On the Thermodynamics of Ionized Gases

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The transport coefficients of a completely ionized gas are determined from an extended thermodynamic theory of mixtures of ideal gases in the presence of external electromagnetic fields. The Onsager relations for the transport coefficients in the presence of external magnetic flux density are also discussed.

I. Introduction

The extended thermodynamic theory for a mixture of ν ideal gases was first formulated by Kremer [1] as a field theory whose objective was the determination of 13 ν fields of partial mass densities, partial velocities, partial pressure tensors and partial heat fluxes.

Recently Pennisi and Trovato [2] analised the same problem and incorporated the electric charge of the constituents and the influence of electromagnetic fields, but they have not obtained the dependence of the transport coefficients on the external magnetic flux density. It is well known in the literature of ionized gases that the transport coefficients do depend on the external magnetic flux density (see for example Balescu [3] or Rodbard, Bezerra Jr. and Kremer [4]).

The purpose of this work is to obtain - from a phenomenological extended thermodynamic theory of a fully ionized gas - the laws of Navier-Stokes, Fourier and Ohm and to find the dependence of the transport coefficients on the external magnetic flux density. In Section II we base on [1] and remind the principal features of extended thermodynamics of mixtures of ν ideal gases, while in Section III we present the so-called one-fluid theory of an ionized gas whose objective is the determination of the five fields of density, velocity and temperature. The constitutive equations for the partial pressure deviators and partial heat fluxes are calculated in Section IV by the use of a method akin to the Maxwellian iteration method of kinetic theory of gases (see [5]). The laws of Navier-Stokes, Fourier and Ohm are obtained in Section V and the transport coefficients are calculated as functions of the external magnetic flux density. In Section VI we discuss the Onsager reciprocity relations in the presence of external axial fields. The representation and the inverse of second- and fourth-order tensors that depend on an axial field are given in the Appendix.

Cartesian notation for tensors with the usual summation convention is used. Parentheses denote symmetrization of all indices while angular parentheses indicate traceless symmetrization.

II. A reminder of extended thermodynamics of mixtures

We may say that objective of extended thermodynamics of ionized gases is the determination of 13ν fields of

ϱ_α - partial mass densities,
v_i^α - partial velocities,

p_{ij}^α - partial pressure tensors, and

q_i^α - partial heat flux vectors, (2.1)

where $\alpha = 1, 2, \ldots, \nu$ denotes the constituents of a mixture of electrons, ions and neutral particles.

To achieve this objective, we need 13 ν field equations that are based on the following balance equations for the moments (2.1):

$$\frac{\partial \varrho_\alpha}{\partial t} + \frac{\partial \varrho_\alpha v_i^\alpha}{\partial x_i} = 0, \tag{2.2}$$

$$\varrho_\alpha \left(\frac{\partial v_i^\alpha}{\partial t} + \frac{\partial v_i^\alpha}{\partial x_j} v_j^\alpha \right) + \frac{\partial p_{ij}^\alpha}{\partial x_j} = P_i^\alpha + \frac{e_\alpha \varrho_\alpha}{m_\alpha} \left[E_i + (\mathbf{v}^\alpha \times \mathbf{B})_i \right], \tag{2.3}$$

$$\frac{\partial p_{ij}^\alpha}{\partial t} + \frac{\partial p_{ij}^\alpha v_k^\alpha}{\partial x_k} + \frac{\partial p_{\langle ijk\rangle}^\alpha}{\partial x_k} + \frac{4}{5}\frac{\partial q_{(i}^\alpha}{\partial x_{j)}} + \frac{2}{5}\frac{\partial q_k^\alpha}{\partial x_k}\delta_{ij} + 2p_{k(i}^\alpha \frac{\partial v_{j)}^\alpha}{\partial x_k}$$

$$= \mathcal{P}_{ij}^\alpha + 2\frac{e_\alpha}{m_\alpha} p_{k(i}^\alpha \epsilon_{j)kl} B_l, \tag{2.4}$$

$$\frac{\partial q_i^\alpha}{\partial t} + \frac{\partial q_i^\alpha v_j^\alpha}{\partial x_j} + \frac{1}{2}\frac{\partial p_{ijjk}^\alpha}{\partial x_k} + p_{\langle ijk\rangle}^\alpha \frac{\partial v_j^\alpha}{\partial x_k} + \frac{2}{5} q_j^\alpha \frac{\partial v_j^\alpha}{\partial x_i} + \frac{7}{5} q_j^\alpha \frac{\partial v_i^\alpha}{\partial x_j}$$

$$+\frac{2}{5} q_i^\alpha \frac{\partial v_k^\alpha}{\partial x_k} - \frac{p_{ij}^\alpha}{\varrho_\alpha}\frac{\partial p_{jk}^\alpha}{\partial x_k} - \frac{p_{rr}^\alpha}{2\varrho_\alpha}\frac{\partial p_{ik}^\alpha}{\partial x_k} = \mathcal{P}_i^\alpha - \frac{3}{2}\frac{p_{(ij}^\alpha}{\varrho_\alpha} P_{j)}^\alpha + \frac{e_\alpha}{m_\alpha}\epsilon_{ijk} q_j^\alpha B_k. \tag{2.5}$$

In the above equations $p_{\langle ijk\rangle}^\alpha = p_{ijk}^\alpha + \frac{2}{5}\left(q_i^\alpha \delta_{jk} + q_j^\alpha \delta_{ik} + q_k^\alpha \delta_{ij}\right)$ and p_{ijjk}^α are higher-order moments while P_i^α, $\mathcal{P}_{ij}^\alpha$ and $\mathcal{P}_i^\alpha$ are production terms. The production term P_i^α is constrained by the relation

$$\sum_{\alpha=1}^{\nu} P_i^\alpha = 0, \tag{2.6}$$

which expresses the conservation of the total momentum density. Moreover, e_α and m_α are the electric charge and the mass of a particle of constituent α in the mixture, E_i is the external electric field and B_i the external magnetic flux density.

To close the system of equations (2.2)-(2.5) we assume $p_{\langle ijk\rangle}^\alpha$, p_{ijjk}^α, P_i^α, $\mathcal{P}_{ij}^\alpha$ and $\mathcal{P}_i^\alpha$ as constitutive quantities that depend on the basic fields (2.1). Through the explotation of the principle of material frame-indifference and of the entropy principle we get the following linearized constitutive equations:[1]

$$p_{\langle ijk\rangle}^\alpha = 0, \qquad p_{ijjk}^\alpha = 5\frac{p_\alpha^2}{\varrho_\alpha}\delta_{ik} + 7\frac{p_\alpha}{\varrho_\alpha} p_{\langle ik\rangle}^\alpha, \tag{2.7}$$

$$P_i^\alpha = \sum_{\beta=1}^{\nu} M_{\alpha\beta}^q q_i^\beta + \sum_{\beta=1}^{\nu-1} M_{\alpha\beta}^V V_i^\beta, \tag{2.8}$$

$$\mathcal{P}_{ii}^\alpha = R_\alpha, \qquad \mathcal{P}_{\langle ij\rangle}^\alpha = \sum_{\beta=1}^{\nu} \sigma_{\alpha\beta} p_{\langle ij\rangle}^\beta, \tag{2.9}$$

$$\mathcal{P}_i^\alpha = \sum_{\beta=1}^{\nu} Q_{\alpha\beta}^q q_i^\beta + \sum_{\beta=1}^{\nu-1} Q_{\alpha\beta}^V V_i^\beta. \tag{2.10}$$

We have introduced above the velocity difference $V_i^\alpha = v_i^\alpha - v_i^\nu$.

III. The one-fluid theory

It is usual in the literature (see for example [3]) to describe an ionized gas by the so-called one-fluid theory whose objective is the determination of five fields of mass density ϱ, velocity v_i and temperature T. These fields are defined with respect to the partial fields as:

[1] We have restricted to a classical ideal gas where $p_\alpha = \varrho_\alpha \frac{k}{m_\alpha} T_\alpha$ and dropped out all constants of integrations.

$$\varrho = \sum_{\alpha=1}^{\nu} \varrho_\alpha, \qquad v_i = \frac{1}{\varrho} \sum_{\alpha=1}^{\nu} \varrho_\alpha v_i^\alpha, \quad T = \frac{1}{n} \sum_{\alpha=1}^{\nu} n_\alpha T_\alpha, \tag{3.1}$$

where $n_\alpha = \varrho_\alpha / m_\alpha$ is the number density of constituent α and $n = \sum_{\alpha=1}^{\nu} n_\alpha$. Here we shall suppose that all constituents have the same temperature, which is the temperature of the mixture T.

To determine the five fields above we base on the following balance equations for the mass density ϱ, momentum density ϱv_i and internal energy density $\varrho\varepsilon$:

$$\frac{\partial \varrho}{\partial t} + \frac{\partial \varrho v_i}{\partial x_i} = 0, \tag{3.2}$$

$$\varrho \left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right) + \frac{\partial p_{ij}}{\partial x_j} = \sum_{\alpha=1}^{\nu} \frac{e_\alpha \varrho_\alpha}{m_\alpha} E_i^* + (\mathbf{I} \times \mathbf{B})_i. \tag{3.3}$$

$$\frac{\partial \varrho\varepsilon}{\partial t} + \frac{\partial}{\partial x_i} (\varrho\varepsilon v_i + q_i) + p_{ij} \frac{\partial v_i}{\partial x_j} = \mathbf{I} \cdot \mathbf{E}^*, \tag{3.4}$$

where $\mathbf{E}^* = \mathbf{E} + (\mathbf{v} \times \mathbf{B})$. The above equations were obtained by summing over all constituents of the mixture the equations (2.2), (2.3) and the trace of equation (2.4). Moreover, we have introduced the electric current density I_i, the pressure tensor p_{ij}, the heat flux vetor q_i and the internal energy density through the relationships:

$$I_i = \sum_{\alpha=1}^{\nu} e_\alpha n_\alpha u_i^\alpha, \qquad p_{ij} = \sum_{\alpha=1}^{\nu} (p_{ij}^\alpha + \varrho_\alpha u_i^\alpha u_j^\alpha), \tag{3.5}$$

$$q_i = \sum_{\alpha=1}^{\nu} \left[q_i^\alpha + \varrho_\alpha \left(\varepsilon_\alpha + \frac{1}{2} u_\alpha^2 \right) u_i^\alpha + p_{ji}^\alpha u_j^\alpha \right], \qquad \varrho\varepsilon = \sum_{\alpha=1}^{\nu} \varrho_\alpha \left(\varepsilon_\alpha + \frac{1}{2} u_\alpha^2 \right), \tag{3.6}$$

where u_i^α and $\varrho_\alpha \varepsilon_\alpha$ are the diffusion velocity and the internal energy density of constituent α in the mixture, respectively, which are defined by

$$u_i^\alpha = v_i^\alpha - v_i, \qquad \varrho_\alpha \varepsilon_\alpha = \frac{3}{2} p_\alpha = \frac{1}{2} p_{rr}^\alpha. \tag{3.7}$$

To get a system of field equations from (3.2)-(3.4) one has to consider ε, p_{ij}, q_i and I_i as constitutive quantities that depend on the basic fields ϱ, v_i, T. Since we are dealing with an ideal gas and interested only in a linear theory, ε and $p = \sum_{\alpha=1}^{\nu} p_\alpha$ are known functions of ϱ and T. In the next section we shall show how to get the constitutive equations for $p_{\langle ij \rangle}$, q_i and I_i from the system of field equations of extended thermodynamics.

IV. The Maxwellian iteration method

In most applications of plasma physics the ionized gas is in a state of complete ionization, i.e., it is considered as a binary mixture of ions ($\alpha = I$) and eletrons ($\alpha = E$). Here we shall restrict ourselves to this case.

First we note that there are only $\nu - 1$ linearly independent diffusion velocities, since

$$\sum_{\alpha=1}^{\nu} \varrho_\alpha u_i^\alpha = 0. \tag{4.1}$$

For the binary mixture in study we have $u_i^I = -\varrho_E u_i^E / \varrho_I$ and the following relationship hold:

$$I_i = \left(\frac{e_E}{m_E} - \frac{e_I}{m_I} \right) \varrho_E u_i^E, \qquad V_i^E = -V_i^I = \frac{\varrho}{\varrho_I} u_i^E. \tag{4.2}$$

On the other hand, in a linearized theory equations $(3.5)_2$ and $(3.6)_2$ reduce to:

$$p_{ij} = p_{ij}^E + p_{ij}^I, \qquad q_i = q_i^E + q_i^I + \frac{5}{2} kT \frac{m_I - m_E}{e_E m_I - e_I m_E} I_i, \tag{4.3}$$

since $p_\alpha = \frac{2}{3}\varrho_\alpha\varepsilon_\alpha = \varrho_\alpha \frac{k}{m_\alpha}T$.

To get the constitutive equations for $p^\alpha_{\langle ij\rangle}$ and q^α_i we proceed as follows: we insert the constitutive equations (2.7)-(2.10) into the equations (2.3), (2.5), and into the traceless part of equation (2.4), consider only linear terms in the resulting equations. Hence

$$\frac{m_E m_I}{e_E m_I - e_I m_E}\frac{\partial}{\partial t}\left(\frac{\varrho}{\varrho_I\varrho_E}I_i\right) + \frac{1}{\varrho_E}\frac{\partial p^E_{\langle ij\rangle}}{\partial x_j} - \frac{1}{\varrho_I}\frac{\partial p^I_{\langle ij\rangle}}{\partial x_j} + \frac{1}{\varrho_E}\frac{\partial p_E}{\partial x_i} - \frac{1}{\varrho_I}\frac{\partial p_I}{\partial x_i}$$

$$-\left(\frac{e_E m_I - e_I m_E}{m_E m_I}\right)[E_i + (\mathbf{v}\times\mathbf{B})_i] = \frac{\varrho}{\varrho_E\varrho_I}M^q_{EE}q^E_i + \frac{\varrho}{\varrho_E\varrho_I}M^q_{EI}q^I_i$$

$$+\left[\left(\frac{\varrho}{\varrho_E\varrho_I}\right)^2\frac{m_E m_I}{e_E m_I - e_I m_E}M^V_{EE}\delta_{ij} + \frac{e_E\varrho_I m_I + e_I\varrho_E m_E}{\varrho_E\varrho_I(e_E m_I - e_I m_E)}\epsilon_{ijk}B_k\right]I_j, \tag{4.4}$$

$$\frac{\partial p^\alpha_{\langle ij\rangle}}{\partial t} + \frac{4}{5}\frac{\partial q^\alpha_{\langle i}}{\partial x_{j\rangle}} + 2p_\alpha\frac{\partial u^\alpha_{\langle i}}{\partial x_{j\rangle}} + 2p_\alpha\frac{\partial v_{\langle i}}{\partial x_{j\rangle}} = \sigma_{\alpha E}p^E_{\langle ij\rangle} + \sigma_{\alpha I}p^I_{\langle ij\rangle} + 2\frac{e_\alpha}{m_\alpha}p^\alpha_{\langle k(i\rangle}\epsilon_{j)kl}B_l, \qquad (\alpha = E, I) \tag{4.5}$$

$$\frac{\partial q^\alpha_i}{\partial t} + \frac{5}{2}\frac{p^2_\alpha}{\varrho_\alpha}\frac{1}{T}\frac{\partial T}{\partial x_i} + \frac{p_\alpha}{\varrho_\alpha}\frac{\partial p^\alpha_{\langle ij\rangle}}{\partial x_j} = \left(Q^q_{\alpha E} - \frac{5}{2}\frac{p_\alpha}{\varrho_\alpha}M^q_{\alpha E}\right)q^E_i + \left(Q^q_{\alpha I} - \frac{5}{2}\frac{p_\alpha}{\varrho_\alpha}M^q_{\alpha I}\right)q^I_i$$

$$+\left(Q^V_{\alpha E} - \frac{5}{2}\frac{p_\alpha}{\varrho_\alpha}M^V_{\alpha E}\right)\frac{\varrho}{\varrho_E\varrho_I}\frac{m_E m_I}{e_E m_I - e_I m_E}I_i + \frac{e_\alpha}{m_\alpha}\epsilon_{ijk}q^\alpha_j B_k, \qquad (\alpha = E, I) \tag{4.6}$$

by performing some rearrangements. Equation (4.4) follows by subtraction of the ion equation from the electron equation. Now we use an iteration method akin to the Maxwellian iteration of kinetic gas theory (see [5]). For the first iteration step we insert the equilibrium values $I^{(0)}_i = 0$, $p^{\alpha(0)}_{\langle ij\rangle} = 0$ and $q^{\alpha(0)}_i$ $(\alpha = E, I)$ into the left hand side of equations (4.4)-(4.6) and obtain the first iterated values $I^{(1)}_i$, $p^{\alpha(1)}_{\langle ij\rangle}$ and $q^{\alpha(1)}_i$ on the right hand side:

$$\frac{e_I m_E - e_E m_I}{m_E m_I}\mathcal{E}_i + \frac{5}{2}k\frac{(m_I - m_E)}{m_E m_I}\frac{\partial T}{\partial x_i} = \frac{\varrho}{\varrho_E\varrho_I}M^q_{EE}q^{E(1)}_i + \frac{\varrho}{\varrho_E\varrho_I}M^q_{EI}q^{I(1)}_i$$

$$+\left[\left(\frac{\varrho}{\varrho_E\varrho_I}\right)^2\frac{m_E m_I}{e_E m_I - e_I m_E}M^V_{EE}\delta_{ij} + \frac{e_E\varrho_I m_I + e_I\varrho_E m_E}{\varrho_E\varrho_I(e_E m_I - e_I m_E)}\epsilon_{ijk}B_k\right]I^{(1)}_j, \tag{4.7}$$

$$2p_\alpha\frac{\partial v_{\langle i}}{\partial x_{j\rangle}} = \sigma_{\alpha E}p^{E(1)}_{\langle ij\rangle} + \sigma_{\alpha I}p^{I(1)}_{\langle ij\rangle} + 2\frac{e_\alpha}{m_\alpha}p^{\alpha(1)}_{\langle k(i\rangle}\epsilon_{j)kl}B_l, \qquad (\alpha = E, I) \tag{4.8}$$

$$\frac{5}{2}\frac{p^2_\alpha}{\varrho_\alpha}\frac{1}{T}\frac{\partial T}{\partial x_i} = H^q_{\alpha E}q^{E(1)}_i + H^q_{\alpha I}q^{I(1)}_i + H^V_{\alpha E}I^{(1)}_i + \frac{e_\alpha}{m_\alpha}\epsilon_{ijk}q^{\alpha(1)}_j B_k, \qquad (\alpha = E, I). \tag{4.9}$$

In the above equations we have introduced

$$H^q_{\alpha\beta} = \left(Q^q_{\alpha\beta} - \frac{5p_\alpha}{2\varrho_\alpha}M^q_{\alpha\beta}\right), \qquad H^V_{\alpha E} = \left(Q^V_{\alpha E} - \frac{5p_\alpha}{2\varrho_\alpha}M^V_{\alpha E}\right)\frac{\varrho}{\varrho_E\varrho_I}\frac{m_E m_I}{e_E m_I - e_I m_E}, \tag{4.10}$$

$$\mathcal{E}_i = E^*_i + \frac{m_E m_I T}{e_I m_E - e_E m_I}\frac{\partial}{\partial x_i}\left(\frac{\mu_E - \mu_I}{T}\right), \tag{4.11}$$

where μ_α $(\alpha = E, I)$ is the chemical potential of constituent α. For simplicity from now on we shall drop the index (1) that denotes the iterated value.

From the system of equations (4.9) we obtain:

$$q^\alpha_i = -K^\alpha_{ij}\frac{\partial T}{\partial x_j} + L^\alpha_{ij}I_j, \qquad (\alpha = E, I) \tag{4.12}$$

by using equations $(B.4)$ and $(B.5)$ of Appendix B, where

$$K^\alpha_{ij} = -(a^\alpha_1\delta_{ij} + a^\alpha_2\epsilon_{ijk}B_k + a^\alpha_3 B_i B_j), \qquad L^\alpha_{ij} = a^\alpha_4\delta_{ij} + a^\alpha_5\epsilon_{ijk}B_k + a^\alpha_6 B_i B_j. \tag{4.13}$$

The coefficients a_1^α through a_6^α are given in Appendix A.

Now by the use of equation $(B.9)$ of Appendix B, it follows from the system of equations (4.8):

$$p^E_{\langle ij\rangle} = -2\eta^\alpha_{\langle ij\rangle\langle kl\rangle}\frac{\partial v_{\langle k}}{\partial x_{l\rangle}}, \qquad p^I_{\langle ij\rangle} = -2\eta^I_{\langle ij\rangle\langle kl\rangle}\frac{\partial v_{\langle k}}{\partial x_{l\rangle}}, \tag{4.14}$$

where

$$\eta^E_{\langle ij\rangle\langle kl\rangle} = b_1^\alpha\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-\frac{2}{3}\delta_{ij}\delta_{kl}\right)$$
$$+b_2^\alpha\left(\epsilon_{jlr}B_r\delta_{ik}+\epsilon_{jkr}B_r\delta_{il}+\epsilon_{ilr}B_r\delta_{jk}+\epsilon_{ikr}B_r\delta_{jl}\right)$$
$$+b_3^\alpha\left(\delta_{ik}B_jB_l+\delta_{il}B_jB_k+\delta_{jk}B_iB_l+\delta_{jl}B_iB_k-\frac{4}{3}\delta_{kl}B_iB_j-\frac{4}{3}\delta_{ij}B_kB_l+\frac{4}{9}B^2\delta_{ij}\delta_{kl}\right)$$
$$+b_4^\alpha\left(\epsilon_{ikr}B_rB_jB_l+\epsilon_{ilr}B_rB_jB_k+\epsilon_{jkr}B_rB_iB_l+\epsilon_{jlr}B_rB_iB_k\right)$$
$$+b_5^\alpha\left(B_iB_jB_kB_l-\frac{1}{3}B^2B_iB_j\delta_{kl}-\frac{1}{3}B^2B_kB_l\delta_{ij}+\frac{1}{9}B^4\delta_{ij}\delta_{kl}\right). \tag{4.15}$$

The coefficients b_1^α through b_5^α are also given in Appendix A.

V. The laws of Navier-Stokes, Fourier and Ohm

In a linearized theory the presure tensor and the heat flux vector of the mixture are given by equations (4.3). Hence by the use of equations (4.14) and $(4.3)_1$, one can get

$$p_{\langle ij\rangle} = -2\eta_{\langle ij\rangle\langle kl\rangle}\frac{\partial v_{\langle k}}{\partial x_{l\rangle}}, \tag{5.1}$$

which is the mathematical expression of the law of Navier-Stokes. The fourth-order tensor $\eta_{\langle ij\rangle\langle kl\rangle}$ is identified with the coefficient of shear viscosity and it is given by

$$\eta_{\langle ij\rangle\langle kl\rangle} = \eta^E_{\langle ij\rangle\langle kl\rangle} + \eta^I_{\langle ij\rangle\langle kl\rangle}. \tag{5.2}$$

On the other hand, the law of Fourier is obtained from equations (4.12) and $(4.3)_2$:

$$q_i = -K_{ij}\frac{\partial T}{\partial x_j} + L_{ij}I_j. \tag{5.3}$$

K_{ij} denotes the tensor of thermal conductivity and L_{ij} the tensor of a electrical thermal effect. They are given by

$$K_{ij} = K^E_{ij} + K^I_{ij}, \qquad L_{ij} = L^E_{ij} + L^I_{ij} + \frac{5}{2}kT\frac{m_I - m_E}{e_Im_E - e_Em_I}\delta_{ij}. \tag{5.4}$$

Finally the law of Ohm follows from equations (4.7) and (4.12):

$$\mathcal{E}_i = \Sigma_{ij}I_j - L^*_{ij}\frac{\partial T}{\partial x_j}. \tag{5.5}$$

In the above equation we identify Σ_{ij} as the electrical resistivity tensor and L^*_{ij} the tensor of a thermal electrical effect. The expressions for Σ_{ij} and L^*_{ij} read:

$$\Sigma_{ij} = \frac{m_Em_I}{e_Im_E - m_Em_I}\left\{\left(\frac{\varrho}{\varrho_E\varrho_I}\right)^2\frac{m_Em_I}{e_Em_I - e_Im_E}M^V_{EE}\delta_{ij} + \frac{e_E\varrho_Im_I + e_I\varrho_Em_E}{\varrho_E\varrho_I(e_Em_I - e_Im_E)}\epsilon_{ijk}B_k\right.$$

$$+\frac{\varrho}{\varrho_I\varrho_E}\left[M^q_{EE}L^E_{ij}+M^q_{EI}L^I_{ij}\right]\Bigg\}, \tag{5.6}$$

$$L^*_{ij}=\frac{m_E m_I}{e_I m_E - e_E m_I}\left\{\frac{\varrho}{\varrho_E\varrho_I}\left[M^q_{EE}K^E_{ij}+M^q_{EI}K^I_{ij}\right]+\frac{5}{2}k\frac{m_I-m_E}{m_E m_I}\delta_{ij}\right\}. \tag{5.7}$$

VI. The Onsager relations

In a linearized theory the entropy flux is given by (see equation $(9.2)_3$ of [1]):

$$\phi_i=\frac{q_i}{T}+\frac{m_E m_I}{e_I m_E - e_E m_I}\frac{\mu_E-\mu_I}{T}I_i, \tag{6.1}$$

or by the use of equation (5.3):

$$\phi_i=-\frac{K_{ij}}{T}\frac{\partial T}{\partial x_i}+D_{ij}I_j, \tag{6.2}$$

where according to $(5.4)_2$, D_{ij} is given by

$$D_{ij}=\frac{1}{T}\left\{L^E_{ij}+L^I_{ij}+\left[\frac{5}{2}kT\frac{m_I-m_E}{m_E m_I}+(\mu_E-\mu_I)\right]\frac{m_E m_I}{e_I m_E - e_E m_I}\delta_{ij}\right\}. \tag{6.3}$$

On the other hand, one can obtain from equations (5.5) and (4.11) that

$$E^*_i+\frac{m_E m_I}{e_I m_E - e_E m_I}\frac{\partial(\mu_E-\mu_I)}{\partial x_i}=\Sigma_{ij}I_j-D^*_{ij}\frac{\partial T}{\partial x_j}, \tag{6.4}$$

where the coefficient D^*_{ij} is given, in conformity with (5.7), by:

$$D^*_{ij}=\frac{m_E m_I}{e_I m_E - e_E m_I}\left\{\frac{\varrho}{\varrho_E\varrho_I}\left[M^q_{EE}K^E_{ij}+M^q_{EI}K^I_{ij}\right]+\left[\frac{5}{2}k\frac{m_I-m_E}{m_E m_I}+\frac{\mu_E-\mu_I}{T}\right]\delta_{ij}\right\}. \tag{6.5}$$

In a linear irreversible thermodynamics (see for example [6])

$$\phi_i \qquad \text{and} \qquad E^*_i+\frac{m_E m_I}{e_I m_E - e_E m_I}\frac{\partial(\mu_E-\mu_I)}{\partial x_i}$$

are identified as thermodynamic fluxes while I_i and $-\frac{\partial T}{\partial x_i}$ as thermodynamic forces. Besides, the following symmetric relations are postulated for the coefficients in the presence of a magnetic flux density $\mathbf{B}$:

$$K_{ij}(\mathbf{B})=K_{ji}(-\mathbf{B}),\quad \Sigma_{ij}(\mathbf{B})=\Sigma_{ji}(-\mathbf{B}),\quad D_{ij}(\mathbf{B})=D_{ji}(-\mathbf{B}),\quad D^*_{ij}(\mathbf{B})=D^*_{ji}(-\mathbf{B}), \tag{6.6}$$

$$D_{ij}=D^*_{ij} \tag{6.7}$$

which are known as the Onsager reciprocity relations.

The relationships (6.6) are satisfied since the coefficients are expressed in a form like the one given by equation (4.13). However, the relationship (6.7) is not satisfied in general as it can be seen from equations (6.3) and (6.5). In order to get such a relationship we base on [1] and assume that:

- the production term of the partial heat fluxes do not depend on the diffusion fluxes, so that $H^V_{EE}=H^V_{IE}=0$;
- the production term of the partial momentum density do not depend on the partial heat fluxes, so that $M^q_{EE}=M^q_{EI}=0$.

With the two above assumptions it is easy to show from equations (A.4), (A.5), (A.6), $(A.8)_2$ and (A.9) from Appendix A that $a^\alpha_4=a^\alpha_5=a^\alpha_6=0$ for $\alpha=E,I$, and from equation $(4.13)_2$ thet $L^E_{ij}=L^I_{ij}=0$. Hence, it follows

$$D_{ij} = D^*_{ij} = \frac{m_E m_I}{e_I m_E - e_E m_I}\left[\frac{5}{2}k\frac{m_I - m_E}{m_E m_I} + \frac{\mu_E - \mu_I}{T}\right]\delta_{ij}. \tag{6.8}$$

Appendix A: Scalar Coefficients

The scalar coefficients of q_i^α are given by:

$$a_1^E = \frac{1}{D}\frac{5}{2}k^2T\left[\left(\frac{n_I}{m_I}H^q_{EI} - \frac{n_E}{m_E}H^q_{II}\right)\left(H^q_{IE}H^q_{EI} - H^q_{EE}H^q_{II} + \frac{e_E}{m_E}\frac{e_I}{m_I}B^2\right)\right.$$
$$\left. - \left(\frac{e_E}{m_E}H^q_{II} + \frac{e_I}{m_I}H^q_{EE}\right)\frac{e_I}{m_I}\frac{n_E}{m_E}B^2\right](H^q_{IE}H^q_{EI} - H^q_{EE}H^q_{II}), \tag{A.1}$$

$$a_2^E = \frac{1}{D}\frac{5}{2}k^2T\left[\left(\frac{n_I}{m_I}H^q_{EI} - \frac{n_E}{m_E}H^q_{II}\right)\left(\frac{e_E}{m_E}H^q_{II} + \frac{e_I}{m_I}H^q_{EE}\right)\right.$$
$$\left. -\frac{e_I}{m_I}\frac{n_E}{m_E}\left(H^q_{IE}H^q_{EI} - H^q_{EE}H^q_{II} + \frac{e_E}{m_E}\frac{e_I}{m_I}B^2\right)\right](H^q_{IE}H^q_{EI} - H^q_{EE}H^q_{II}), \tag{A.2}$$

$$a_3^E = \frac{1}{D}\frac{5}{2}k^2T\left\{\left(\frac{n_I}{m_I}H^q_{EI} - \frac{n_E}{m_E}H^q_{II}\right)\left[\left(\frac{e_E}{m_E}H^q_{II} + \frac{e_I}{m_I}H^q_{EE}\right)^2 + \left(H^q_{IE}H^q_{EI} - H^q_{EE}H^q_{II}\right.\right.\right.$$
$$\left.\left.\left. + \frac{e_E}{m_E}\frac{e_I}{m_I}B^2\right)\frac{e_E}{m_E}\frac{e_I}{m_I}\right] + \left(\frac{e_E}{m_E}H^q_{II} + \frac{e_I}{m_I}H^q_{EE}\right)(H^q_{IE}H^q_{EI} - H^q_{EE}H^q_{II})\frac{e_I}{m_I}\frac{n_E}{m_E}\right\}, \tag{A.3}$$

$$a_4^E = \frac{1}{D}\left[\left(H^q_{IE}H^q_{EI} - H^q_{EE}H^q_{II} + \frac{e_E}{m_E}\frac{e_I}{m_I}B^2\right)(H^q_{II}H^V_{EE} - H^V_{IE}H^q_{EI})\right.$$
$$\left. + \frac{e_I}{m_I}H^V_{EE}\left(\frac{e_E}{m_E}H^q_{II} + \frac{e_I}{m_I}H^q_{EE}\right)B^2\right](H^q_{IE}H^q_{EI} - H^q_{EE}H^q_{II}), \tag{A.4}$$

$$a_5^E = \frac{1}{D}\left[\left(H^q_{IE}H^q_{EI} - H^q_{EE}H^q_{II} + \frac{e_E}{m_E}\frac{e_I}{m_I}B^2\right)\frac{e_I}{m_I}H^V_{EE}\right.$$
$$\left. + \left(\frac{e_E}{m_E}H^q_{II} + \frac{e_I}{m_I}H^q_{EE}\right)(H^q_{II}H^V_{EE} - H^V_{IE}H^q_{EI})\right](H^q_{IE}H^q_{EI} - H^q_{EE}H^q_{II}), \tag{A.5}$$

$$a_6^E = \frac{1}{D}\left\{(H^q_{II}H^V_{EE} - H^V_{IE}H^q_{EI})\left[\left(\frac{e_E}{m_E}H^q_{II} + \frac{e_I}{m_I}H^q_{EE}\right)^2 + \left(H^q_{IE}H^q_{EI} - H^q_{EE}H^q_{II}\right.\right.\right.$$
$$\left.\left.\left. + \frac{e_E}{m_E}\frac{e_I}{m_I}B^2\right)\frac{e_E}{m_E}\frac{e_I}{m_I}\right] - \frac{e_I}{m_I}H^V_{EE}\left(\frac{e_E}{m_E}H^q_{II} + \frac{e_I}{m_I}H^q_{EE}\right)(H^q_{IE}H^q_{EI} - H^q_{EE}H^q_{II})\right\}, \tag{A.6}$$

$$a_1^I = \frac{1}{H^q_{EI}}\left[\frac{5}{2}k^2T\frac{n_E}{m_E} - H^q_{EE}a_1^E + \frac{e_E}{m_E}a_2^E B^2\right], \qquad a_2^I = -\frac{1}{H^q_{EI}}\left[\frac{e_E}{m_E}a_1^E + H^q_{EE}a_2^E\right], \tag{A.7}$$

$$a_3^I = -\frac{1}{H^q_{EI}}\left[\frac{e_E}{m_E}a_2^E + H^q_{EE}a_3^E\right], \qquad a_4^I = \frac{1}{H^q_{EI}}\left[\frac{e_E}{m_E}a_5^E B^2 - H^q_{EE}a_4^E - H^V_{EE}\right], \tag{A.8}$$

$$a_5^I = -\frac{1}{H^q_{EI}}\left[\frac{e_E}{m_E}a_4^E + H^q_{EE}a_5^E\right], \qquad a_6^I = -\frac{1}{H^q_{EI}}\left[\frac{e_E}{m_E}a_5^E + H^q_{EE}a_6^E\right], \tag{A.9}$$

provided $H^q_{EI} \neq 0$, and where

$$D = \left[\left(H^q_{IE}H^q_{EI} - H^q_{EE}H^q_{II} + \frac{e_E}{m_E}\frac{e_I}{m_I}B^2\right)^2 + \left(\frac{e_E}{m_E}H^q_{II} + \frac{e_I}{m_I}H^q_{EE}\right)^2 B^2\right](H^q_{IE}H^q_{EI} - H^q_{EE}H^q_{II}). \qquad (A.10)$$

The scalar coefficients of $p^\alpha_{\langle ij\rangle}$ are:

$$b^E_1 = c_1(p_E\sigma_{II} - p_I\sigma_{EI}) - 4c_2\frac{e_I}{m_I}p_E B^2, \qquad (A.11)$$

$$b^E_2 = c_2(p_E\sigma_{II} - p_I\sigma_{EI}) + c_1\frac{e_I}{m_I}p_E, \qquad (A.12)$$

$$b^E_3 = c_3(p_E\sigma_{II} - p_I\sigma_{EI}) + 3c_2\frac{e_I}{m_I}p_E - c_4\frac{e_I}{m_I}p_E B^2, \qquad (A.13)$$

$$b^E_4 = c_4(p_E\sigma_{II} - p_I\sigma_{EI}) + c_3\frac{e_I}{m_I}p_E, \qquad (A.14)$$

$$b^E_5 = c_5(p_E\sigma_{II} - p_I\sigma_{EI}) + 4c_4\frac{e_I}{m_I}p_E, \qquad (A.15)$$

$$b^I_1 = -\frac{1}{2}\frac{p_E}{\sigma_{EI}} - \frac{\sigma_{EE}}{\sigma_{II}}b^E_1 + 4\frac{e_E}{m_E}\frac{b^E_2}{\sigma_{EI}}B^2, \qquad (A.16)$$

$$b^I_2 = -\frac{e_E}{m_E}\frac{b^E_1}{\sigma_{EI}} - \frac{\sigma_{EE}}{\sigma_{II}}b^E_2, \qquad (A.17)$$

$$b^I_3 = -\frac{\sigma_{EE}}{\sigma_{II}}b^E_3 - 3\frac{e_E}{m_E}\frac{b^E_2}{\sigma_{EI}} + \frac{1}{2}\frac{e_E}{m_E}\frac{b^E_4}{\sigma_{EI}}B^2, \qquad (A.18)$$

$$b^I_4 = -\frac{\sigma_{EE}}{\sigma_{II}}b^E_4 - \frac{e_E}{m_E}\frac{b^E_3}{\sigma_{EI}}, \qquad (A.19)$$

$$b^I_5 = -\frac{\sigma_{EE}}{\sigma_{II}}b^E_5 - 4\frac{e_E}{m_E}\frac{b^E_4}{\sigma_{EI}}, \qquad (A.20)$$

where:

$$c_1 = \frac{d_1}{4(d_1^2 + 4d_2^2B^2)}, \qquad c_2 = \frac{d_2}{4(d_1^2 + 4d_2^2B^2)}, \qquad (A.21)$$

$$c_3 = \frac{3d_1d_2^2 - d_1^2d_3 + 4d_2^2d_3B^2 - d_1d_3^2B^2}{4(d_1^2 + 4d_2^2B^2)(d_1^2 + d_2^2B^2 + 2d_1d_3B^2 + d_3^2B^4)} \qquad (A.22)$$

$$c_4 = \frac{d_2(3d_2^2 - 2d_1d_3 - d_3^2B^2)}{4(d_1^2 + 4d_2^2B^2)(d_1^2 + d_2^2B^2 + 2d_1d_3B^2 + d_3^2B^4)} \qquad (A.23)$$

$$c_5 = \frac{(9d_2^4 - 9d_1d_2^2d_3 + d_1^2d_3^2 - 7d_2^2d_3^2B^2 + d_1d_3^3B^2)}{(d_1^2 + 4d_2^2B^2)(3d_1 + 4d_3B^2)(d_1^2 + d_2^2B^2 + 2d_1d_3B^2 + d_3^2B^4)} \qquad (A.24)$$

with

$$d_1 = \frac{\sigma_{IE}\sigma_{EI}}{2} - \frac{\sigma_{II}\sigma_{EE}}{2} + 2\frac{e_E}{m_E}\frac{e_I}{m_I}B^2, \qquad (A.25)$$

$$d_2 = -\frac{1}{2}\left(\sigma_{II}\frac{e_E}{m_E} + \sigma_{EE}\frac{e_I}{m_I}\right), \qquad d_3 = -\frac{3}{2}\frac{e_E}{m_E}\frac{e_I}{m_I}. \qquad (A.26)$$

Appendix B: Representation and Inverse of Tensors

Let T_{ij} be an isotropic second-order tensor that is a function of the axial vector $\mathbf{B}$. The representation of T_{ij} is given by:

$$T_{ij} = a\delta_{ij} + b\epsilon_{ijk}B_k + cB_iB_j, \qquad (B.1)$$

where a, b and c are scalar coefficients that depend on B^2. The inverse of T_{ij} is obtained from the Cayley-Hamilton theorem, which can be written as:

$$(\mathbf{T}^{-1})_{ij} = \frac{1}{I_3}\left[\left(\mathbf{T}^2\right)_{ij} - I_1T_{ij} + I_2\delta_{ij}\right] \qquad (B.2)$$

with I_1, I_2 and I_3 denoting the scalar invariants

$$I_1 = T_{ii}, \qquad I_2 = \frac{1}{2}\left[T_{ii}T_{jj} - (\mathbf{T}^2)_{ii}\right], \qquad I_3 = \det(\mathbf{T}). \tag{B.3}$$

Hence for T_{ij} given by (B.1) we have

$$(\mathbf{T}^{-1})_{ij} = \frac{1}{I_3}\left[a(a+cB^2)\delta_{ij} - b(a+cB^2)\epsilon_{ijk}B_k + (b^2-ac)B_iB_j\right] \tag{B.4}$$

where

$$I_3 = (a^2+b^2B^2)(a+cB^2). \tag{B.5}$$

Let T_{ijkl} be and isotropic fourth-order tensor symmetric in the indices (i,j) and (k,l) that is a function of the axial vector $\mathbf{B}$, and let S_{ij} be an arbitrary symmetrical tensor. According to the tables of Smith [7] the representation of the tensor $T_{ijkl}S_{kl}$, which is linear in $\mathbf{S}$ and depends on the skew-symmetric tensor $W_{ij} = \varepsilon_{ijk}B_k$, is given by

$$\begin{aligned} T_{ijkl}S_{kl} = {}& \alpha_1 S_{ij} + \alpha_2[(\mathbf{SW})_{ij} - (\mathbf{WS})_{ij}] + \alpha_3(\mathbf{WSW})_{ij} + \alpha_4[(\mathbf{WSW^2})_{ij} - (\mathbf{W^2SW})_{ij}] \\ & + \alpha_5(\mathbf{SW^2})_{rr}(\mathbf{W^2})_{ij} + \alpha_6 S_{rr}(\mathbf{W^2})_{ij} + \alpha_7 S_{rr}\delta_{ij} + \alpha_8(\mathbf{SW^2})_{rr}\delta_{ij}. \end{aligned} \tag{B.6}$$

where α_1 through α_8 depend on $(\mathbf{W^2})_{rr}$. By taking the derivative of equation (B.6) with respect to $\mathbf{S}$ and returning to the axial vector $\mathbf{B}$, we get the desidered representation for T_{ijkl}. Since we are interested only in the fourth-order tensor $T_{\langle ij\rangle\langle kl\rangle}$, which is symmetric and traceless in (i,j) and (k,l), by performing the symmetrization it reduces to:

$$\begin{aligned} T_{\langle ij\rangle\langle kl\rangle} = {}& a\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}\right) \\ & + b\left(\epsilon_{jlr}B_r\delta_{ik} + \epsilon_{jkr}B_r\delta_{il} + \epsilon_{ilr}B_r\delta_{jk} + \epsilon_{ikr}B_r\delta_{jl}\right) \\ & + c\left(\delta_{ik}B_jB_l + \delta_{il}B_jB_k + \delta_{jk}B_iB_l + \delta_{jl}B_iB_k - \frac{4}{3}\delta_{kl}B_iB_j - \frac{4}{3}\delta_{ij}B_kB_l + \frac{4}{9}B^2\delta_{ij}\delta_{kl}\right) \\ & + d\left(\epsilon_{ikr}B_rB_jB_l + \epsilon_{ilr}B_rB_jB_k + \epsilon_{jkr}B_rB_iB_l + \epsilon_{jlr}B_rB_iB_k\right) \\ & + e\left(B_iB_jB_kB_l - \frac{1}{3}B^2B_iB_j\delta_{kl} - \frac{1}{3}B^2B_kB_l\delta_{ij} + \frac{1}{9}B^4\delta_{ij}\delta_{kl}\right), \end{aligned} \tag{B.7}$$

where a through e are scalar coefficients tha depend on $\mathbf{B}^2$.

The inverse of $T_{\langle ij\rangle\langle kl\rangle}$ is found by the use of the relationship

$$(\mathbf{T}^{-1})_{\langle ij\rangle\langle kl\rangle}T_{\langle kl\rangle\langle mn\rangle} = \frac{1}{2}\left(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm} - \frac{2}{3}\delta_{ij}\delta_{mn}\right), \tag{B.8}$$

and it reads

$$\begin{aligned} (\mathbf{T}^{-1})_{<ij><kl>} = {}& \frac{1}{4(a^2+4\,b^2\,B^2)}\Big[a\Big(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk} - \frac{2}{3}\,\delta_{ij}\,\delta_{kl}\Big) \\ & + b\Big(\delta_{ik}\,\epsilon_{jlr}\,B_r + \delta_{il}\,\epsilon_{jkr}\,B_r + \delta_{jk}\,\epsilon_{ilr}\,B_r + \delta_{jl}\,\epsilon_{ikr}\,B_r\Big) \\ & + \frac{(3\,a\,b^2 - a^2\,c + 4\,b^2\,B^2\,c - a\,B^2\,c^2 - 2\,a\,b\,B^2\,d - a\,B^4\,d^2)}{(a^2+b^2\,B^2+2\,a\,B^2\,c+B^4\,c^2+2\,b\,B^4\,d+B^6\,d^2)}\Big(\delta_{ik}\,B_j\,B_l \\ & + \delta_{il}\,B_j\,B_k + \delta_{jk}\,B_i\,B_l + \delta_{jl}\,B_i\,B_k - \frac{4}{3}\,B_i\,B_j\,\delta_{kl} - \frac{4}{3}\,B_k\,B_l\,\delta_{ij} + \frac{4}{9}\,B^2\,\delta_{ij}\,\delta_{kl}\Big) \\ & + \frac{(3\,b^3 - 2\,a\,b\,c - b\,B^2\,c^2 + a^2\,d + 2\,b^2\,B^2\,d - b\,B^4\,d^2)}{(a^2+b^2\,B^2+2\,a\,B^2\,c+B^4\,c^2+2\,b\,B^4\,d+B^6\,d^2)} \\ & \times\Big(\epsilon_{ikr}\,B_r\,B_j\,B_l + \epsilon_{ilr}\,B_r\,B_j\,B_k + \epsilon_{jkr}\,B_r\,B_i\,B_l + \epsilon_{jlr}\,B_r\,B_i\,B_k\Big) \end{aligned}$$

$$+\frac{36\,b^4-36\,a\,b^2\,c+4\,a^2\,c^2-28\,b^2\,B^2\,c^2+4\,a\,B^2\,c^3+24\,a^2\,b\,d}{(3\,a+4\,B^2\,c+B^4\,e)(a^2+b^2\,B^2+2\,a\,B^2\,c+B^4\,c^2+2\,b\,B^4\,d+B^6\,d^2)}$$
$$+\frac{72\,b^3\,B^2\,d+8\,a\,b\,B^2\,c\,d+12\,a^2\,B^2\,d^2+36\,b^2\,B^4\,d^2+4\,a\,B^4\,c\,d^2-3\,a^3\,e}{(3\,a+4\,B^2\,c+B^4\,e)(a^2+b^2\,B^2+2\,a\,B^2\,c+B^4\,c^2+2\,b\,B^4\,d+B^6\,d^2)}$$
$$+\frac{a\,B^6\,d^2\,e-15\,a\,b^2\,B^2\,e-2\,a^2\,B^2\,c\,e-16\,b^2\,B^4\,c\,e+a\,B^4\,c^2\,e+2\,a\,b\,B^4\,d\,e}{(3\,a+4\,B^2\,c+B^4\,e)(a^2+b^2\,B^2+2\,a\,B^2\,c+B^4\,c^2+2\,b\,B^4\,d+B^6\,d^2)}$$
$$\times\Big(B_i\,B_j\,B_k\,B_l-\frac{1}{3}\,B^2\,B_i\,B_j\,\delta_{kl}-\frac{1}{3}\,B^2\,B_k\,B_l\,\delta_{ij}+\frac{1}{9}\,B^4\,\delta_{ij}\,\delta_{kl}\Big)\Big]. \qquad (B.9)$$

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