# Analysis of the Relativistic Brownian Motion in Momentum Space

Kwok Sau Fa
*Departamento de Física, Universidade Estadual de Maringá, Av. Colombo 5790, 87020-900, Maringá-PR, Brazil*

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We investigate the relativistic Brownian motion in the context of Fokker-Planck equation. Due to the multiplicative noise term of the corresponding relativistic Langevin equation many Fokker-Planck equations can be generated. Here, we only consider the Ito, Stratonovich and Hänggi-Klimontovich approaches. We analyze the behaviors of the second moment of momentum in terms of temperature. We show that the second moment increases with the temperature $T$ for all three approaches. Also, we present differential equations for more complicated averages of the momentum. In a specific case, in the Ito approach, we can obtain an analytical solution of the temporal evolution of an average of the momentum. We present approximate solutions for the probability density for all three cases.

Keywords: Relativistic Brownian motion; Langevin equation; Fokker-Planck equation

## I. INTRODUCTION

Nonrelativistic diffusion processes have been the subject of intense investigations in the last century. For instance, the well-known Brownian motion can be described by the usual nonrelativistic diffusion equation [1, 2]. However, recently, several approaches have been used to model different kinds of anomalous diffusion processes [3, 4]. In particular, the extension of the Brownian motion to the relativistic regime has been tackled by using several approaches [5–10]. In the case of using the Langevin approach two types of Langevin equations have been used [7–10]. One of them [7] the authors have postulated a constant diffusion coefficient and the corresponding Langevin equation generates only one Fokker-Planck equation. Whereas, the other one [9] the authors have used the relativistic equation motions and as a consequence the relativistic Langevin equation has a multiplicative noise term. In this last case, the one to one correspondence between the Fokker-Planck equation and the Langevin equation can not be assured due to the different order of prescription in stochastic calculus. In fact, this Langevin equation can generate many different Fokker-Planck equations. In the paper [9] the authors have considered three different prescriptions which we follow them in our work. We analyze the behaviors of the second moment of momentum in terms of the temperature, for the stationary solutions. For the nonstationary solutions, the second moment is difficult to be obtained analytically. In this case, we investigate some more complicated averages of the momentum. We also present approximate solutions for the probability densities. These approximate solutions are used to investigate the second moments.

## II. FOKKER-PLANCK EQUATIONS OF THE RELATIVISTIC BROWNIAN MOTION

In this section we recall the Fokker-Planck equations and their stationary solutions obtained from the relativistic Brownian motion [9, 10], in the laboratory frame, which are given by

$$\frac{\partial\rho(\mathbf{p},t)}{\partial t}=\frac{\partial j^i_{S/I/HK}(\mathbf{p},t)}{\partial p^i}\ , \tag{1}$$

where $t$ is the time, $p^i$ are the relativistic momenta, $j^i_{S/I/HK}(\mathbf{p},t)$ are the probability currents and the indices are the abbreviations of Stratonovich, Ito and Hänggi-Klimontovich, respectively. For one-dimensional case the probability currents are given by

$$j_S=-\left[\nu p\rho(p,t)+D\sqrt{\gamma(p)}\frac{\partial}{\partial p}\left(\sqrt{\gamma(p)}\rho(p,t)\right)\right]\ , \tag{2}$$

$$j_I=-\left[\nu p\rho(p,t)+D\frac{\partial}{\partial p}\left(\gamma(p)\rho(p,t)\right)\right] \tag{3}$$

and

$$j_{HK}=-\left[\nu p\rho(p,t)+D\gamma(p)\frac{\partial}{\partial p}\left(\rho(p,t)\right)\right]\ , \tag{4}$$

where $\gamma(p)=(1+p^2/(mc)^2)^{1/2}$. For three-dimensional case the probability currents are given by

$$j^i_S=-\left[\nu p^i\rho(p,t)+D(\mathbf{L}^{-1})^i_k\frac{\partial}{\partial p_j}\left(((\mathbf{L}^{-1})^T)^k_j\rho(p,t)\right)\right]\ , \tag{5}$$

$$j^i_I=-\left[\nu p^i\rho(p,t)+D\frac{\partial}{\partial p_j}\left((\mathbf{A}^{-1})^i_j\rho(p,t)\right)\right] \tag{6}$$

and

$$j^i_{HK}=-\left[\nu p^i\rho(p,t)+D(\mathbf{A}^{-1})^i_j\frac{\partial}{\partial p_j}\left(\rho(p,t)\right)\right]\ , \tag{7}$$

where **A** and **L** are the matrices associated with the momentum components [10].

The stationary solutions of Eq.(1) are obtained by setting $\frac{\partial\rho}{\partial t}=0$ and $j^i_{S/I/HK}=0$. For one-dimensional case we have

$$\rho_S(p)=\frac{C_{1S}\exp\left(-\beta\sqrt{1+\frac{p^2}{m^2c^2}}\right)}{\left(1+\frac{p^2}{m^2c^2}\right)^{1/4}}\,, \tag{8}$$

$$\rho_I(p)=\frac{C_{1I}\exp\left(-\beta\sqrt{1+\frac{p^2}{m^2c^2}}\right)}{\left(1+\frac{p^2}{m^2c^2}\right)^{1/2}} \tag{9}$$

and

$$\rho_{HK}(p)=C_{1HK}\exp\left(-\beta\sqrt{1+\frac{p^2}{m^2c^2}}\right)\,, \tag{10}$$

where $C_{1S}$, $C_{1I}$ and $C_{1HK}$ are the normalization factors, $\beta=\nu m^2c^2/D=mc^2/(kT)$, $k$ is the Boltzmann constant and $T$ is the temperature. In the three-dimensional case the stationary solutions are given by

$$\rho_S(\mathbf{p})=\frac{C_{3S}\exp\left(-\beta\sqrt{1+\frac{\mathbf{p}^2}{m^2c^2}}\right)}{\left(1+\frac{\mathbf{p}^2}{m^2c^2}\right)^{\frac{3}{4}}}\,, \tag{11}$$

$$\rho_I(\mathbf{p})=\frac{C_{3I}\exp\left(-\beta\sqrt{1+\frac{\mathbf{p}^2}{m^2c^2}}\right)}{\left(1+\frac{\mathbf{p}^2}{m^2c^2}\right)^{\frac{3}{2}}} \tag{12}$$

and

$$\rho_{HK}(\mathbf{p})=C_{3HK}\exp\left(-\beta\sqrt{1+\frac{\mathbf{p}^2}{m^2c^2}}\right)\,, \tag{13}$$

where $C_{3S}$, $C_{3I}$ and $C_{3HK}$ are the normalization factors.

## III. NORMALIZATION AND SECOND MOMENT FOR STATIONARY SOLUTIONS

The above stationary solutions can be normalized analytically. The normalization of the stationary solutions permit us to analyze the behaviors of the second moments in terms of the temperature. For one-dimensional case we obtain the following normalized solutions:

$$\rho_S(p)=\frac{\sqrt{2\pi}\exp\left(-\beta\sqrt{1+\frac{p^2}{m^2c^2}}\right)}{2mc\sqrt{\beta}K_{\frac{3}{4}}(\frac{\beta}{2})K_{\frac{1}{4}}(\frac{\beta}{2})\left(1+\frac{p^2}{m^2c^2}\right)^{1/4}}\,, \tag{14}$$

$$\rho_I(p)=\frac{\exp\left(-\beta\sqrt{1+\frac{p^2}{m^2c^2}}\right)}{2mcK_0(\beta)\left(1+\frac{p^2}{m^2c^2}\right)^{1/2}} \tag{15}$$

and

$$\rho_{HK}(p)=\frac{\exp\left(-\beta\sqrt{1+\frac{p^2}{m^2c^2}}\right)}{2mcK_1(\beta)}\,, \tag{16}$$

where $K_\nu(z)$ denotes the modified Hankel function. The corresponding second moments are given by

$$\left\langle\frac{p^2}{m^2c^2}\right\rangle_S=-\frac{1}{2\sqrt{\beta}K_{\frac{3}{4}}(\frac{\beta}{2})K_{\frac{1}{4}}(\frac{\beta}{2})}\frac{\partial}{\partial\beta}\left\{\sqrt{\beta}\left[K_{\frac{5}{4}}(\frac{\beta}{2})K_{\frac{3}{4}}(\frac{\beta}{2})-K_{\frac{1}{4}}(\frac{\beta}{2})K_{-\frac{1}{4}}(\frac{\beta}{2})\right]\right\}\,, \tag{17}$$

$$\left\langle\frac{p^2}{m^2c^2}\right\rangle_I=\frac{K_1(\beta)}{\beta K_0(\beta)} \tag{18}$$

and

$$\left\langle\frac{p^2}{m^2c^2}\right\rangle_{HK}=\frac{K_2(\beta)}{\beta K_1(\beta)}\,. \tag{19}$$

In Fig. 1 we show the second moment $\left\langle\frac{p^2}{m^2c^2}\right\rangle$, in one dimension, in function of $\beta$. We see that all three cases decrease with $\beta$. This means that they increase with the temperature, just as in the classical case. Moreover, in the Hänggi-Klimontovich approach the second moment has the highest value among them, whereas in the classical case has the lowest value. For large values of β all three cases converge to the classical one.

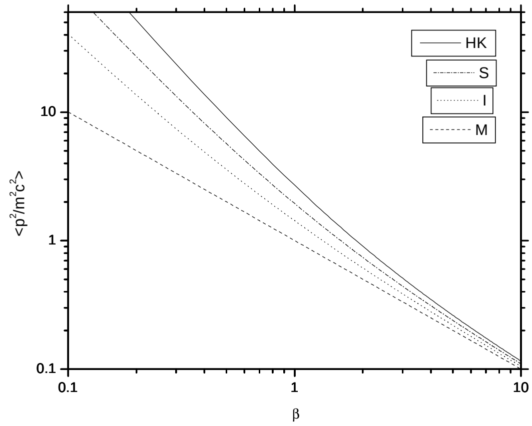


FIG. 1: Plots of the second moments $\left\langle \frac{p^2}{m^2c^2} \right\rangle_{HK,S,I,M}$ in function of β, for one-dimensional stationary processes. The symbols HK, S, I, M correspond to the Hänggi-Klimontovich, Stratonovich, Ito and Maxwell distributions.

In the case of three-dimensional processes we can also obtain the analytical normalization factors which are given by

$$C_{3S}^{-1} = (mc)^3\sqrt{2\pi\beta}\left[K_{\frac{5}{4}}(\frac{\beta}{2})K_{\frac{3}{4}}(\frac{\beta}{2}) - K_{\frac{1}{4}}(\frac{\beta}{2})K_{-\frac{1}{4}}(\frac{\beta}{2})\right], \tag{20}$$

$$C_{3I}^{-1} = 2\pi(mc)^3\left\{\beta\pi - 2\,{}_2F_3\left[-\frac{1}{2},-\frac{1}{2};\,\frac{1}{2},\frac{1}{2},1;\,\frac{\beta^2}{4}\right] + \right.$$

$$\left. + \ln\left(\frac{4}{\beta^2}\right)\,{}_1F_2\left[-\frac{1}{2};\frac{1}{2},1;\frac{\beta^2}{4}\right] - \sum_{n=0}^{\infty}\frac{2^{1-2n}\beta^{2n}\psi(1+n)}{(-1+2n)\left(\Gamma[1+n]\right)^2}\right\} \tag{21}$$

and

$$C_{3HK}^{-1} = \frac{4\pi(mc)^3K_2(\beta)}{\beta}, \tag{22}$$

where $\psi(z)$ is the Psi function, ${}_pF_q\left[a_{1,\ldots,a_p};\, b_1,\ldots,b_q;\, z\right]$ is the generalized hypergeometric function and $\Gamma[z]$ is the Gamma function.

The corresponding second moments of the momentum are given by

$$\left\langle \frac{\mathbf{p}^2}{m^2c^2} \right\rangle_S = C_{3S}\frac{\partial^2}{\partial\beta^2}\left(\frac{1}{C_{3S}}\right) - 1, \tag{23}$$

$$\left\langle \frac{\mathbf{p}^2}{m^2c^2} \right\rangle_I = \frac{4\pi(mc)^3C_{3I}K_1(\beta)}{\beta} - 1, \tag{24}$$

and

$$\left\langle \frac{\mathbf{p}^2}{m^2c^2} \right\rangle_{HK} = \frac{1}{2}\left[\frac{K_4(\beta)}{K_2(\beta)} - 1\right]. \tag{25}$$

We note that the relations (22) and (25) have been obtained in [7].

In Fig. 2 we show the second moment $\left\langle \frac{\mathbf{p}^2}{m^2c^2} \right\rangle$, in three dimensions, in function of β. All three cases also decrease with β, just as in one-dimensional cases. For large value of β the three cases converge to the classical one. We note that in the Ito approach the second moment has close values to those of the classical case, except for small β.

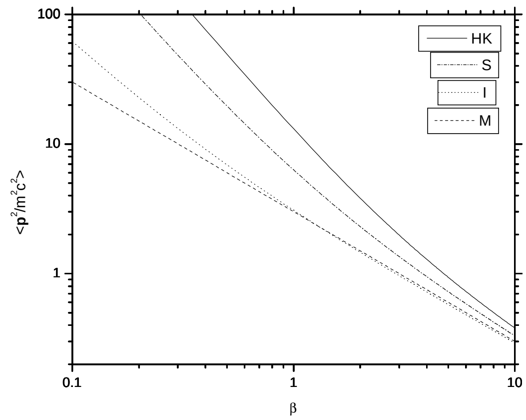


FIG. 2: Plots of the second moments $\left\langle \frac{\mathbf{p}^2}{m^2c^2} \right\rangle_{HK,S,I,M}$ in function of β, for three-dimensional stationary processes. The symbols HK, S, I, M correspond to the Hänggi-Klimontovich, Stratonovich, Ito and Maxwell distributions.

## IV. AVERAGES OF THE MOMENTUM FOR ONE-DIMENSIONAL CASE

The n-moments are important physical quantities for the analysis of the stochastic processes. However, they are difficult to be obtained for the relativistic Brownian motion. In the above section we have only obtained the second moments for the stationary solutions. In order to obtain some average for the whole processes, we have found more complicated averages of the momentum by using the differential equation (1). For the average $\langle\gamma^a\rangle$, where $a$ is a real number, we can combine to obtain, by choosing different values of $a$, the differential equations for the averages of momentum. For the Stratonovich approach we have

$$\frac{d\left\langle \gamma^{\frac{3}{2}} \right\rangle_S}{dt} = -\frac{3\nu}{2}\left[\left\langle \gamma^{\frac{3}{2}} \right\rangle_S - \left\langle \gamma^{-\frac{1}{2}} \right\rangle_S - \frac{1}{\beta}\left\langle \gamma^{\frac{1}{2}} \right\rangle_S\right] \tag{26}$$

and

$$\frac{d\left\langle \gamma^{\frac{5}{2}} \right\rangle_S}{dt} = \frac{5\nu}{2}\left[\left\langle \gamma^{\frac{1}{2}} \right\rangle_S - \left\langle \gamma^{\frac{5}{2}} \right\rangle_S + \frac{1}{\beta}\left\langle \gamma^{\frac{3}{2}} \right\rangle_S + \frac{1}{\beta}\left\langle \gamma^{-\frac{1}{2}} \right\rangle_S\right] . \tag{27}$$

For $\beta = 1$ we can obtain the following differential equation:

$$\frac{d\left\langle \gamma^{\frac{5}{2}} \right\rangle_S}{dt} - \frac{5}{3}\frac{d\left\langle \gamma^{\frac{3}{2}} \right\rangle_S}{dt} + \frac{5\nu}{2}\left\langle \gamma^{\frac{5}{2}} \right\rangle_S - 5\nu\left\langle \gamma^{\frac{3}{2}} \right\rangle_S = 0 \ . \tag{28}$$

In the case of the Ito approach we have the following differential equations:

$$\frac{d\left\langle \gamma^2 - 1 \right\rangle_I}{dt} = -2\nu\left\langle \gamma^2 - 1 \right\rangle_I + \frac{2\nu}{\beta}\left\langle \gamma \right\rangle_I \tag{29}$$

and

$$\frac{d\left\langle \gamma^3 \right\rangle_I}{dt} = 3\nu\left[-\left\langle \gamma^3 \right\rangle_I + \left\langle \gamma \right\rangle_I + \frac{2}{\beta}\left\langle \gamma^2 - 1 \right\rangle_I + \frac{1}{\beta}\right] \ . \tag{30}$$

They can be combined to form the expression

$$\frac{d\left\langle \gamma^3 \right\rangle_I}{dt} + 3\nu\left\langle \gamma^3 \right\rangle_I = \frac{3\beta}{2}\left[\frac{d\left\langle \gamma^2 - 1 \right\rangle_I}{dt} + 2\nu\left(1 + \frac{2}{\beta^2}\right)\left\langle \gamma^2 - 1 \right\rangle_I + \frac{2\nu}{\beta^2}\right] . \tag{31}$$

We note that for $\beta = 2$ we can obtain the analytical solution which is given by

$$\left|\left\langle \gamma^3 \right\rangle_I - 3\left\langle \gamma^2 - 1 \right\rangle_I - \frac{1}{2}\right| = A\exp\left(-3\nu t\right) \ , \tag{32}$$

where $A$ is an integration constant. We see that the above average gives a simple exponential decay in time. Even the solution (32) is just valid for $\beta = 2$, it can be applied to a wide range of diffusion coefficient $D$ due to the relation $\beta = \nu m^2c^2/D$. However, it ($\beta = 2$) determines the value of temperature which is given by $T = mc^2/(2k)$.

For the Hanggi-Klimontovich approach we consider the following averages:

$$\frac{d\left\langle \gamma \right\rangle_{HK}}{dt} = \nu\left[-\left\langle \frac{\gamma^2 - 1}{\gamma} \right\rangle_{HK} + \frac{1}{\beta}\right] \tag{33}$$

and

$$\frac{d\left\langle \gamma^2 - 1 \right\rangle_{HK}}{dt} = \frac{2\nu}{\beta}\left[\left\langle \frac{\gamma^2 - 1}{\gamma} \right\rangle_{HK} + \left\langle \gamma \right\rangle_{HK} - \beta\left\langle \gamma^2 - 1 \right\rangle_{HK}\right] . \tag{34}$$

These equations can be combined to form the expression

$$\frac{d\left\langle \gamma^2 - 1 \right\rangle_{HK}}{dt} + 2\nu\left\langle \gamma^2 - 1 \right\rangle_{HK} + \frac{2}{\beta}\left[\frac{d\left\langle \gamma \right\rangle_{HK}}{dt} - \nu\left\langle \gamma \right\rangle_{HK}\right] = \frac{2\nu}{\beta^2} \ . \tag{35}$$

Unfortunately, only in the Ito approach one can solve the differential equation for the averages. We can check the solution (32) by using the stationary solution (15). Numerical calculation yields

$$\left\langle \gamma^3 \right\rangle_I - 3\left\langle \gamma^2 - 1 \right\rangle_I = 0.5 \tag{36}$$

which is in excellent agreement with the analytical result (32).

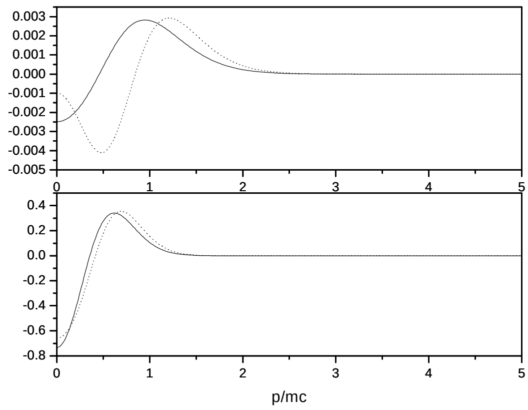


FIG. 3: Plots of the variations of $\frac{\partial \rho_S}{\partial(\nu t)}$ and $\frac{\partial j_S}{\partial(p/mc)}$ in function of the nondimensional variable $\frac{p}{mc}$ by using the approximate solution Eq.(37). The solid lines correspond to the variations of $\frac{\partial \rho_S}{\partial(\nu t)}$, whereas the dotted lines correspond to the variations of $\frac{\partial j_S}{\partial(p/mc)}$. The lower figure corresponds to $\nu t = 0.5$ and the upper figure corresponds to $\nu t = 3$.

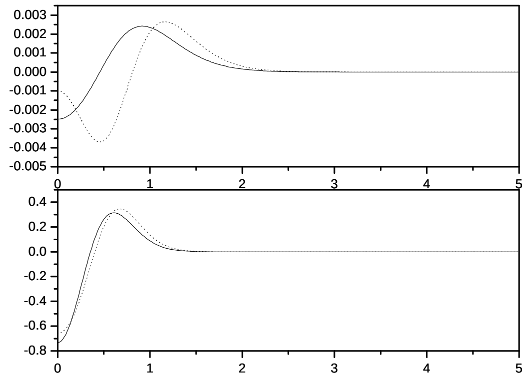


FIG. 4: Plots of the variations of $\frac{\partial \rho_I}{\partial(\nu t)}$ and $\frac{\partial j_I}{\partial(p/mc)}$ in function of the nondimensional variable $\frac{p}{mc}$ by using the approximate solution Eq.(38). The solid lines correspond to the variations of $\frac{\partial \rho_I}{\partial(\nu t)}$, whereas the dotted lines correspond to the variations of $\frac{\partial j_I}{\partial(p/mc)}$. The lower figure corresponds to $\nu t = 0.5$ and the upper figure corresponds to $\nu t = 3$.

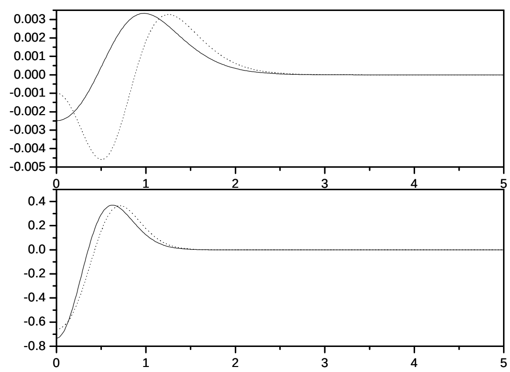


FIG. 5: Plots of the variations of $\frac{\partial \rho_{HK}}{\partial(\nu t)}$ and $\frac{\partial j_{HK}}{\partial(p/mc)}$ in function of the nondimensional variable $\frac{p}{mc}$ by using the approximate solution Eq.(39). The solid lines correspond to the variations of $\frac{\partial \rho_{HK}}{\partial(\nu t)}$, whereas the dotted lines correspond to the variations of $\frac{\partial j_{HK}}{\partial(p/mc)}$. The lower figure corresponds to $\nu t = 0.5$ and the upper figure corresponds to $\nu t = 3$.

## V. ONE-DIMENSIONAL APPROXIMATE SOLUTIONS

We note that Eqs. (1), (2), (3) and (4) are difficult to be solved analytically. In this case, approximate solutions can be useful to the analysis of the behaviors of probability densities. Also, they can be used to check the numerical results. In order to obtain our approximate solutions we have guided by the asymptotic solutions for $|p| \ll 1$ and $|p| \gg 1$. The approximate

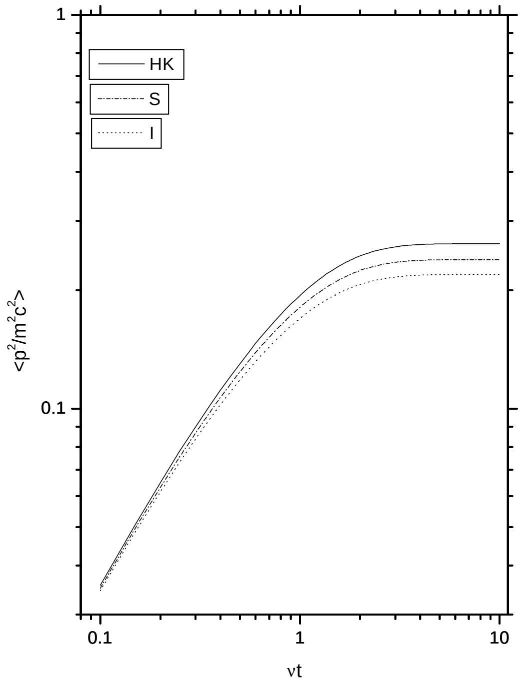


FIG. 6: Plots of the second moments $\left\langle \frac{\mathbf{p}^2}{m^2c^2} \right\rangle_{HK,S,I}$ in function of the nondimensional time $\nu t$ by using the approximate solutions (37), (38) and (39), for $\beta = 5$.

solutions are given by

$$\rho_S(p,t) = \frac{A_S \exp\left( -\beta \frac{\left(\sqrt{1+\frac{p2}{m^2c^2}}-1\right)}{1-\exp\left(-\nu\left(\frac{2+\frac{p2}{m^2c^2}}{1+\frac{p2}{m^2c^2}}\right)t\right)} \right)}{\sqrt{1-\exp\left(-\nu\left(\frac{2+\frac{p2}{m^2c^2}}{1+\frac{p2}{m^2c^2}}\right)t\right)}\left(1+\frac{p2}{m^2c^2}\right)^{1/4}}, \tag{37}$$

$$\rho_I(p,t) = \frac{A_I \exp\left( -\beta \frac{\left(\sqrt{1+\frac{p2}{m^2c^2}}-1\right)}{1-\exp\left(-\nu\left(\frac{2+\frac{p2}{m^2c^2}}{1+\frac{p2}{m^2c^2}}\right)t\right)} \right)}{\sqrt{1-\exp\left(-\nu\left(\frac{2+\frac{p2}{m^2c^2}}{1+\frac{p2}{m^2c^2}}\right)t\right)}\left(1+\frac{p2}{m^2c^2}\right)^{1/2}} \tag{38}$$

and

$$\rho_{HK}(p,t)=\frac{A_{HK}\exp\left(-\beta\frac{\left(\sqrt{1+\frac{p^2}{m^2c^2}}-1\right)}{1-\exp\left(-\nu\left(\frac{2+\frac{p^2}{m^2c^2}}{1+\frac{p^2}{m^2c^2}}\right)t\right)}\right)}{\sqrt{1-\exp\left(-\nu\left(\frac{2+\frac{p^2}{m^2c^2}}{1+\frac{p^2}{m^2c^2}}\right)t\right)}}, \quad (39)$$

where $A_S$, $A_I$ and $A_{HK}$ are the normalization factors.

In Figs. 3, 4 and 5 we show the variations of $\frac{\partial\rho_{I/S/HK}}{\partial(\nu t)}$ and $\frac{\partial j_{I/S/HK}}{\partial(p/mc)}$ obtained from the Stratonovich, Ito and Hänggi-Klimontovich approaches, respectively. We note that the behaviors of these quantities are similar for these three approaches. Also, these quantities are close together, specially for small $\nu t$. This means that our approximate solutions are in good approximations to the exact solutions. Moreover, we have analyzed numerically that the variations of $\frac{\partial\rho_{I/S/HK}}{\partial(\nu t)}$ and $\frac{\partial j_{I/S/HK}}{\partial(p/mc)}$ can maintain close together for not large $t$, $\beta$ greater than 3 and $\nu$ less than 1. For large $\nu t$ the variation between these two quantities is very small for $\frac{p}{mc}<2$, and, in general, they maintain very close together for $\frac{p}{mc}>2$.

From these approximate solutions we can calculate the second moments. In Fig. 6 we show the second moment $\left\langle\frac{p^2}{m^2c^2}\right\rangle$ for the Ito, Stratonovich and Hänggi-Klimontovich approaches, for $\beta=5$. All three cases have similar behaviors to that of the classical Brownian motion. We have also checked for $\beta=20$, and all three cases approximate closely to the result of classical theory. As has been numerically demonstrated in [9], the probability densities in velocity space approach a common Gaussian shape for large $\beta$.

## VI. CONCLUSION

We have analyzed the relativistic Brownian motion in the context of Fokker-Planck equations. We have put forward some analytical results in terms of the averages of momentum. We have shown an interesting analytical result in the Ito approach, i.e., the average $\left\langle\gamma^3\right\rangle_I-3\left\langle\gamma^2-1\right\rangle_I$ has a simple exponential decay in time. Further, approximate solutions for the probability densities have also been given for one-dimensional cases. We have used our approximate solutions to study the temporal behaviors of the second moments $\left\langle\frac{p^2}{m^2c^2}\right\rangle_{S/I/HK}$, and the results of these three approaches converge to the classical result for large $\beta$. On the other hand, it has been emphasized in [9] that the three prescriptions discussed in this work, only the stationary solution of the Hänggi-Klimontovich approach is consistent with the relativistic Maxwell distribution. However, this last result is not the only one obtained by using the Langevin equation. In fact, the relativistic Maxwell distribution can also be obtained by using a different Langevin equation as that used by Debbasch [7, 8]. Therefore, in order to choose which of the above approaches is the correct one, we need to know detailed information about the microscopic structure of the system. Finally, we hope that these analyses may contribute to a broad investigation of the relativistic Brownian motion.


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