On the Trace Anomaly and the Energy-Momentum Conservation of Quantum Fields at D=2 in Classical Curved Backgrounds

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We study the conformal symmetry and the energy-momentum conservation of scalar field interacting with a curved background at $D = 2$. We avoid to incorporate the metric determinant into the measure of the scalar field to explain the conformal anomaly and the consequent energy-momentum conservation. Contrarily, we split the scalar field in two other fields, in such a way that just one of them can be quantized. We show that the same usual geometric quantities of the anomaly are obtained, which are accompanied by terms containing the new field of the theory.

1 Introduction

The interaction between quantum fields and classical gravity has been intensively studied for a long time [1]. One of the main motivations is that this procedure is considered to be the first initial step to understand the full quantum theory including the gravitatonal field itself. It resembles the old formalism of dealing with quantum fields interacting with the classical external electromagnetic background before the advent of the quantum electrodynamics.

However, even though these are aparently similar procedures, the former is much more involved. For example, the concept of particle is not well defined. Consequently, the definition of $|in\rangle$ and $|out\rangle$ states cannot be done in a clear way and the definition S-matrix becomes meaningless. So, instead os particles, it is the energy-momentum tensor that plays the fundamental role, due to its local nature and being the source of the curvature in the general relativity theory. There is an important property that the energy-momentum tensor has to satisfy, the (covariant) divergenceless, which corresponds to the conservation of energy and momentum of the theory. This is a kind of symmetry that cannot be modified by quantum corrections.

On the other hand, we have an interesting symmetry related to massless theories, called conformal symmetry, which means, in a broader sense, the absence of scales. This symmetry is manifested in the traceless of the energy-momentum tensor. Contrarily to previous case, it is not necessarily kept in the quantum scenario (trace anomaly). This occurs because the quantum formalism naturally introduces scale parameters in order to deal with infinities during the regularization procedures.

In this paper, we consider a quantum massless scalar field interacting with a classical curved background. We mention that this theory exhibits the conformal symmetry, where for $D > 2$ it is necessary to couple the scalar field to the classical curvature (nominimal coupling) [1]. The particular case of $D = 2$, that is the subject of the present paper,

has an interesting feature. The conformal symmetry is verified without necessity of coupling the scalar field to the curvature, because there is no conformal transformation for it. This result may lead to a wrong conclusion that there is no trace anomaly for scalar fields at $D = 2$ because the absence of conformal transformation for them leads to an invariance of the corresponding measure in the path integral formalism.

The above reasoning and conclusion, which there is no trace anomaly for $D = 2$, cannot be true because it is accompanied by an unpleasant absence of energy and momentum conservation (after quantum corrections). A more carefull study of the conformal symmetry shows that the conformal anomaly does actually exist and the energy and momentum are actually conserved, as they should be [2, 3]. We mention that this problem can be circumvented by splitting the $\sqrt{-g}$ of the action as $\sqrt{-g} = (-g)^{-\frac{1}{4}}(-g)^{-\frac{1}{4}}$ and incorporating each one of these factors to the scalar field [2]. In this way, the new scalar field acquires a convenient conformal transformation [4], whose noninvariance of the measure renders the expected trace anomaly and the energy-momentum conservation.

The purpose of the present paper is to display a different alternative of dealing with this problem. We avoid to incorporate any factor involving the metric tensor to the scalar field because, since we intend to work with the path integral formalism, this would be inconsistent with the initial assumption that the gravitational field is classical. Our proposal consists in splitting the scalar field in a product of two fields with different conformal transformations. The classical conformal symmetry is not modified, but the measure of one of them is. We show that this leads to a trace anomaly, that has the same geometrical terms of the usual case, plus other ones related to the new field. We also show that, eventhough more involved, the energy momentum conservation is also achieved quantically.

Our paper is organized as follows. In Sec. II we brief discuss how this this problem can be solved by incorporating a factor inovolving the metric tensor into the scalar field.

Eventhough this is a section review, we follow the lines that we shall used into the next section, where our formalism is presented. We left Sec. IV for some concluding remarks.

2 Trace anomaly and energy-momentum conservation

Let us start from the action

$$S = \frac{1}{2} \int d^2 x \sqrt{-g} \, g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi \qquad (2.1)$$

The classical energy-momentum tensor is

$$
\begin{aligned}
T_{\mu\nu} &= \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} \\
&= \partial_\mu \phi \, \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \, \partial^\rho \phi \, \partial_\rho \phi \qquad (2.2)
\end{aligned}
$$

which has the following properties

$$g^{\mu\nu} T_{\mu\nu} = 0 \qquad (2.3)$$
$$\nabla^\mu T_{\mu\nu} = \partial_\nu \phi \, \Box \, \phi = 0 \qquad (2.4)$$

where ∇^μ is the covariante derivative and $\Box$ is the Laplace-Beltrami operator, $\Box \phi = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \partial^\mu \phi) = g^{\mu\nu} \nabla_\mu \nabla_\nu \phi$.

Expression (2.3) tell us that the theory exhibits the conformal symmetry. The conformal transformation for the metric tensor is $\tilde{g}_{\mu\nu} = e^{2\alpha} g_{\mu\nu}$ and, consequently, $\sqrt{-g} \, g^{\mu\nu}$ is conformally invariant. This means that the conformal symmetry is verified at $D = 2$ without any transformation for the scalar field ϕ. The meaning of expression (2.4) is that the energy and momentum are conserved. It is opportune to mention that this expression was obtained by using the equation of motion for ϕ.

In the quantum scenario (with a classical background metric) the energy momentum tensor can be obtained by means of the vacuum functional as

$$\langle T_{\mu\nu} \rangle = \frac{2}{\sqrt{-g}} \frac{\delta Z}{\delta g^{\mu\nu}} \qquad (2.5)$$

where

$$Z = \int [d\phi] \exp \left(\frac{i}{2} \int d^2 x \sqrt{-g} \, g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right) \qquad (2.6)$$

Since the scalar field does not change under conformal transformation, we have that the measure $[d\phi]$ also remains invariant. Consequently, the semiclassical expression for the energy-momentum tensor reads

$$\langle T_{\mu\nu} \rangle = \langle \partial_\mu \phi \, \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \, \partial^\rho \phi \, \partial_\rho \phi \rangle \qquad (2.7)$$

The curved background is considered to be classical, so we may have

$$g^{\mu\nu} \langle T_{\mu\nu} \rangle = \langle g^{\mu\nu} T_{\mu\nu} \rangle = 0 \qquad (2.8)$$
$$\nabla^\mu \langle T_{\mu\nu} \rangle = \langle \nabla^\mu T_{\mu\nu} \rangle = \langle \partial_\nu \phi \Box \phi \rangle \qquad (2.9)$$

It is not possible to conclude that expression (2.9) is zero because the equation of motion cannot be used in the quantum scenario. Of course, the results above do not merit confidence because there is no reason to believe that energy and momentum are not conserved after quantum effects are taken into account.

To circumvent this problem, the action (2.1) can be rewriten as

$$S = \frac{1}{2} \int d^2 x \, g^{\mu\nu} \partial_\mu \Phi \, \partial_\nu \Phi \qquad (2.10)$$

where

$$\Phi = (-g)^{\frac{1}{4}} \phi \qquad (2.11)$$

Of course, the classical $T_{\mu\nu}$ is precisely the previous one given by (2.2) (which can be rewritten in terms of Φ) and the action, S given by (2.10), is still conformally invariant. The conformal transformation for the new scalar field Φ is

$$\tilde{\Phi} = e^{\alpha} \Phi \qquad (2.12)$$

However, for the vacuum functional, the measure $[d\Phi]$ is not invariant under conformal transformation. In a general way, we have [5]

$$[d\tilde{\Phi}] = \exp \left(i \int d^2 x \sqrt{-g} \, \alpha(x) \, A(x) \right) [d\Phi] \qquad (2.13)$$

where $A(x)$ is a badly divergent quantity that can be regularized by means of the zeta function technique leading to [6]

$$
\begin{aligned}
A(x) &= \lim_{s \to 0} \operatorname{tr} \zeta(x, s) \\
&= \frac{[a_1(x)]}{4\pi} \qquad (2.14)
\end{aligned}
$$

The last step of the expression above is restrict to $D = 2$, and the coefficient $a_1(x, x')$ is related to heat kernel expansion. The notation $[a_1(x)]$ means $[a_1(x)] = \lim_{x' \to x} a_1(x, x')$. These coefficients can be obtained by means of a recursion relation that depends on the kind of operator that acts on the field [7]. For the present case, $a_1(x) = -\frac{1}{6} R$, where R is the Ricci scalar curvature

Now, the corresponding energy-momentum tensor obtained by means of expression (2.6), which we shall denote by $\tilde{T}_{\mu\nu}$, is not traceless. The trace $\langle T^\mu{}_\mu \rangle$ can be directly obtained by

$$
\begin{aligned}
\langle \tilde{T}^\mu{}_\mu \rangle &= -\frac{i}{\sqrt{-g}} \frac{\delta Z}{\delta \alpha} \\
&= A(x) \\
&= -\frac{1}{24\pi} R \qquad (2.15)
\end{aligned}
$$

This result embodies the trace anomaly. Since in two space-time dimensions we have the identity $R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}R$, one may say that the full expression for the energy-momentum tensor $\langle \tilde{T}_{\mu\nu}\rangle$ should be

$$\langle \tilde{T}_{\mu\nu}\rangle = \langle T_{\mu\nu}\rangle - \frac{1}{48\pi}g_{\mu\nu}R \qquad (2.16)$$

where $\langle T_{\mu\nu}\rangle$ is the one given by (2.7) (it is indifferent to write it in terms of ϕ or Φ). Acting the covariant derivative in both sides of the expression above, we get

$$\nabla^\mu \langle \tilde{T}_{\mu\nu}\rangle = \langle \partial_\nu\phi \,\Box\, \phi\rangle - \frac{1}{48\pi}\partial_\nu R \qquad (2.17)$$

Expanding the field ϕ in terms of eigenfunctions of the operator $\Box$, one can show that [2]

$$\langle \partial_\nu\phi \,\Box\, \phi\rangle = \frac{1}{2}\partial_\nu\langle\phi\Box\phi\rangle \qquad (2.18)$$

and the quantity $\langle\phi \,\Box\, \phi\rangle$ can be regularized and leads to [2]

$$\langle\phi \,\Box\, \phi\rangle = \frac{1}{24\pi}\,R \qquad (2.19)$$

So,

$$\nabla^\mu \langle \tilde{T}_{\mu\nu}\rangle = 0 \qquad (2.20)$$

as it should be.

3 Alternative procedure

Now, instead of incorporating the factor $(-g)^{-\frac{1}{4}}$ to the scalar field, we go in a opposite direction by splitting the field ϕ as

$$\phi = e^\theta\,\varphi \qquad (3.1)$$

where θ and φ are considered to be two independent quantities with the following conformal tranformations

$$\tilde{\varphi} = e^{-\alpha}\varphi \qquad (3.2)$$
$$\tilde{\theta} = \theta + \alpha \qquad (3.3)$$

The field φ remains quantum, but θ can be quantum or not. It is important the field θ appears in a exponential term and, consequently, with a conformal transformation like (3.3). This is so because, in the hypothesis that θ is also quantum, its corresponding measure, $[d\theta]$, remains unchanged (the jacobian is trivial). In the developments which follow, we shall consider θ classical. At the end, we briefly talk on the possibility of θ being quantum.

Replacing ϕ given by (3.1) into the initial expression for S, (2.1), we have

$$S = -\frac{1}{2}\int d^2x\,\sqrt{-g}\,\varphi\left[e^{2\theta}\big(\Box - \partial_\mu\theta\partial^\mu\theta\big)\right]\varphi \qquad (3.4)$$

We have just done a change of variables and, consequently, there is no changing into the classical case. But,

in the path integral, the measure $[d\varphi]$ is not invariant under conformal transformation. Considering θ classical, se have the vacuum functional

$$Z = \int[d\varphi]\exp\left\{-\frac{i}{2}\int d^2x\sqrt{-g}\varphi\left[e^{2\theta}\big(\Box - \partial_\mu\theta\partial^\mu\theta\big)\right]\varphi\right\} \qquad (3.5)$$

Now, the coefficient $[a_1]$, related to the operator that is acting on φ, is

$$[a_1] = -e^{2\theta}\left(\frac{1}{6}R + \partial_\mu\theta\partial^\mu\theta\right) \qquad (3.6)$$

So, the trace anomaly reads

$$\langle T^\mu{}_\mu\rangle = -\frac{e^{2\theta}}{4\pi}\left(\frac{1}{6}R + \partial_\mu\theta\partial^\mu\theta\right) \qquad (3.7)$$

Since the field θ is considered to be classical, one may infer that the expression for the energy momentum tensor $\langle \tilde{T}_{\mu\nu}\rangle$ is given by

$$\langle \tilde{T}_{\mu\nu}\rangle = \langle T_{\mu\nu}\rangle - \frac{e^{2\theta}}{4\pi}\left(\frac{1}{12}\,g_{\mu\nu}R + \partial_\mu\theta\partial_\nu\theta\right) \qquad (3.8)$$

where $\langle T_{\mu\nu}\rangle$ is the same one as given by (2.7), with ϕ replaced by $e^\theta\varphi$, i.e.

$$\begin{aligned}
\langle T_{\mu\nu}\rangle = e^{2\theta}&\left[\left(\partial_\mu\theta\partial_\nu\theta - \frac{1}{2}g_{\mu\nu}\,\partial_\rho\theta\partial^\rho\theta\right)\langle\varphi^2\rangle\right.\\
&+ \frac{1}{2}\,\partial_\mu\theta\langle\partial_\nu\varphi^2\rangle + \frac{1}{2}\,\partial_\nu\theta\langle\partial_\mu\varphi^2\rangle\\
&- \frac{1}{2}\,g_{\mu\nu}\,\partial_\rho\theta\langle\partial^\rho\varphi^2\rangle\\
&\left.+ \langle\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}g_{\mu\nu}\,\partial_\rho\varphi\partial^\rho\varphi\rangle\right] \qquad (3.9)
\end{aligned}$$

Now, let us verify the consistency with respect the energy and momentum conservation. First, we consider $\nabla^\mu\langle \tilde{T}_{\mu\nu}\rangle$. This can be done by directly acting the operator ∇^μ on $\langle \tilde{T}_{\mu\nu}\rangle$, given by the expression (3.9), or by using (2.9), where we should replace ϕ by $e^\theta\varphi$ [notice, however, that this replacement cannot be done into (2.18), because just part of ϕ is quantum]. We thus have

$$\begin{aligned}
\nabla^\mu\langle T_{\mu\nu}\rangle = e^{2\theta}&\left[\left(\partial_\rho\theta\partial^\rho\theta + \Box\,\theta\right)\partial_\nu\theta\langle\varphi^2\rangle\right.\\
&+ \partial_\nu\theta\partial^\rho\theta\,\langle\partial_\rho\varphi^2\rangle + \partial_\nu\theta\langle\varphi \,\Box\, \varphi\rangle\\
&+ \frac{1}{2}\big(\partial_\rho\theta\partial^\rho\theta + \Box\,\theta\big)\,\langle\partial_\nu\varphi^2\rangle\\
&\left.+ 2\partial^\rho\theta\langle\partial_\rho\varphi\partial_\nu\varphi\rangle + \langle\partial_\nu\varphi \,\Box\, \varphi\rangle\right] \qquad (3.10)
\end{aligned}$$

For the second term of (3.8) we have

$$-\frac{1}{4\pi}\nabla^\mu\left[e^{2\theta}\left(\frac{1}{12}\,g_{\mu\nu}\,R+\partial_\mu\theta\partial_\nu\theta\right)\right]$$

$$=-\frac{1}{4\pi}\,e^{2\theta}\left(\frac{1}{6}\,R\,\partial_\nu\theta+\frac{1}{12}\,\partial_\nu R\right.$$

$$+2\partial_\nu\theta\partial_\rho\theta\partial^\rho\theta$$

$$+\Box\,\theta\partial_\nu\theta+\partial^\rho\theta\partial_\rho\partial_\nu\theta\Big)\qquad(3.11)$$

Now, since just φ is quantum, we can write

$$\langle\partial_\nu\varphi\,\Box\,\varphi\rangle=\frac{1}{2}\partial_\nu\langle\varphi\,\Box\,\varphi\rangle\qquad(3.12)$$

Considering the action involving φ and with the term $e^{2\theta}$ factorized, as well as the renormalization factor we are using into (2.14), we can obtain the regularized quantities

$$\langle\varphi\,\Box\,\varphi\rangle=\frac{1}{4\pi}\left(\frac{1}{6}\,R+\partial_\rho\theta\partial^\rho\theta\right)\qquad(3.13)$$

$$\langle\varphi^2\rangle=\frac{1}{4\pi}\qquad(3.14)$$

Finally, since φ is a scalar quantity and $\langle\partial_\nu\varphi^2\rangle$ is an average involving all directions aleatory, and the same occurs with $\langle\partial_\rho\varphi\partial_\nu\varphi\rangle$, we have that these quantities are null. Replacing all these results into (3.10) and (3.11), we get

$$\nabla^\mu\langle\tilde{T}_{\mu\nu}\rangle=0\qquad(3.15)$$

which express the consistency with the energy-momentum conservation.

In the case that θ is also quantum, the measure $[d\theta]$ does not change under conformal transformation, but the problem is much more involved and difficult to solve. Just formally, one may write that the trace anomaly reads

$$\langle\tilde{T}^\mu{}_\mu\rangle=-\frac{1}{4\pi}\,\langle e^{2\theta}\left(\frac{1}{6}R+\partial_\mu\theta\partial^\mu\theta\right)\rangle\qquad(3.16)$$

From which one cannot either infer an expression similar as the second term of (3.8) or try to regularize it because the bad divergencies occuring in the exponential $e^{2\theta}$.

4   Conclusion

In this paper we have study the problem of conformal anomaly and the energy-momentum conservation for a quantum scalar field interacting with a classical curved background in the spacetime dimension D=2. We have considered the scalar field as split in two other scalar fields, and kept just of them quantum. This procedure is in opposite direction to what is done in literature, where a factor containing the metric determinant is absorbed by the scalar field, leading to a new field with a convenient conformal transformation. We have shown that our procedure is consistent with the geometric terms of the usual treatment of the conformal anomaly and also consistent with the expected result that energy and momentum should be conserved after quantum effects are taken into account.

References

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