Inverse Dynamics in the Classical Fermi Accelerator

M.S. Hussein*, and M.P. Pato
Nuclear Theory and Elementary Particle Phenomenology Group,
Departamento de Física Nuclear, Instituto de Física,
Universidade de São Paulo. C.P. 66318, 05389-970, São Paulo, SP, Brazil

V. Kharchenko†
Institute for Theoretical Atomic and Molecular Physics at the
Harvard-Smithsonian Center for Astrophysics
60 Garden Street
Cambridge, Massachusetts 02138 U.S.A.

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The motion of a particle constrained to move between two walls one of which is moveable is solved. We determine analytically the effective forces that act on the particle or on the moveable wall. The quantal version of the solution is briefly discussed.

I. Introduction

Classical Mechanics text books discuss friction forces without paying much attention to the dvnamics behind their origin. It would be useful for didatical purposes, to develop simple models that allows one to "derive" friction forces and thus enables one to discuss their properties.

In this paper we fully solve a simple one-dimentional problem for this purpose. The problem at hand is that of the motion of a particle of mass m constrained within a region bounded by two walls one with infinite mass and the other with arbitrary mass $M > m$.

When the wall with mass M is oscillating the system is called a Fermi Accelerator [1] and has been investigated in the past by Ulam [2]. Usually Ulam's model has been used for he investigation of the stochastic acceleration of the light particles in collisions with the moving heavy obJects [2-5]. In this paper we study a simpler problem of a free movable wall. The model can be schematized as in Fig. 1. In contrast to the Fermi accelerator, we study the inverse process, where the kinetic energy of the light particle is ransferred to the heavy particle - the movable wall. Upon hitting the fixed wall the particle loses momentum but not energy whereas collision with the movable wall allows for loss of momentum and energy. Our model is relevant to studies involvin the confinement of ultracold neutrons in bottles [6]. Further, it is analytically fully soluble. In Section II, for a given value $\zeta \equiv m/M$, we solve, fully the problem and obtain the value of the velocities of the particle, ν_n, and the movable wall, V_n, after their n^{th} collision. In Section III, we discuss the effective force that acts on the light particle due to the collisions with the wall. In Section IV we give a brief discussion of the quantum mechanical solution recently obtained by Hussein and Kharchenko [7]. Finally, in Section V we present our concluding remarks.

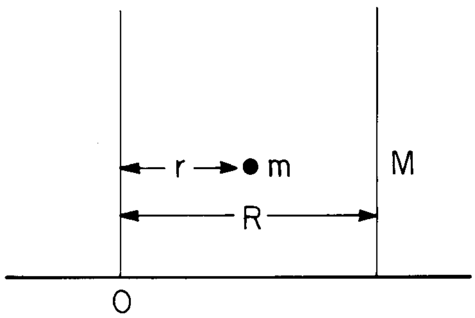

Figure 1. A schematic figure showing the Inverse Fermi Accelerator.

*Supported in part by CNPq (Brazil). e-mail: hussein@if.usp.br
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II. Exact solution of the Inverse Fermi Accelerator

The conservation of momentum and energy after the n collision is given by the following recursive formula

$$m\nu_n + MV_n = -m\nu_{n+1} + MV_{n+1} \qquad (1)$$
$$\frac{1}{2}m\nu_n^2 + \frac{1}{2}MV_n^2 = \frac{1}{2}m\nu_{n+1}^2 + \frac{1}{2}MV_{n+1}^2$$

The solution of the above equations can be written in a matrix form

$$\begin{pmatrix} \nu_{n+1} \\ V_{n+1} \end{pmatrix} = \hat{A}\begin{pmatrix} \nu_n \\ V_n \end{pmatrix} = \begin{pmatrix} \frac{1-\zeta}{1+\zeta} & \frac{-2}{1+\zeta} \\ \frac{2\zeta}{1+\zeta} & \frac{1-\zeta}{1+\zeta} \end{pmatrix}\begin{pmatrix} \nu_n \\ V_n \end{pmatrix}, \qquad (2)$$

where we have introduced the parameter

$$\zeta = \frac{m}{M} \qquad (3)$$

Recursion relation given by Eq.2 allows to evaluate the velocities of the light and heavy particles after $n--th$ collisions, if their initial velocities are known:

$$\nu_{n=0} = \nu_o \;\; ; \;\; V_{n=0} = V_0 \qquad (4)$$

Momentum and energy transfer between heavy and light sub-systems is stopped after some number of collisions. i. e. there is a critical n, n_c, after which the particle always lags behind the now moving wall.

To obtain a closed expression for V_n and ν_n we first recognize that Eq.(2) can be thought of as an evolution equation for a two-dimensional velocity vector which we call

$$\vec{V}_n \equiv \hat{x}\nu_n + \hat{y}V_n. \qquad (5)$$

This vector is not conserved, neither its length nor its direction, and the velocity evolution matrix $\hat{A}$ describes these changes.

It is possible however, to define a constant-length vector in the phase space form the energy conservation equation Eq.1b. vis [8]

$$\vec{Q}_n \equiv \hat{x}\sqrt{m}\nu_n + \hat{y}\sqrt{M}V_n \qquad (6)$$

Clearly

$$\vec{Q}_n^2 = m\nu_n^2 + MV_n^2 = m\nu_0^2 + MV_0^2 \equiv 2E_0, \qquad (7)$$

which is a constant. In this case the collisional evolution of the system is described by the phase Φ_n of the $\vec{Q}_n$ vector

$$\Phi_n = \arctan\left(\sqrt{\frac{M}{m}}\frac{V_n}{\nu_n}\right) \qquad (8)$$

Thus $\vec{Q}_n$ evolves in a circle and the matrix that relates the components $\sqrt{m}\nu_{n+1}$ and $\sqrt{M}V_{n+1}$ to $\sqrt{m}\nu_n$ and $\sqrt{M}V_n$ is an elementary rotation

$$\begin{pmatrix} \sqrt{m}\nu_{n+1} \\ \sqrt{M}V_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{1-\zeta}{1+\zeta} & \frac{-2\zeta^{1/2}}{1+\zeta} \\ \frac{2\zeta^{1/2}}{1+\zeta} & \frac{1-\zeta}{1+\zeta} \end{pmatrix}\begin{pmatrix} \sqrt{m}\nu_n \\ \sqrt{M}V_n \end{pmatrix} \equiv B\begin{pmatrix} \sqrt{m}\nu_n \\ \sqrt{M}V_n \end{pmatrix}. \qquad (9)$$

Call

$$\frac{1-\zeta}{1+\zeta} \equiv \cos\alpha \quad \text{and} \quad \frac{2\zeta^{1/2}}{1+z}\sin\alpha\,. \qquad (10)$$

Thus

$$B = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \qquad (11)$$

which is the rotation matrix in two dimensions through angle α. Clearly, after n collisions the velocity evolution matrix is given by B^n

$$B^n = \begin{pmatrix} \cos n\alpha & -\sin n\alpha \\ \sin n\alpha & \cos n\alpha \end{pmatrix} \qquad (12)$$

Thus

$$\begin{pmatrix} \sqrt{m}\,\nu_n \\ \sqrt{M}V_n \end{pmatrix} = \begin{pmatrix} \cos n\alpha & -\sin n\alpha \\ \sin n\alpha & \cos n\alpha \end{pmatrix}\begin{pmatrix} \sqrt{m}\,\nu_0 \\ \sqrt{M}V_0 \end{pmatrix} \qquad (13)$$

Or finally the velocity vector projections from the Eq.(5) satisfy the eliptic equations

$$\nu_n = \sqrt{\frac{2E_0}{m}}\cos\Phi_n \quad , \quad V_n = \sqrt{\frac{2E_0}{M}}\sin\Phi_n \ , \tag{14}$$
$$\Phi_n = n\alpha + \Phi_0 \ ,$$

and

$$\alpha = \arctan(2\zeta^{1/2}/1-\zeta), \quad \Phi_0 = \arctan\left(\frac{V_0}{\nu_0\zeta^{1/2}}\right) \ , \tag{15}$$

where the initial phase angle Φ_0 can be calculated via the initial velocities of the Eq.(4). The magnitude of the total phase $\Phi_n = n\alpha + \Phi_0$ is restricted: $|\Phi_n| \leq \frac{\pi}{2}$ and it allows the evaluation of the maximum number of collisions $N_{\max}$, for a system with arbitrary initial conditions and mass ratio $\zeta = m/M$. Indeed, according to Eq.(14)

$$n = Int\left[\frac{\Phi_n - \Phi_0}{\alpha}\right] \leq N_{\max} = \frac{\pi}{\alpha}. \tag{16}$$

where the function $Int[x]$ gives the greatest integer less than or equal to x. The evolution of velocities, Eq.(14), is more simple in the case of zero initial velocity of the moveable wall, $V_0 = 0$. The initial phase $\Phi_0 = O$ and

$$\nu_n = \nu_0\cos n\alpha \quad , \quad V_n = \nu_0\zeta^{1/2}\sin n\alpha. \tag{17}$$

The total number of collisions $n = n_c^{(0)}$ between particle and wall is restricted by the conditions $V_n \leq \nu_n$, or $tg(n_c^{(0)}) \leq 1/\zeta^{1/2}$. The last inequality can be written as the restriction of the total angle of rotation in the phase space $n\alpha$:

$$n_c^{(0)} = max[n] = Int\left[\frac{1}{\alpha}\arctan\left(\frac{1}{\zeta^{1/2}}\right)\right] \tag{18}$$

For the adiabatic limit $\zeta = m/M \ll 1$ the elementary phase angle α is very small ($\alpha \simeq 2\zeta^{1/2}$), $\Phi_{n_c^{(0)}} \simeq \frac{\pi}{2}$ and critical value $n_c^{(0)}$ can be estimated as

$$n_c^{(0)} \simeq \frac{1}{2\zeta^{1/2}}\arctan\left[\frac{1}{\zeta^{1/2}}\right] = \frac{\pi}{4}\sqrt{\frac{M}{m}} \tag{19}$$

To verify the above findings, we have numerically solved Eq.(2), taking for initial conditions $\nu(0) = \nu_0 = 1$, $V(0) = 0$, $r(0) = 0$, $R(0) = R_0 = 1$ (see Fig. 1). The results are shown in Figs. 2 and 3 for $\zeta = 4\times10^{-2}$ and 10^{-4}, respectively. The value of n_c in Figs. 2 is 5, close to $\frac{\pi}{4\zeta^{1/2}} \simeq 4$. In Fig. 3, we have $n_c^{(0)} \simeq 78$. compared to $\frac{\pi}{4\zeta^{1/2}} \simeq 78.5$. Further, Fig. 3 shows a rather smooth function of time. The agreement with the analytical results, Eq.(19), is excellent. This would allow the extraction of an effective friction force which we discuss in the following section.

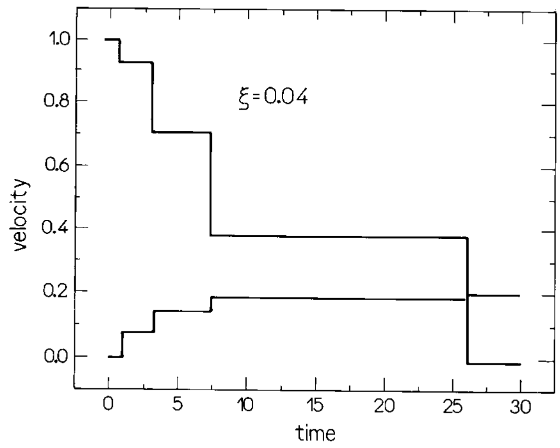

Figure 2. The velocities ν_n and V_n vs. t for $\zeta = 4\times10^{-2}$, $m = \nu_0 = R_0 = 1$.

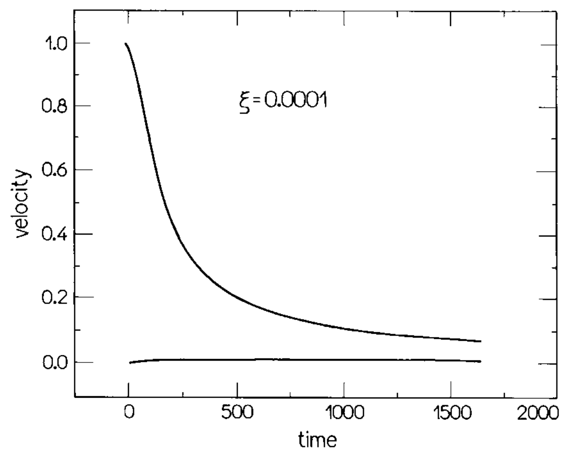

Figure 3. Same as Fig.2 for $\zeta = 10^{-4}$.

For a system with arbitrary initial velocities the critical (or total) number of collisons between particle and moveable wall n_c is defined as a maximal value of n satisfying velocity conditions $\nu_n \geq V_n$, or $tg\ \Phi_n \leq \frac{1}{\sqrt{\zeta}}$:

$$n_c = max[n] = Int\left[\frac{\arctan\left(\frac{1}{\sqrt{\zeta}}\right) - \arctan\left(\frac{V_0}{\nu_0\zeta^{1/2}}\right)}{\alpha}\right] \tag{20}$$

For the adiabatic limit $\zeta = m/M \leq 1$ the elementary phase angle α is very small ($\alpha \simeq 2\zeta^{1/2}$), and one can derive the asymptotic formula

$$n_c \simeq \frac{1}{2\zeta^{1/2}}\left(\frac{\pi}{2} - \Phi_0\right) \tag{21}$$

It is important to note, that the intial phase angle Φ_0 can be positive or negative, depending on the relative orientation of the particle and the wall velocities. If the wall initially moves to the left, $\Phi_0 < O$ and the number of collisions n_s, needed to stop the wall or to change the wall velocity direction is

$$n_s \simeq \frac{|\Phi_0|}{2\zeta^{1/2}} = \frac{1}{2}\sqrt{\frac{M}{m}}\arctan\left[\frac{|V_0|}{\nu_0}\sqrt{\frac{M}{m}}\right] = \frac{1}{2}\sqrt{\frac{M}{m}}\arctan\left[\sqrt{\frac{K_0}{k_0}}\right] \tag{22}$$

where the initial kinetic energies of the light particle and the moveable wall are denoted by k_0 and K_0 respectively. The situation during these n_s, collisions can be described as an acceleration of the particle as the kinetic energy of the wall is transferred to it until the wall comes to a halt, followed by the inverse Fermi acceleration when the particle losses energy. The critical number of collisions for arbitrary initial velocities is

$$n_c^{(\mp)} = (n_c^{(0)} \pm n_s) \tag{23}$$

where $n_c^{(-)}$ and $n_c^{(+)}$ are critical number of collisions for the initial movable wall velocity towards and away from the stationary wall. Phase space diagrams for the different initial conditions are shown in the Fig. 4. From the Eqs. (22), (23) it is not difficult to extract the collisionless condition $n^{(+)} = (n_c^{(0)} - n_s) \leq 0$, that is identical to the velocity relation $V_0 \geq \nu_0$. According to Eq. (22) the maximal number of the collisions neded to stop the wall is $n_s \simeq \frac{1}{2}\sqrt{\frac{M}{m}}\frac{\pi}{2} \simeq n_c^{(0)}$ and thus $n_c^{(-)} = 2n_c^{(0)} \simeq \sqrt{\frac{M}{m}}\frac{\pi}{2}$. This result shows a good agreement with our general estimate of the maximal number of collisions from Eq.(16): $N_{max} = \frac{\pi}{\alpha} \simeq \frac{\pi}{2}\sqrt{\frac{M}{m}}$, which is given for the particular case of asymptotically small value of ζ. The method of finite rotation in phase space, developed in his Section, provides the exact evaluation of the velocity evolution matrix. Natural generalization of this method for the completely quantal Fermi accelerator system allows also to find the exact quantum solution, as has been shown in [7].

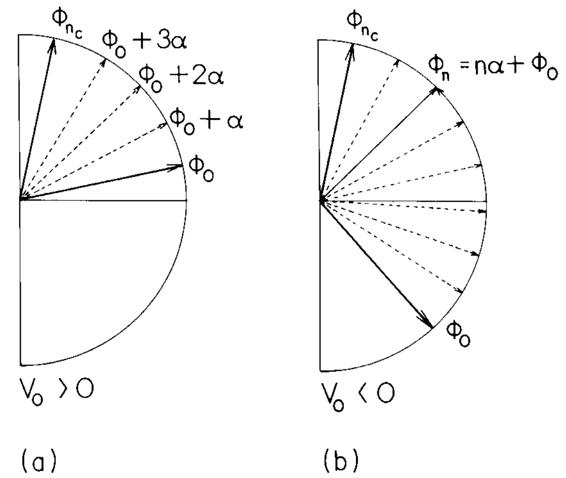

Figure 4. The phase space diagram for the positive (a) and negative (b) velocity projection of moveable wall.

III. The particle-wall effective force

In this section the particle wall effective force is studied for the strong adiabatic condition, when the parameter $\zeta << 1$ and elementary rotational angle $\alpha \simeq 2\zeta^{1/2}$ is very small. The critical number of collisions in this case is very large (see Eqs.(19),(23)) and for the description of the velocity evolution of both the particle and the wall we can use a quasi-continuous variable n related with the time

$$dn = \frac{dt}{\frac{2R_n}{\nu_n}} = \frac{dt\nu_n}{2R_n} \,. \tag{24}$$

We write for the time variation of the kinetic energy of m the following rate equation

$$\frac{d}{dt}\left(\frac{1}{2}m\nu^2\right) = -\nu \cdot F_{ef} \tag{25}$$

$$F_{ef} = m\frac{d\nu}{dt}$$

where F_{ef}, represents the effective force felt by m due to its collisions with the moveable wall. Clearly, from conservation of total energy,

$$F_{ef} = -M\frac{V}{\nu}\frac{dV}{dt} \tag{26}$$

$$dt = \frac{dR_n}{V_n} \tag{27}$$

Since, from Eq. (17)

$$\begin{aligned}\frac{dV_n}{dn} &= \zeta^{1/2}\nu_0\alpha \cos n\alpha \\ &= \alpha\zeta^{1/2}\nu_n \ .\end{aligned} \tag{28}$$

Or with Eq. (24)

$$2R_n\frac{dV_n}{\nu_n dt} = \nu_n\zeta^{1/2}\alpha \ , \tag{29}$$

thus

$$\frac{dV_n}{dt} = \frac{\nu_n^2\zeta^{1/2}\alpha}{2R_N} = \frac{\zeta^{-1/2}\alpha\left(\frac{2E_0}{M} - V_n^2\right)}{R_n} \ . \tag{30}$$

and

$$\frac{dV_n}{V_\infty^2 - V_n^2} = \zeta^{-1/2}\alpha\frac{dt}{R_N} = \zeta^{-1/2}\alpha\frac{1}{V_n}\frac{dR_n}{R_n} \ , \tag{31}$$

which gives

$$V_n = V_\infty\sqrt{1-\left(\frac{R_0}{R}\right)^{\alpha\zeta^{-1/2}}} \equiv \frac{dR_n}{dt} \tag{32}$$

or

$$R_n = R_0\left(\frac{V_\infty^2}{V_\infty^2 - V_n^2}\right)^{\zeta^{1/2}\alpha^{-1}} \tag{33}$$

Now $R_n(t)$ can be obtained from Eq.(32) by simple integration. A closed expression is obtained by setting $\alpha\zeta^{-1/2} \sim 2$ which is an excellent approximation for $\frac{m}{M} << 1$. We thus find

$$\sqrt{R_n^2 - R_0^2} = V_\infty t = \zeta^{1/2}\nu_0 t \tag{34}$$

$$R^2(t) = R_0^2 + V_\infty^2 t^2$$

Putting things together, we finally obtain for the effective force the following expression

$$F_{ef}(\nu_n) = -\frac{m}{R_0}\frac{(\nu_0^2 - \nu_n^2)^{1/2}\nu_n^2}{\nu_0} \tag{35}$$

or

$$F_{ef}(V_n) = -\frac{\sqrt{mM}}{R_0}\frac{(V_\infty^2 - V_n^2)V_n}{V_\infty} \tag{36}$$

and

$$F_{ef}(t) = -(2E_0\zeta^{1/2}R_0)\frac{V_\infty t}{(R_0^2 + V_\infty^2 t^2)^{3/2}} \tag{37}$$

The above functions peak at $\nu_0/\sqrt{3}$, $\zeta^{1/2}\nu_0/\sqrt{3}$ and $R_0/\sqrt{2}\zeta^{1/2}\nu_0$ respectively. Before we end this section, we give in the following the expression for the acceleration of the movable wall, d^2R/dt^2. From Eq.(34), we find

$$\frac{d^2R}{dt^2} = \frac{V_\infty^2 R_0^2}{(R_0^2 + V_\infty^2 t^2)^{3/2}} \tag{38}$$

The numerically generated effective force is shown as functions of $\nu_0 - \nu$, V and t, in Figs. 5, 6 and 7, respectively, for $\zeta = 10^{-4}$. The agreement with our analytical formulae, Eqs. (35), (36) and (37) is excellent. We should mention, however, that Eqs. (35-38) are approximate formulae valid in the limit of a very large number of collisions and at times shorter than the time, t_c, at which the velocities attain their terminal values. Clearly, at longer times the acceleration is identically zero. Before we end this section, we comment on the difference between the effective "friction" force, given by $m\frac{d\nu}{dt}$ and the acceleration force $M\frac{dV}{dt}$. Whereas the movable wall of mass M receives energy through energy - and momentum-conserving collisions with m, the particle suffers also momentum-nonconserving collisions with the fixed wall. In each of these collisions, say the n^{th} one, a quantity of momentum, $2m\nu_n$, is lost to the fixed wall. This is basically the reason why F_{ef}/m, Eq.(37), is qualitativelv different from $\frac{d^2R}{dt^2}$ of Eq.(38).

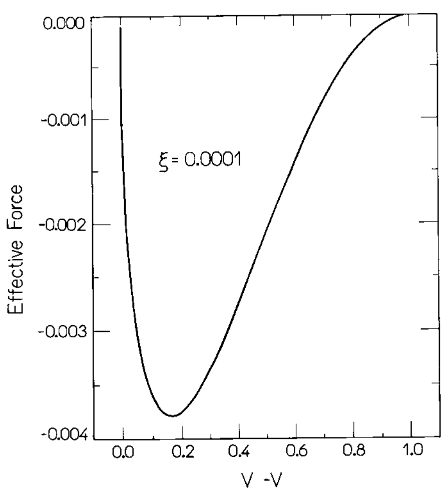

Figure 5. The friction force corresponding to the case of $\zeta = 10^{-4}$ (Fig.3).

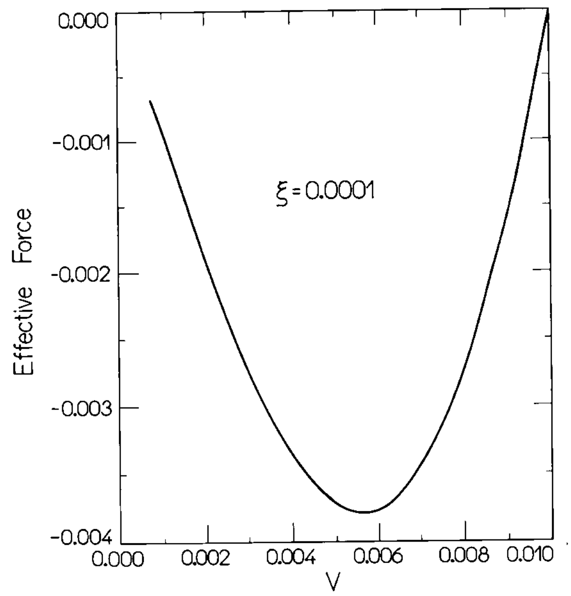

Figure 6. Same as Fig.5 but vs. V.

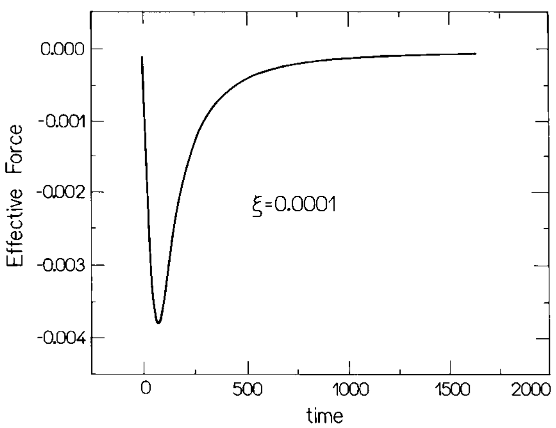

Figure 7. Same as Fig.6 but vs. time.

IV. Semiquantal treatment of the inverse Fermi accelerator

In a recent work [7] two of us soived the problem of the inverse Fermi accelerator quantum mechanically for the light and heavy particles. The solution was obtained for different limits: the semi-quantal, the full Born-Oppenheimer and the exact solutions were presented. It is instructive to compare the classical results obtained in the present paper. with the semi-quantal solution. where the moving wall is treated classically while the particle quantum mechanically. The quantized motion of the particle is fully described by the normalized wave function

$$\phi_n(r,R)=\sqrt{\frac{2}{R}}\sin\left(\frac{n\pi r}{R}\right) \tag{39}$$

$n=1,2....$

The eigenenergies, ϵ_1 are

$$\epsilon_n=\frac{\hbar^2}{2m}\left(\frac{n\pi}{R}\right)^2 \tag{40}$$

The classical "Hamiltonian" that describes the motion of the wall is accordingly given by

$$H=\frac{P^2}{2M}+\epsilon_n(R) \tag{41}$$

It is clear that there are infinite number of Hamiltonians corresponding to the infinite number of eigenstates of m described by (39). From the equation of motion follows

$$M\frac{d^2R}{dt^2}=-\frac{\partial\epsilon_n}{\partial R}=\frac{\hbar^2n^2\pi^2}{mR^3} \tag{42}$$

On the other hand, the conservation of total energy gives

$$\frac{dR}{dt}=\frac{n\pi\hbar}{\sqrt{mM}R_0}\sqrt{1-\frac{R_0^2}{R^2}} \tag{43}$$

Thus

$$R(t)=(R_0^2+V_\infty^2t^2)^{1/2},\ V_\infty=\frac{n\pi\hbar}{\sqrt{mM}R_0} \tag{44}$$

With Eq.(44), Eq.(42) becomes

$$\frac{d^2R}{dt^2}=\frac{V_\infty^2R_0^2}{(R_0^2+V_\infty^2t^2)^{3/2}}\ , \tag{45}$$

Notice, that all reference to quantum mechanics in (45) is contained in the definition of the terminal velocity V_∞, Eq. (44). We see from (38) and (45) an apparent equality between the classical and semiquantal treatments of the inverse Fermi accelerator. There is, however, an important difference between the two. As we have discussed in Section III, Eq.(38) is valid only at times shorter than the limiting time t_c. On the other hand, when treated quantum mechanically, the particle will have continuous distribution of velocities. Accordingly the interaction between wall and the particle may be considered "continuous" thus making Eq.(45) valid for all times.

V. Conclusions

In this paper the classical solution of the inverse Fermi accelerator is presented in details. The effective "friction" force that acts on the particle due to its collisions with the moveable wall in derived. A semiquantal solution of the problem is also presented. The classical and quantal accelerating forces that act on the movable wall were found to be identical at times shorter than the critical time at which classically the particle and wall attain their terminal velocities.

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