Langevin Simulations with Colored Noise and Non-Markovian Dissipation

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The nonequilibrium dynamics of an homogeneous scalar field is studied using Langevin equations. Microscopic derivations based on quantum field theory methods can lead to complicated nonlocal equations of motion. Here we study, numerically, the results obtained by appropriately approximating these equations in a local form (the Markovian approximation) and compare with results obtained with suitable prescriptions for accounting for the nonlocal terms, *i.e.* the non-Markovian form. We use a prescription for the nonlocal equations motivated by the results obtained from previous derivations using nonequilibrium quantum field theory methods.

Keywords: Langevin equation; Non-Markovian; Colored noise

1. INTRODUCTION

Nonequilibrium dynamics is expected to happen in many important physical problems. Systems of particular interest are in the physics of heavy-ion collisions, cosmology and condensed matter physics. In the context of cosmology, we have interest in early universe scenarios. Nonequilibrium methods are being applied to get a quantitative understanding of the theory of reheating with the aim of explaining the change to radiation phase of the universe after inflation [1].

In the context of heavy-ion collisions, with the recent experiments in the RHIC concerning the possibility of formation of a quark-gluon plasma [2], it is expected that the chiral fields should evolve under extreme conditions of temperature and energy density during the QCD phase transition and the out-equilibrium evolution for the fields becomes an important issue. To have a clear understanding of data coming from BNL-RHIC, and especially of data that will be produced at CERN-LHC, one needs a realistic description of the hierarchy of scales associated with dissipation, noise and radiation. Also, the expansion and finite size of the system must be considered. The study and understanding of all these processes mentioned above require the use of nonequilibrium quantum field theory methods [3].

In the context of condensed matter physics, many efforts have been devoted to get a better understanding of quantum many body dynamics, e.g in laboratory experiments of ultracold quantum Bose/Fermi gases [4]. One interesting question in that context is about the role of quantum fluctuations on the dynamics of scalar fields, which are usually neglected when we apply the classical field theory approximation given by the Gross-Pitaevskii equation [5].

Recently, a nonperturbative description of nonequilibrium quantum fields based on the two-particle irreducible (2PI) effective action formalism has proven very powerful with a wide range of applications [6]. In this work we study the nonequilibrium dynamics making use of numerical methods based on simulations of Langevin equations in a lattice. In our approach, the interaction with the environment is modeled by noise and dissipation terms and these terms are considered as local (Markovian) ones. However, microscopic derivations based on quantum field theory methods lead to complicated nonlocal equations of motion [7, 8].

Here, we study numerically the Langevin dynamics for an homogeneous $\lambda\phi^4$ theory. Since there is an immense saving of effort as well as much more transparent understanding of the physics from a local equation as opposed to a nonlocal one, since the former can generally be analyzed with much less numerical treatment than the latter, thus it is a very important question when and how accurately the generally nonlocal effective equations can be approximated by a local form. Numerical results from simulations for a specific model are shown here and the local (Markovian) and nonlocal (non-Markovian) equations are then compared for different region of parameters. We expect that these results and methods presented here can be useful in the problems being considered in the context of the RHIC physics.

2. CLASSICAL LANGEVIN EQUATION

Typical stochastic evolution is well represented by the problem of the classical brownian motion, whose properties can be represented by a phenomenological equation of the form

$$\ddot{\phi}(t) + \eta\dot{\phi}(t) + V'(\phi) = \xi(t) , \qquad (1)$$

where ϕ is a variable of the system (for example the coordinate of a particle) in interaction with a thermal bath, effectively modeled by a friction term, of intensity η, and stochastic noise $\xi(t)$. The stochastic noise $\xi(t)$, in its simplest realization, is considered as Gaussian and white and two-point correlation function given according to the classical Fluctuation-Dissipation theorem:

$$\langle\xi(t)\rangle = 0 , \quad \langle\xi(t)\xi(t')\rangle = 2T\eta\delta(t-t') . \qquad (2)$$

Approaches with Langevin equations like Eq. (1) and its generalizations, are used in different contexts, e.g. in classical statistical mechanics to study problems with dissipation and noise, to determine how order parameters equilibrate and to understand the dynamics in critical phenomena.

3. THE NONLOCAL NOISE AND DISSIPATION KERNELS IN QUANTUM FIELD THEORY

In the realm of quantum field theory, explicit derivations of effective equations of motion for background fields (used for example in the determination of the dynamics of an order parameter in the theory, e.g. for studying phase transitions), show that the effective dynamics for fields are determined by complicate integro-differential equations [7, 9, 10], where both dissipation and noise are determined by nonlocal (non-Markovian) kernels. This is exemplified by a result obtained in a scalar field theory with a $\lambda\phi^4$ interacting potential and in a thermal bath at temperature T. For an homogeneous field configuration, computed in a two-loop calculation in a quasi-particle approach for the scalar field propagators, it can be shown that the noise correlation function is found to be given by [7, 8]

$$\langle \xi_1(\mathbf{x},t)\xi_1(\mathbf{x}',t')\rangle = K(t-t')\,\delta(\mathbf{x}-\mathbf{x}')\,, \qquad (3)$$

where

$$
\begin{aligned}
K(t-t') &= \frac{\lambda^2}{2}\int \frac{d^3q}{(2\pi)^3}\frac{e^{-2\Gamma(q)|t-t'|}}{4\omega^2(q)}\Big\{\,2n(\omega)\,[1+n(\omega)] \\
&\quad + \,[1+2n(\omega)+2n^2(\omega)]\cos[2\omega|t-t'|] \\
&\quad + \,2\beta\Gamma(q)n(\omega)[1+n(\omega)][1+2n(\omega)] \\
&\quad \times\,\sin[2\omega|t-t'|]\,\Big\}\,,
\end{aligned}
\qquad (4)
$$

with $n(\omega) = [\exp(\beta\omega)-1]^{-1}$ is the Bose-Einstein distribution, $\beta = 1/T$, ω is the dispersion relation given in terms of the momentum $q \equiv |\mathbf{q}|$ and $\Gamma(q)$ is the thermal decay width for the scalar field ϕ. Differently than the simplest Langevin equation, where noise is considered white (Markovian), like in Eq. (2), here, we see from Eq. (4) that noise is non-Markovian in time.

4. NON-MARKOVIAN EQUATION OF MOTION

Our goal is to simulate a non-Markovian Langevin-like equation of motion, whose noise kernel is in a form similar to Eq. (4). We consider an homogeneous scalar field configuration, $\phi \equiv \phi(t)$, with equation of motion

$$\ddot{\phi}+V'(\phi)+\int_{t_0}^{t}dt'K(t-t')\dot{\phi}(t') = \xi(t)\,, \qquad (5)$$

where t_0 is the initial time,

$$V'(\phi) = \Omega_0^2\phi+\frac{\lambda}{3!}\phi^3\,, \qquad (6)$$

and the nonlocal kernel is given by

$$
\begin{aligned}
K(t-t') &= e^{-\frac{\gamma}{2}(t-t')}\frac{Q\Omega_0^2}{\gamma}\Big\{\cos[\Omega_1(t-t')]+ \\
&\quad + \frac{\gamma}{2\Omega_1}\sin[\Omega_1(t-t')]\Big\}\,,
\end{aligned}
\qquad (7)
$$

where γ, Q and Ω_0 are parameters and $\Omega_1^2 = \Omega_0^2 - \gamma^2/4 > 0$.

In Eq. (5) $\xi(t)$ is a non-Markovian Gaussian noise satisfying

$$\langle\xi(t)\rangle = 0\,, \quad \langle\xi(t)\xi(t')\rangle = TK(t-t')\,. \qquad (8)$$

It can be easily shown that this noise can be generated by the following differential equation [11]:

$$\ddot{\xi}(t)+\gamma\dot{\xi}(t)+\Omega_0^2\xi(t) = \Omega_0^2\sqrt{2TQ}\zeta(t)\,, \qquad (9)$$

$\zeta(t)$ in Eq. (9) is a white Gaussian noise satisfying

$$\langle\zeta(t)\rangle = 0\,, \quad \langle\zeta(t)\zeta(t')\rangle = \delta(t-t')\,. \qquad (10)$$

It can be noted that the general solution of Eq. (9) is given by

$$
\begin{aligned}
\xi(t) &= C_1 e^{-\frac{\gamma}{2}t}\cos(\Omega_1 t)+C_2 e^{-\frac{\gamma}{2}t}\sin(\Omega_1 t) \\
&\quad + \frac{e^{-\frac{\gamma}{2}t}}{\Omega_1}\Omega_0^2\sqrt{2QT}\Big\{\Big[\int dt'\cos(\Omega_1 t') \\
&\quad \times\,\zeta(t')\,e^{-\frac{\gamma}{2}t'}\Big]\sin(\Omega_1 t)-\Big[\int dt''\sin(\Omega_1 t'') \\
&\quad \times\,\zeta(t'')\,e^{-\frac{\gamma}{2}t''}\Big]\cos(\Omega_1 t)\Big\}\,.
\end{aligned}
\qquad (11)
$$

It can easily be shown that the stationary part of Eq. (11), when used to compute the two-point correlation function, gives the kernel $K(t-t')$, Eq. (7). We can then solve Eq. (9) numerically for some time so to assure to get the stationary solutions. After this time, we solve the remaining equations of the system altogether. To built this system of equations, it is useful to do the following tricks that allow us to write Eq. (5) in a completely local form. Firstly, we define two new variables, $w(t)$ and $u(t)$, given, respectively, by

$$w(t) = -\int_{t_0}^{t}dt'K(t-t')\dot{\phi}(t')+\xi(t)\,, \qquad (12)$$

and

$$u(t) = \int_{t_0}^{t}dt'\left[\dot{K}(t-t')-\gamma K(t-t')\right]\dot{\phi}(t')\,, \qquad (13)$$

where the time derivatives inside the time integral are (always) with respect to t'. After a little algebra, we find that the equations of motion for w and u are

$$\dot{w}(t) = u(t)-\gamma[w(t)-\xi(t)]-K(0)\dot{\phi}(t)+\dot{\xi}(t)\,, \qquad (14)$$

and

$$\dot{u}(t) = -\Omega_0^2[w(t)-\xi(t)]+\dot{K}(0)\dot{\phi}(t)-\gamma K(0)\dot{\phi}(t)\,. \qquad (15)$$

Putting Eqs. (5), (9), (14) and (15) together, we obtain a sixth order dynamical system that is completely local in time,

$$
\begin{aligned}
\dot{\phi} &= y, \\
\dot{y} &= -V'(\phi) + w, \\
\dot{w} &= u - \gamma(w - \xi) - K(0)y + z, \\
\dot{u} &= -\Omega_0^2 (w - \xi) + \dot{K}(0)y - \gamma K(0)y, \\
\dot{\xi} &= z, \\
\dot{z} &= -\gamma z - \Omega_0^2 \xi + \Omega_0^2 \sqrt{2TQ}\zeta.
\end{aligned}
\tag{16}
$$

Therefore, starting with a non-Markovian Langevin like equation of motion like Eq. (5), we are able to rewrite it in terms of a sixth order dynamical system of local equations, which are more feasible to treat numerically.

5. NUMERICAL RESULTS

We now show the results of our simulations for the time evolution of $\phi(t)$. Results can be presented for different choices of parameters. For example, we can show the results varying the parameter γ in the kernel (7), which sets the time scale for the thermal bath. Results can also be presented for different values of Ω_0, the frequency of the system and also chosen for the thermal bath, thus simulating the original derivation of Eq. (4), where the thermal bath was generated by the system own fluctuations. Since we want to more closely follow this last point of view, here we study the results for different choices of Ω_0, chosen so as to mimic closely the results e.g. of Ref. [7], and we also consider the cases where $T > \Omega_0$, again mimicking the derivation in that reference.

In Figs. 1-3 we show the numerical results coming from the simulation of the system of equations (16) for ϕ, obtained as an average over $100,000$ realizations over the noise. The step size in time considered was 0.01. In all cases we checked for the numerical stability of the results. The results from the nonlocal equation of motion are then compared with the ones obtained in the local approximation, obtained from Eq. (1), with η defined by

$$
\eta \equiv \int_0^\infty dt K(t) = Q,
\tag{17}
$$

which corresponds to the local limit for the kernel given by Eq. (7). The initial conditions considered for all simulations were $\xi(t_0) = 1$, $\dot{\xi}(t_0) = 0$, $\phi(t_0) = 1$, $\dot{\phi}(t_0) = 0$, $w(t_0) = 1$ and $u(t_0) = 0$. The interaction parameter λ in Eq. (6) is chosen as $\lambda = 1$.

From the results shown in Figs. 1-3 we can notice that for values of $\Omega_0 \lesssim 1.5$ the Markovian (local) approximation tends to overestimate the dissipation in the system, while for values of $\Omega_0 \gtrsim 1.5$ the local approximation tends to underestimate the true dissipation as would come from the nonlocal kernel. These same figures also show that the larger is Ω_0, the faster is the thermalization of the system. This shows that as Ω_0 increases, which is also associated to the frequency of the memory kernel chosen, the larger is the fluctuations of the kernel,
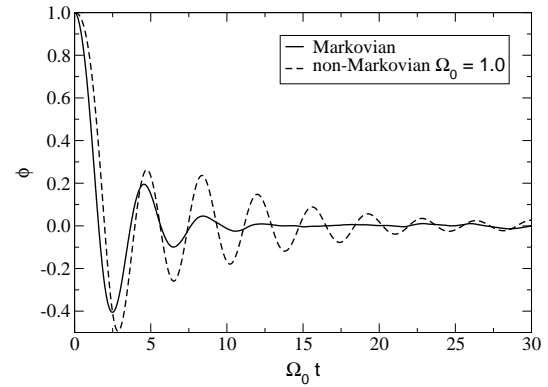

FIG. 1: Time evolution for $\phi(t)$ in both Markovian and non-Markovian regimes and $\Omega_0 = 1.0$. The other parameters are taken as $\gamma = 1.0$, $Q = 0.5$, $T = 10.0$ and $\lambda = 1.0$.
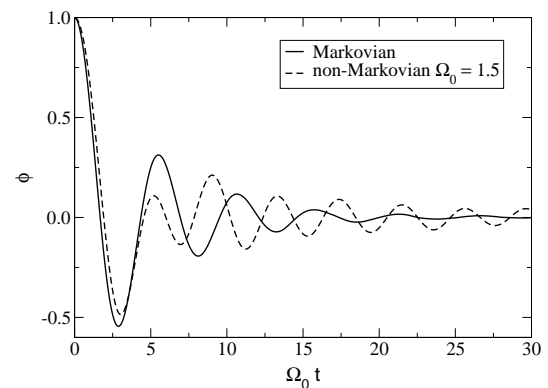

FIG. 2: Time evolution for $\phi(t)$ in both Markovian and non-Markovian regimes and $\Omega_0 = 1.5$. All the other parameters are taken the same as in Fig. 1

which then tend to cancel in the characteristic time scale for the kernel, thus making the nonlocal dissipation larger than the local one.

It is useful to define a time dependent effective temperature T_{eff} through the equipartition of the kinetic energy term, $T_{\text{eff}}(t) = \langle \dot{\phi}^2(t) \rangle$, where the average is over the number of realization in the noise. From the equipartition theorem, at equilibrium this must correspond to the equilibrium temperature T. This is checked in Figs. 4 and 5, for the Markovian and non-Markovian simulations, respectively. These same figures also show that the larger is Ω_0, the faster is the thermalization of the system for the nonlocal dissipation case, in accordance to the discussed in the paragraph above.

6. CONCLUSIONS

In this work we have performed an appropriate numerical approach to deal with a particular non-Markovian Langevin-like equation of motion, with a memory kernel. The non-
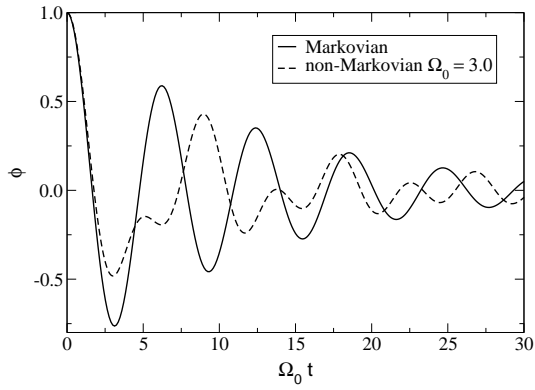
FIG. 3: Time evolution for $\phi(t)$ in both Markovian and non-Markovian regimes and $\Omega_0 = 3.0$. All the other parameters are taken the same as in Fig. 1
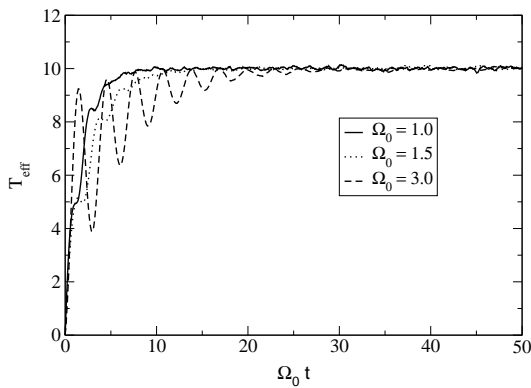

FIG. 4: The effective temperature for the case of Markovian evolution.

Markovian equation of motion (5) is rewritten in terms of a sixth order dynamical system (16) that is completely local in time. We have performed numerical simulations in order to compare the nonlocal dynamics with the one coming from the local approximation.

From the results of our numerical simulations, it can be seen that probably only for some very restrict region of pa-

rameters and time interval the Markovian approximation can be comparable to the non-Markovian time evolution of $\phi(t)$. In general, we note from the results obtained that either the local approximation understimates the dissipation, or overstimates it in most of the region of parameters. The same can be verified varying instead of Ω_0, the parameter γ of the dissipation kernel [12]. The behavior of the scalar field plotted in the figures 1-4 show that the local approximation overestimates the dissipation of the physical system for small values of Ω_0, while for larger values of Ω_0 it tends to underestimate the dissipation at short times. This fact can, of course, conduce to important modifications in the dynamics of models that make use of the local approximation (such as in the inflaton dynamics during inflation). A more complete analysis
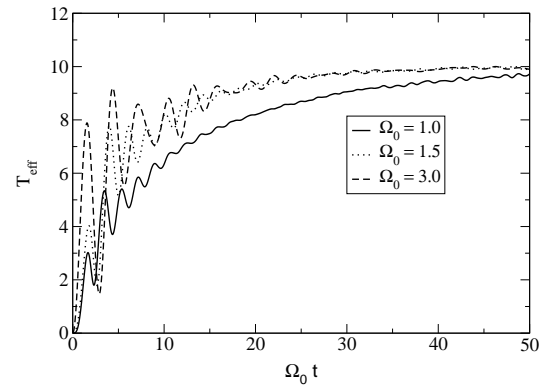

FIG. 5: The effective temperature for the case of non-Markovian evolution.

with more details and including multiplicative noise terms will appear elsewhere [12]. In conclusion, we have seen that for the model studied with nonlocal dissipation, the local approximation does not give a good representation of the dynamics.

[1] L. Kofman, A. D. Linde and A. A. Starobinsky, Phys. Rev. Lett. **73**, 3195 (1994).

[2] Proceedings of Quark Matter 2005, Nucl. Phys. A **774**, 1-968 (2006).

[3] N. C. Cassol-Seewald, R. L. S. Farias, E. S. Fraga, G. Krein, and R. O. Ramos, e-Print: arXiv:0711.1866 [hep-ph].

[4] T. Gasenzer, J. Berges, M. G. Schmidt, and M. Seco, Nucl. Phys. **A785**, 214 (2007).

[5] E. P. Gross, Nuovo Cim. **20**, 454 (1961).

[6] For a recent review, see J. Berges, AIP Conf. Proc. **739**, 3 (2005).

[7] M. Gleiser and R. O. Ramos, Phys. Rev. D **50**, 2441 (1994).

[8] A. Berera, M. Gleiser, and R. O. Ramos, Phys. Rev. D **58** , 123508 (1998).

[9] A. Berera, I. G. Moss, and R. O. Ramos, Phys. Rev. D **76** , 083520 (2007).

[10] R. L. S. Farias, PhD Thesis 2007, São Paulo State University (IFT-UNESP), São Paulo, Brasil.

[11] R. Bartussek, P. Hanggi, B. Lindner, and L. S. Geier, Physica. D **109**, 17 (1997).

[12] R. L. S. Farias, R. O. Ramos, and L.A. da Silva, in preparation.