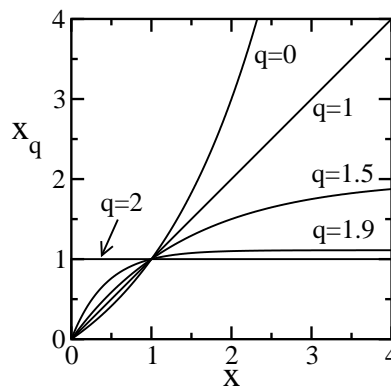


FIG. 1: q -Real number x_q versus x for some typical values of q .

It is amazing to note that in the study of the q -analogues of the hypergeometric series [13], at the second half of the nineteenth century, Heine introduced the deformed number [10]

$$[n]_H = \frac{H^n - 1}{H - 1}, \quad (11)$$

known as the q -analogue of n . The number deformation plays a fundamental role in combinatorics, but also has applications

in the study of fractals, hyperbolic geometry, chaotic dynamical systems, quantum groups, etc. There are also many physical applications, for instance, in exact models in statistical mechanics. It is interesting that the deformed q -number (9) is exactly the Heine number, by the simple change of variables $q = 2 - H$. The connection between nonextensive statistical mechanics and the Heine number (and quantum groups) was already pointed out in [14]. It is worth to note that the coincidence of the symbol q in all these different contexts (q -series, q -analogues, quantum groups, and q -entropy) occurs just by chance.

It is possible to define other generalized numbers, based on the algebraic structures proposed on [1, 9]. In [1], two operations a -sum ($+^a$ and $+_a$) and a -product (\times^a and \times_a) were introduced. The a -sums are, respectively,

$$x +_a y := x \oplus_q y \text{ with } q = 1 - a, \tag{12}$$

$$x +^a y := \left\{ a \ln \left[\exp \left(\frac{x^a}{a} \right) + \exp \left(\frac{y^a}{a} \right) \right] \right\}^{1/a}. \tag{13}$$

The a -products are, respectively,

$$x \times_a y := x \otimes_q y \text{ with } q = 1 - a, \tag{14}$$

$$x \times^a y := \frac{\exp[\ln(1+ax)\ln(1+ay)/a] - 1}{a}. \tag{15}$$

Based on (13), we obtain the deformed $x^{(a)}$ number with generator g :

$$x^{(a)} = [a \ln x + g^a]^{1/a}. \tag{16}$$

In [9], two operations k -sum (\boxplus^k and \boxplus_k) and k -product (\boxtimes^k and \boxtimes_k) were proposed. The k -sums are, respectively,

$$x \boxplus_k y := x \otimes_q y \text{ with } q = 1 - k, \tag{17}$$

$$x \boxplus^k y := \frac{[(1+kx)^{1/k} + (1+ky)^{1/k}]^k - 1}{k}. \tag{18}$$

The k -products are, respectively

$$x \boxtimes_k y := x \oplus_{q=k-1} y \text{ with } q = 1 - k, \tag{19}$$

$$x \boxtimes^k y := \left[\frac{(xy)^k - x^k - y^k + (k+1)}{k} \right]^{1/k}. \tag{20}$$

Based on (17) and (18), the deformed numbers with generator g , $x^{[k]}$ and $x_{[k]}$, associated to \boxplus^k and \boxplus_k , respectively, are:

$$x^{[k]} = \frac{x^k(1+kg) - 1}{k}, \tag{21}$$

$$x_{[k]} = [xg^k - (x-1)]^{1/k}. \tag{22}$$

3. DISTRIBUTIVE PROPERTY

The q -product is non-distributive, i.e.,

$$x \otimes_q (y+z) \neq (x \otimes_q y) + (x \otimes_q z), \forall x \neq 0, 1, \forall q \in \mathbb{R} - \{1\}. \tag{23}$$

As an essential result for our work, we observe that, assuming a set with more than one element and keeping reasonable properties such as the additive neutral element and cancellation to sum, then there is no deformed sum that is distributed by the q -product. In fact:

Let t be the neutral element of such a sum. If we impose the distributive property:

$$x \otimes_q (y \oplus t) = (x \otimes_q y) \oplus (x \otimes_q t)$$

$$x \otimes_q y = (x \otimes_q y) \oplus (x \otimes_q t),$$

thus $t = x \otimes_q t$, using (2), we obtain

$$x^{1-q} = 1,$$

i.e., x has to be one of the complex roots $1^{1/(1-q)}$; so, restricted to real numbers, x has to be 1. Since x is any element, the set has just one element.

Therefore the non-distributivity is an intrinsic property of the q -product. Some authors [1, 9] tried to obtain distributive structures based on q -operations. For instance, note that, although the operation \times_a is distributive over $+^a$, shown in (13), $+^a$ does not have neutral element, as it was consistent with the above result. Moreover \times_a , shown in (15), is distributive over $+_a$.

Concerning the k -sums and the k -products, \boxtimes_k is distributive over \boxplus^k , shown in (18), as well as \boxtimes^k , shown in (20), is distributive over \boxplus_k . Note that the distributivity results from the curious exchange of roles of the operations: the k -sum \boxplus_k is indeed a q -product, and the k -product \boxtimes_k is a q -sum.

Since there is no deformed sum that is distributed by the q -product, we propose a new product, signed \diamond_q , that emerges naturally from the classical construction of the numerical set, just mentioned. This new product is different from equations (15) and (20), and distributes over the q -sum. It is defined as

$$x \diamond_q y := \frac{(2-q)^{\frac{\{\ln[(1+(1-q)x]\ln[1+(1-q)y]\}}{\ln(2-q)^2\}} - 1}}{1-q}. \tag{24}$$

Moreover the q -sum and the q -product obey, respectively, (5) and (6). For the q -sum, we have:

$$x_q \oplus_q y_q = \frac{(2-q)^x + (2-q)^y - 2}{1-q} + \frac{[(2-q)^x - 1][(2-q)^y - 1]}{1-q} \tag{25}$$

$$= \frac{(2-q)^{x+y} - 1}{1-q} = (x+y)_q. \tag{26}$$

For the q -product, we have

$$x_q \diamond_q y_q = \frac{(2-q)^{\frac{\{\ln[(2-q)^x]\ln[(2-q)^y]\}}{\ln(2-q)^2\}} - 1}{1-q} = \frac{(2-q)^{xy} - 1}{1-q} = (xy)_q. \tag{27}$$

It is obvious that, when $q \rightarrow 1$, $x_1 \diamond_1 y_1 = (xy)_1 = xy$.

Using (25) and (27), it is easy to prove that the \diamond_q product distributes over the q -sum when applied to q -numbers, i.e.:

$$x_q \diamond_q (y_q \oplus_q z_q) = [x_q \diamond_q y_q] + [x_q \diamond_q z_q]. \quad (28)$$

In other words,

$$\begin{aligned} x_q \diamond_q (y_q \oplus_q z_q) &= x_q \diamond_q (y + z)_q \\ &= [x(y + z)]_q = [xy + xz]_q \\ &= (xy)_q \oplus_q (xz)_q \\ &= (x_q \diamond_q y_q) \oplus_q (x_q \diamond_q z_q). \end{aligned} \quad (29)$$

4. q -PASCAL'S TRIANGLES

The deformations of operations and numbers open questions about other mathematical objects derived from them. An interesting class of those objects is set up by the Pascal's triangles. Recently, some works have connected nonextensive statistical mechanics with Leibnitz [15] and Pascal's triangles derived from the q -product [16].

In order to exhibit some simple applications of such deformations, in this section we construct q -Pascal's triangles using q -sum as the deformed operation. In this way, their elements are q -numbers. Different patterns are illustrated for

different values of q . For example, we present q -Pascal's triangles for $q = 0$, $q = 1.5$, $q = 2$ and $q = 3$:

a) Increasing pattern

For $q = 0$, we obtain:

| | | | | | | | | | | | | | | | | | | | | | | |
|--|--|--|--|---|---|---|---|---|---|----|----|----|------|-------|------|---------|----|-------|---|----|--|---|
| | | | | 1 | | | | | | | | | | | | | | | | | | |
| | | | | | 1 | | 1 | | | | | | | | | | | | | | | |
| | | | | | | 1 | | 3 | | 1 | | | | | | | | | | | | |
| | | | | | | | 1 | | 7 | | 7 | | 1 | | | | | | | | | |
| | | | | | | | | 1 | | 15 | | 63 | | 15 | | 1 | | | | | | |
| | | | | | | | | | 1 | | 31 | | 1023 | | 1023 | | 31 | | 1 | | | |
| | | | | | | | | | | 1 | | 63 | | 32767 | | 1048575 | | 32767 | | 63 | | 1 |

Note that this increasing pattern occurs for any value of $q \leq 1$. For $q = 1$, we recover the usual Pascal's triangle. It is compatible with the divergent curve shown in figure 1 for natural values of x .

b) Asymptotical pattern

For $q = 1.5$, we obtain:

| | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|--|--|--|--|--|--|--|--|--|---|---|---|---|-----|------|-------|-------|-------|-------|-------|-------|-------|-------|-------|---|---|-------|---|-------|---|---|-------|--|---|
| | | | | | | | | | 1 | | | | | | | | | | | | | | | | | | | | | | | | |
| | | | | | | | | | | 1 | | 1 | | | | | | | | | | | | | | | | | | | | | |
| | | | | | | | | | | | 1 | | 1.5 | | 1 | | | | | | | | | | | | | | | | | | |
| | | | | | | | | | | | | 1 | | 1.75 | | 1.75 | | 1 | | | | | | | | | | | | | | | |
| | | | | | | | | | | | | | 1 | | 1.875 | | 1.968 | | 1.875 | | 1 | | | | | | | | | | | | |
| | | | | | | | | | | | | | | 1 | | 1.937 | | 1.998 | | 1.998 | | 1.937 | | 1 | | | | | | | | | |
| | | | | | | | | | | | | | | | 1 | | 1.968 | | 1.999 | | 1.999 | | 1.968 | | 1 | | | | | | | | |
| | | | | | | | | | | | | | | | | 1 | | 1.984 | | 1.999 | | 2 | | 2 | | 1.999 | | 1.984 | | 1 | | | |
| | | | | | | | | | | | | | | | | | 1 | | 1.992 | | 2 | | 2 | | 2 | | 2 | | 2 | | 1.992 | | 1 |

c) Fixed pattern

For $q = 2$, we obtain:

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