# Perturbation Theory for the Fermi Liquids in $d > 1$

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We develop a mathematically rigorous momentum space renormalization group formalism for the perturbative study of interacting Fermi systems in spatial dimensions greater than 1. We first show, exploiting the different geometry of renormalizing around a surface, rather around a point, improved bounds with respect to power counting for some Feynman graphs. These bounds show that among all graphs with 4 external legs, only some special ones, called direct and exchange graphs diverge, and they do so only at Cooper pairs or forward scattering configurations of external momenta. Using these bounds, we propose a novel renormalization scheme for the Fermi liquids, based on the physical idea of renormalizing only momentum configurations close (in a precise sense) to these. We write the beta functional in this formalism, which would tell in principle (and in perturbation theory) whether the system is a normal Fermi liquid or not. Our not renormalizing unnecessary terms makes the the structure of the beta functional simpler, but a complete analysis is still complicated. We explain some of its features, which lead us to *conjecture* as the result of a heuristic analysis that Fermi liquids are normal for any dimension $d > 1$ if the interaction potential is repulsive.

## I. Introduction

We are concerned in this paper with the perturbation theory for a system of *spinless* fermions in $d$ spatial dimensions at zero temperature interacting through a rotation invariant pair potential $\lambda_0(\vec{x}-\vec{y})$, which we will study in the grand-canonical ensemble. The (grand-canonical) Hamiltonian for the non-interacting system is $H_0 = T - \mu N$ , where $T = \frac{1}{2m}\int d\vec{x}\,\vec{\partial}\psi^+_{\vec{x}} \cdot \vec{\partial}\psi^-_{\vec{x}}$ is the total kinetic energy, $\mu$ is the chemical potential and $N = \int d\vec{x}\,\psi^+_{\vec{x}}\psi^-_{\vec{x}}$ is the number of fermions operator. In the above formulae $\psi^{\pm}_{\vec{x}}$ are respectively creation and annihilation operators for fermions at position $\vec{x}$. The interaction part to be summed to $H_0$ is $\overline{\nu}_0 N + V$ , where $V$, given by $V = \frac{1}{2}\int d\vec{x}\,d\vec{y}\,\lambda_0(\vec{x}-\vec{y})\;\psi^+_{\vec{x}}\psi^+_{\vec{y}}\psi^-_{\vec{y}}\psi^-_{\vec{x}}$ , is the pair potential interaction and $\overline{\nu}_0$ is the chemical potential counterterm. The Fermi momentum $p_F$ is given by $p_F = \sqrt{2m\mu}$ .

Interest in quantum many-particle systems has increased in the last years because of the discovery of the high-$T_c$ superconductors, not yet understood. As conduction electrons in these materials are concentrated in Cu-O planes, it could be that some phenomenon characteristic of $d = 2$ is the mechanism for the superconductivity. Discovering in $d = 2$ some anomalous Fermi liquid behavior, *i.e.* some behavior different of the one predicted by the phenomenological Landau theory[1], would be a good bet for explaining high-$T_c$ superconductivity,[2].

We note that existence of anomalous Fermi liquids in $d = 1$ was rigourously established already in 65 by Mattis and Lieb,[3],these results being generalized to all weakly interacting Fermi liquids in $d = 1$,[4,5]. In $d > 1$ there exist numerous calculations *via* the Feynman-graphs approach, see [6] or [7] for a complete account mainly in $d = 3$ and [8],[9] for recent calculations in $d = 2$. In spite of great successes, as in the BCS the-

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ory of superconductivity, most calculations use uncontrolled approximations, such as low-order perturbation theory, random phase, Hartree-Fock, resummations of parts of the perturbative series, *etc*, and it is a common feature that results change according to the approximations used. This is why we think a rigorous approach from first principles is necessary.

Our long term objective is to understand in a mathematically rigorous way whether interacting Fermi systems in $d > 1$ are *normal* Fermi liquids in the sense of Landau theory,[1], or not. The Kohn-Luttinger effect[10], theoretically proposed in 1965, is an example of the necessity for rigorous methods. At that time it was well known that any attractive effective interaction between fermions, no matter how small, would make the Landau picture unstable and lead to superconductivity. Kohn and Luttinger argued then, based on a careful examination of a second-order perturbative calculation, that for systems in $d = 3$, superconductivity would occur at small enough temperatures and high enough angular momentum, even if the interaction potential between fermions is repulsive. Since then there have been various detailed calculations within several kinds of approximations. A recent survey,[11], claims the existence of the Kohn-Luttinger effect in $d = 3$ down to angular momentum $l = 1$, whereas 2-dimensional systems would remain normal Fermi liquids even at zero temperature. One year later,[12], one of the authors would affirm that at $d = 2$ there would be a Kohn-Luttinger effect, too. A mathematical proof of either normality or anomaly (due to Kohn-Luttinger or anything else) of interacting Fermi systems, both in $d = 2$ and in $d = 3$ is still lacking.

The idea of using the renormalization group to attack the problem of interacting many-fermion systems is recent[4,13] Renormalization group methods, even non-rigorous ones, produce in principle safer results. One example for fermions in $d = 1$ is noted by Shankar[14]. The mean-field approximation predicts a charge-density wave for weak repulsion and superconductivity for weak attraction, whereas the renormalization group predicts the correct result (rigorously established in [5]), which is neither of them: a Luttinger liquid for weak interaction of either sign.

Our main result is a new rigorous renormalization group formalism with physical ideas simplifying some aspects of the problem for $d > 1$ in the important case of spherical Fermi surface. We hope to explore in forthcoming papers these simplifications. This formalism originated in [15]. For a more readable exposition of these ideas, with emphasis on the different geometry of renormalizing around a surface, see[16].

Our formalism is an extension of the one invented by Benfatto and Gallavotti[4,17], to any spatial dimension $d$ (theirs works only for odd values of $d$) incorporating important ideas already present in a non-rigorous work by Shankar[14]. In the rest of this introduction we present briefly our results and compare them to the works of Benfatto and Gallavotti and Shankar.

In section II we make our version of the formalism of Benfatto and Gallavotti, introducing a multiscale decomposition of the free-fermion propagator which, like theirs, has Landau quasiparticles present from the beginning. Our multiscale decomposition has the desired properties only in momentum space, so we work in this space. Also in section II we present power counting bounds for the Feynman graphs.

Section III presents the main technical results of this work, showing improved bounds for some Feynman graphs with respect to power counting. These are consequence of the different geometry of renormalizing around a surface, instead of around a finite number of points (the case $d = 1$). Without these improvements, the novel renormalization scheme of section IV would not be possible. We first state and prove theorem 1, which exploits integration over angular variables on the Fermi surface to improve power counting for direct and exchange graphs, showing that these graphs diverge only in some special configurations: Cooper pairs for direct and forward scattering for exchange graphs. We continue then with theorem 5, which shows improved bounds for other graphs. In particular, it shows that all the other 4-legged graphs which are neither direct, nor exchange do not diverge in any momentum configuration.

In section IV, we devise a new *momentum-dependent* and *graph-dependent* renormalization scheme which takes advantage of the improvements with respect to power counting to remove all divergences of the theory. This is another difference with respect to [4], which relies on power counting to define relevant and irrelevant couplings. The main idea behind our scheme is to renormalize *only* really divergent graphs, *only* at the momentum configurations in which they diverge. We see then that many of the terms Benfatto and Gallavotti thought were relevant are in fact irrelevant.

This fact was also noted by Shankar in [14], but in his non-rigorous approach he neglects the irrelevant terms. These, even absent from the beta function, contribute indirectly to the running couplings, because one may build relevant terms from irrelevant ones. He thus obtains an oversimplified beta functional, too easy to analyze, concluding that all Fermi liquids in $d > 1$ should be normal if the interaction potential is repulsive. [1]

On the contrary, the beta functional in the works of Benfatto and Gallavotti for $d = 3$ is too complicated. They are forced to make too many conjectures in order to simplify it and conclude, under these conjectures, that Fermi liquids in $d = 3$ are normal if the interaction is repulsive. The complication of their beta functional comes from two sources. The first is the infinite number of running couplings, consequence of the fact that if $d > 1$ the Fermi surface consists of infinite points. This was overcome in Shankar's approach by neglecting the irrelevant terms. The second source of complication is the inclusion in the beta functional of unnecessary terms coming from graphs which seem relevant by power counting, but are in fact irrelevant if one uses better bounds.

In section V we give a definition of normal Fermi liquid and study the beta functional. Our beta functional is in some sense intermediate between the ones of Shankar and Benfatto and Gallavotti. For example, we also have the same complication of an infinite number of running couplings of the latter, but our renormalization scheme incorporating Shankar's ideas has introduced some simplifications with respect to [4]:

1. Like them, we can also, by an angular momentum expansion, diagonalize the beta functional for the 4-legged couplings, not only to second order in perturbation theory, but to any order. The difference is due to non-inclusion in our beta functional of some graphs irrelevant by theorem 5, but appearing in theirs already at third order.

2. They neglect all 4-legged couplings, except Cooper pairs, without justifying it. Our renormalization scheme provides a factor strongly depleting non-Cooper-pairs and non-forward-scattering configurations, see (40).

3. When considering only the Cooper pairs configurations at second order, Benfatto and Gallavotti *neglect* the contribution of the exchange graph, showing that it is small. In our formalism we are able to cope with the exchange graphs and show that they give important contributions at *forward scattering* configurations. In order to separate in an efficient way contributions from direct and exchange graphs, we define two coupling functions: $\lambda$ for direct graphs and $\mu$ for exchange ones. This is also in line with Shankar.

In spite of the simplifications, our beta functional is still complicated, even at only second order, and a completely rigorous analysis of the flow of the running couplings was not possible and has to be postponed to further work. Nevertheless, in section V we explain some properties of the flow of the running couplings determined by the beta functional and perform a partially heuristic analysis of the flow. On the basis of this analysis, as we explain there, we see no important difference between the cases $d = 2$ and $d = 3$ and no clue for the appearance of the Kohn-Luttinger effect or any other anomaly.

So, unless there is some numerical phenomenon hidden in the contribution of irrelevant terms to the beta functional (this might be perhaps checked in a careful numerical experiment), then our renormalization group should not flow away of the trivial fixed point, signaling (in perturbation theory) that all repulsive Fermi liquids in $d > 1$ are *normal*. We should notice that there is in-

[1] By the way, later in [14] he considers the contribution of one irrelevant term at second order in his beta function and concludes for the existence of the Kohn-Luttinger effect in $d = 3$.

deed an interesting rigorous partial result pointing to refutation of this conjecture: Feldman *et al.* show in [18] that for the repulsive delta function interaction in $d = 2$ a Kohn-Luttinger instability occurs within the Bethe-Salpeter approximation at third order.

The case of non-spherical Fermi surface is technically more difficult, but similar arguments should hold as long as the Fermi surface is non-nested,[14]. Feldman *et al.* have also implemented rigorously the same improvements beyond power counting of this paper in the more general case of non-spherical, non-nested Fermi surfaces[19]. With these bounds, they can also prove perturbative renormalizability of the theory, but they cannot answer yet about normality or anomaly.

## II. Multiscale decomposition and power counting

The free-fermion propagator is given[4] by

$$g(x-y) = \int \frac{dk}{(2\pi)^{d+1}} \frac{e^{ik_0(x_0-y_0)+i\vec{k}\cdot(\vec{x}-\vec{y})}}{ik_0 + e(\vec{k})}, \tag{1}$$

where $e(\vec{k}) = (\vec{k}^2 - p_F^2)/2m$. Due to its (infrared) singularity at the Fermi surface $k_0 = 0$, $|\vec{k}| = p_F$, some Feynman graphs may be divergent. More exactly, if $G$ is some graph with $2n$ external legs, then its contribution to the effective potential is of the form

$$\int dk_1 \dots dk_{2n}\, \delta(k_1 + \dots + k_n - k_{n+1} - \dots - k_{2n})\, f(k_1, \dots, k_{2n})\, \psi^+_{k_1} \dots \psi^+_{k_n} \psi^-_{k_{n+1}} \dots \psi^-_{k_{2n}},$$

where $k_1, \dots, k_{2n}$ are the momenta of the external legs and the function $f$, called *form factor* of the graph, can be infinite due to the singularity of the propagator on the Fermi surface. The propagator is also singular in the ultraviolet, but we will apply an UV cut-off in order to eliminate this technical problem which is expected to have nothing to do with the Fermi surface in the interacting case.

The UV-cut-off propagator can be written in momentum space as

$$\begin{aligned} g_{uv}(k) &= \int_{1/4}^{\infty} d\alpha\, [-ik_0 + e(\vec{k})]\, e^{-\alpha(k_0^2+e(\vec{k})^2)} \\ &= \sum_{n=-\infty}^{0} \int_{2^{-2n-2}}^{2^{-2n}} d\alpha\, [-ik_0 + e(\vec{k})]\, e^{-\alpha(k_0^2+e(\vec{k})^2)} \equiv \sum_{n=-\infty}^{0} \hat{g}_n(k)\,. \end{aligned} \tag{2}$$

As each $\hat{g}_n(k)$ is peaked around $|\vec{k}| = p_F$, then $\hat{g}_n(x)$ oscillates spatially with period $p_F^{-1}$, independently on $n$ as explicitly seen in the formulae for $d = 1, 3$ in [4]. In order to have a good multiscale decomposition we further decompose the propagator over the Fermi surface, introducing *quasiparticle* fields similar to those of [4].

By some simple calculations we get to

$$g_{uv}(x) = \sum_{n=-\infty}^{0} \int \frac{dk\, d\vec{\omega}}{(2\pi)^{d+1}}\, 2^{-n}\, g_n(2^{-n}k)\, e^{i[k_0 t + (k_1 + p_F)\vec{\omega}\cdot\vec{x}]}, \tag{3}$$

with

$$\begin{aligned} g_n(k) = \Omega_d p_F^{d-1} \int_{1/4}^{1} & d\alpha\, \theta\,(k_1 + 2^{-n}p_F) \left[-ik_0 + \beta k_1 (1 + \frac{2^{n-1}}{p_F} k_1)\right] \\ & (1 + \frac{2^n}{p_F} k_1)^{d-1} e^{-\alpha\left[k_0^2 + \beta^2 k_1^2 \left(1 + \frac{2^{n-1}}{p_F} k_1\right)^2\right]}, \end{aligned} \tag{4}$$

being $\beta = p_F/m$ the Fermi velocity, $\Omega_d$ the surface of the $d$-dimensional unit sphere, integration in $k$ means integration over real variables $k_0$ and $k_1$, $\vec{\omega}$ is a unitary vector in $d$ dimensions, integration over $\vec{\omega}$ means averaging over the unit sphere and $\theta$ is the step function.

Notice from (3) that the $\vec{\omega}$-variables above introduced act like the direction of a momentum $(k_1+p_F)\vec{\omega}$, which is near the Fermi sphere if $|k_1| \ll p_F$. So, if we decompose our original *particle* fields $\psi_x$ as a sum

$$\psi_x^\pm = \sum_{n=-\infty}^{0} \int \frac{dk d\vec{\omega}}{(2\pi)^{d+1}} \psi_{k,\vec{\omega}}^{\pm(n)} e^{\pm i[k_0 t + (k_1+p_F)\vec{\omega}\cdot\vec{x}]} \theta(k_1+p_F) , \tag{5}$$

of new independent Grassmann fields $\psi_{k,\vec{\omega}}^{(n)}$ distributed with covariance $\langle \psi_{k,\vec{\omega}}^{-(n)} \psi_{k',\vec{\omega}'}^{+(n)} \rangle = \delta(\vec{\omega}-\vec{\omega}')\,\delta(2^{-n}(k-k'))\,2^{-3n} g_n((2^{-n}k))$, then we can interpret the $\psi_{k,\vec{\omega}}^{(n)}$ as *quasiparticle* fields representing momentum scale $2^n$, $n \le 0$, *around* the Fermi surface. As $g_n(k)$ depends weakly on $n$ when $n \to -\infty$, then all fields $\psi_{k,\vec{\omega}}^{(n)}$ are distributed, up to a trivial scaling factor, roughly the same, as required by renormalization group methods.

Differently from the analogous decomposition in [4], we cannot interpret the fields

$$\psi_{x,\vec{\omega}}^{\pm(n)} = \int \frac{dk}{(2\pi)^{d+1}} \psi_{k,\vec{\omega}}^{\pm(n)} e^{\pm i[k_0 t + k_1 \vec{\omega}\cdot\vec{x}]}\, \theta(k_1+p_F)$$

as quasiparticle fields in position space at length scale $2^{-n}$, for their covariance does not decay appropriately due to the $\theta$-function singularity in (4). This fact shows that our decomposition (5) seems not to be suitable for renormalization group methods in *position space*.

The multiscale decomposition of the fields induces a factorization of the free grassmanian measure as $P_0(d\psi) = \prod_{n=-\infty}^{0} P_n(d\psi^{(n)})$, where $P_n(d\psi^{(n)})$ naturally means the grassmanian measure with covariance $\delta(\vec{\omega}-\vec{\omega}')\,\delta(2^{-n}(k-k'))\,2^{-3n} g_n((2^{-n}k))$.

We may now define the effective potential at scale $h$ (more exactly at momentum scale $2^h$, $h \le 0$), as

$$V^{(h)}(\psi^{(\le h)}) = -\log \int \prod_{n=h+1}^{0} P_n(d\psi^{(n)})\, e^{-V^{(0)}(\psi^{(\le h)}+\psi^{(h+1)}+\dots+\psi^{(0)})} , \tag{6}$$

where $V^{(0)}$ is the interaction Hamiltonian rewritten in momentum space and Wick-ordered second-quantized form as

$$\begin{aligned} V^{(0)} &= \nu_0 \int dp_1 dp_2 \left[ :\psi_{k_1,\vec{\omega}_1}^+ \psi_{k_2,\vec{\omega}_2}^- : \delta(p_1-p_2)\theta(k_1^1+p_F)\theta(k_2^1+p_F)\right] \\ &- \int dp_1 \dots dp_4 \,[\lambda_0(k_1,\dots,k_4,\vec{\omega}_1,\dots,\vec{\omega}_4) : \psi_{k_1,\vec{\omega}_1}^+ \psi_{k_2,\vec{\omega}_2}^+ \psi_{k_3,\vec{\omega}_3}^- \psi_{k_4,\vec{\omega}_4}^- : \\ &\quad \delta(p_1+p_2-p_3-p_4)\theta(k_1^1+p_F)\dots\theta(k_4^1+p_F)] , \end{aligned} \tag{7}$$

with

$$\begin{aligned} \lambda_0(k_1^1,\dots,k_4^1,\vec{\omega}_1,\dots,\vec{\omega}_4) &= \frac{1}{4}[\hat{\lambda}_0(\vec{p}_1-\vec{p}_4) - \hat{\lambda}_0(\vec{p}_1-\vec{p}_3) \\ &+ \hat{\lambda}_0(\vec{p}_2-\vec{p}_3) - \hat{\lambda}_0(\vec{p}_1-\vec{p}_3)] \end{aligned} \tag{8}$$

In these formulae $\nu_0$ can be easily obtained from $\overline{\nu}_0$, $\hat{\lambda}_0$ is the Fourier transform of the pair potential $\lambda_0$ and we use the abbreviation $\vec{p}_i$ for the spatial part of the momentum $i$, i.e. $(k_i^1+p_F)\vec{\omega}_i$, $p_i$ for $(k_0^i,\vec{p}_i)$ and $dp_i$ for $dk_i d\vec{\omega}_i$.

In order to generate a formal perturbative expansion for the effective potentials $V^{(h)}$, one may use the algorithm of expansion in Gallavotti-Nicoló trees[20]. In the resulting scale-decomposed Feynman graphs there are two types of internal lines: *hard* and *soft*, see appendix C in [20]. To each internal line of frequency $h_v$ in a scale decomposed graph is associated a propagator, which is $2^{-h_v} g_{h_v}(2^{-h}k)$ if the line is hard, or

$2^{-h_v} g^s_{h_v}(2^{-h}k)$ if the line is soft, where

$$2^{-n} g^s_n(2^{-n}k) = \sum_{h=-\infty}^{n-1} 2^{-h} g_h(2^{-h}k) \; . \qquad (9)$$

An important remark is that the function $g^s_n$ also decays exponentially at infinity and, in spite of its singularity at $k = 0$, possesses an $L_1$-norm smaller than an $n$-independent constant. In the rest of this paper we will almost ignore soft lines, but the reader can verify that their above mentioned properties are sufficient to guarantee that all bounds that follow are unchanged if they are present.

Power counting arguments in momentum space similar to the position space ones,[4], can be used to bound the form factor of any graph. Let $G$ be a scale-decomposed Feynman graph of perturbative order $m$, originated by a certain tree $T$, with $n^e$ external legs. Let $v$ denote a cluster of $T$ having frequency $h_v$. We will denote $v'$ the cluster of $T$ preceding $v$ in the tree order, *i.e.* the first cluster of $T$ we find if we descend from $v$ towards the root. Let also $v_0$ denote the first non-trivial cluster of $T$, *i.e.*, the first branching point of $T$ if we climb it beginning from the root. If we let $f_{h_{v_0}}$ denote the form factor of $G$, then its power counting bound is

$$|f_{h_{v_0}}(x_1, \ldots, x_{n^e}, \vec{\omega}_1, \ldots, \vec{\omega}_{n^e})| < C^m M^m \, 2^{-h_{v_0'}\delta_{v_0}} \prod_{v \leq v_0} 2^{-(h_v - h_{v'})\delta_v} \; , \qquad (10)$$

where

$$\delta_v = -2 + m_{2,v} + n^e_v/2 \qquad (11)$$

is called *dimension of the cluster* $v$, $C$ and $M$ are constants which do not depend neither on the graph nor on the frequencies attached to its clusters, $m_{2,v}$ is the number of vertices of type $\nu_0$ contained in cluster $v$ but not contained in any of its subclusters and $n^e_v$ is the number of lines external to the cluster $v$. The complete proof of this bound can be found in [15].

As for each term of the product we have $h_v > h_{v'}$, then a graph having $\delta_v > 0$ for all its clusters $v$ converges [2] and the bound for a graph in which some cluster $v$ has $\delta_v \leq 0$ diverges. Power counting is thus sufficient to guarantee that some classes of graphs converge. For example, any graph in which all clusters have 6 or more legs is convergent. On the other hand, we see that clusters with 2 or 4 legs may have non-positive dimension. Clusters with non-positive dimension will be either given an improved finite estimate, or renormalized.

## III. Improvements beyond power counting

Up to now all we have done is equally valid for the Fermi liquids in $d = 1$ and $d > 1$, power counting being used also in bosonic problems to identify relevant and irrelevant couplings. In this section we are going to exploit the different geometry of the $d > 1$ case to prove the main technical results in this work, on which the renormalization scheme of section IV is based.

Consider the graph of Fig. 1, for which we have a divergent power counting bound. Its form factor is given by

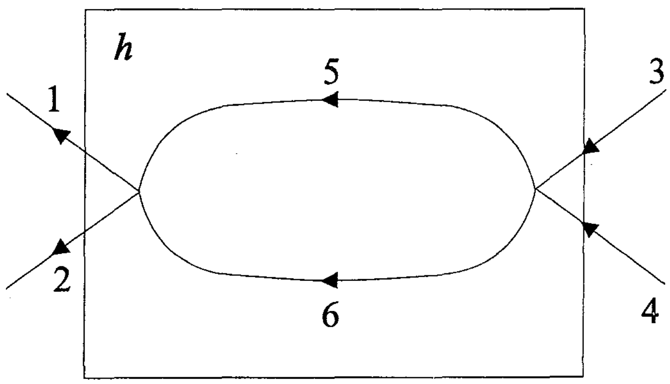


Figure 1. The simplest direct graph.

[2] Of course the dimensional factor $2^{-h_{v_0'}\delta_{v_0}}$ diverges if $\delta_{v_0} < 0$, but the contribution to the *dimensionless* effective potential is convergent.

$$f_h(k_1,\ldots,k_4,\vec{\omega}_1,\ldots,\vec{\omega}_4) = \int \frac{dk_5 d\vec{\omega}_5}{(2\pi)^{d+1}} \lambda_0(1,2,5,6)\,\lambda_0(5,6,3,4)$$
$$2^{-h} g_h(2^{-h}k_5)\, 2^{-h}\overline{g}_h(2^{-h}k_6)\ , \qquad (12)$$

where $\lambda_0(i_1,\ldots,i_4)$ is a short notation for $\lambda_0(\vec{p}_{i_1},\ldots,\vec{p}_{i_4})$ and the variables $k_6$ and $\vec{\omega}_6$ are to be considered as functions of momenta 1,2 and 5 given by

$$\begin{aligned} k_6^0 &= k_1^0 + k_2^0 - k_5^0 \\ k_6^1 &= |(k_1^1+p_F)\vec{\omega}_1 + (k_2^1+p_F)\vec{\omega}_2 - (k_5^1+p_F)\vec{\omega}_5| - p_F \\ \vec{\omega}_6 &= \frac{(k_1^1+p_F)\vec{\omega}_1 + (k_2^1+p_F)\vec{\omega}_2 - (k_5^1+p_F)\vec{\omega}_5}{k_6^1+p_F} \end{aligned} \qquad (13)$$

because of the integration of one momentum-conservation $\delta$-function. Integration of this $\delta$-function also changes the propagator of line 6 to $2^{-h}\overline{g}_h(2^{-h}k)$, where the new function $\overline{g}$ is defined as

$$\overline{g}_n(k) = \int_{1/4}^{1} d\alpha \left[-ik^0 + \beta k^1 (1+\frac{2^{n-1}k^1}{p_F})\right] e^{-\alpha\left[(k^0)^2+\beta^2(k^1)^2\left(1+\frac{2^{n-1}k^1}{p_F}\right)^2\right]}\ , \qquad (14)$$

and is similar to $g_n(k)$. In a general graph there is a *spanning tree*,[4], of lines having $\overline{g}$-propagators, which are all bounded by constants in order to obtain power counting. If we exploit the $\vec{\omega}$ dependence of the propagator we can obtain better bounds. Before stating the theorem which proves it, we introduce some definitions necessary for its full formulation.

A graph is called *direct* if it has 4 external legs, if these external legs originate as pairs from only 2 vertices, if both external legs at the same vertex have arrows pointing in the same direction and if it has no loops sharing an internal line. In Fig. 2 we show some examples of direct graphs.

In Fig. 3 we show some 4-legged graphs which are not direct.

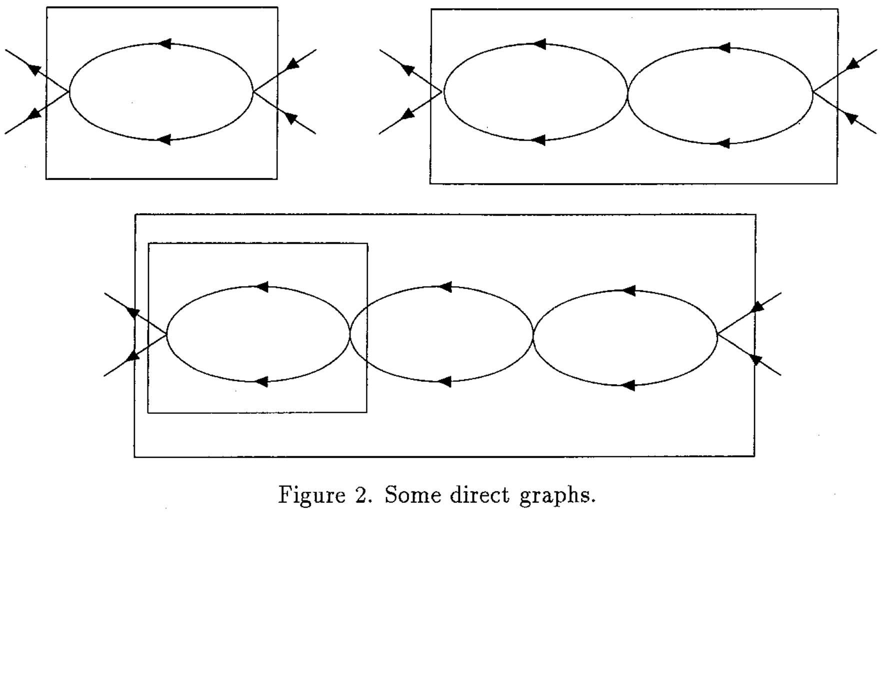

Figure 2. Some direct graphs.

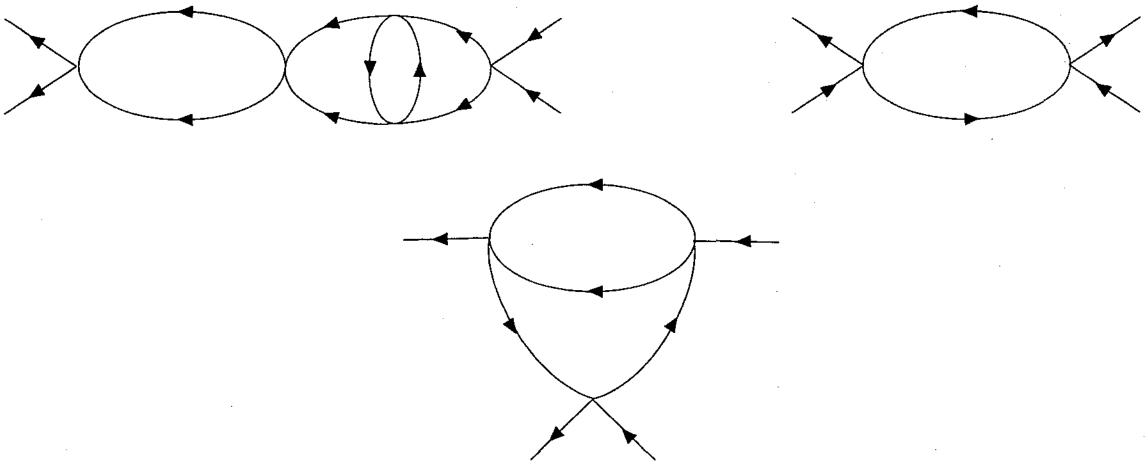

Figure 3. Non-direct graphs.

A graph is called an *exchange graph*, see Fig. 4 for the prototype of them, if it has 4 external legs, if these legs originate as pairs from only 2 vertices, if the external legs originating from the same vertex have arrows pointing in opposite directions and if it has no loops sharing an internal line. Other examples may be obtained from direct graphs by reversing some arrows.

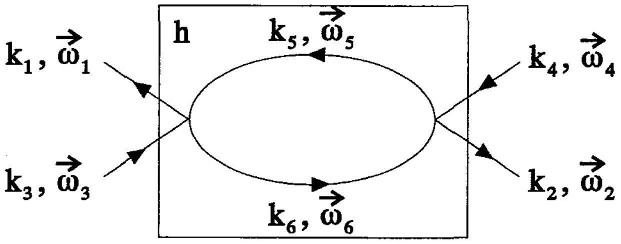


Figure 4. The simplest exchange graph.

If $G$ is a direct graph let us number as 1 and 2 the external lines exiting from one of the two vertices of $G$ from which external lines exit, see Fig. 1. We define some important parameters:

$$\begin{aligned} \Gamma_0 &= k_1^0 + k_2^0 \\ \vec{\Gamma} &= (k_1^1 + p_F)\vec{\omega}_1 + (k_2^1 + p_F)\vec{\omega}_2 \\ \Gamma &= |\Gamma_0| + |\vec{\Gamma}|\,. \end{aligned} \tag{15}$$

Analogously for exchange graphs: if 1 and 3 are external lines exiting from the same vertex, see Fig. 4, we define

$$\Delta = |\Delta_0| + |\vec{\Delta}|\,, \tag{16}$$

where $\Delta_0 = k_1^0 - k_3^0$ and $\vec{\Delta} = (k_1^1 + p_F)\vec{\omega}_1 - (k_3^1 + p_F)\vec{\omega}_3$.

We can now state one of our main results:

**Theorem 1** *Let $G$ be a direct graph with all its internal lines having frequency $h$, $f_h$ be its form factor, $\delta$ its dimension (11) and $\Gamma$ be defined as in (15). Then, if $d > 1$, given any constant $\rho$ such that $0 < \rho < 2p_F$, there exist constants $C$, $C'$ and $N_0$ dependent on $\rho$, but independent of $G$, $\Gamma$ and $h$ such that:*

1. *For $2^h N_0 < \Gamma < 2p_F - \rho$ we have*
$$|f_h| < C^m M^m 2^{-\delta h} \frac{2^h}{\Gamma}\,; \tag{17}$$
2. *For $2p_F - \rho < \Gamma < 2p_F + \rho$ and $2^{-h}\Gamma > N_0$ we have*
$$|f_h| < C^m M^m 2^{-\delta h} 2^{\frac{1}{2}h}\,; \tag{18}$$
3. *For $\Gamma > 2p_F + \rho$ and $2^{-h}\Gamma > N_0$ we have*
$$|f_h| < C^m M^m 2^{-\delta h} 2^h e^{-C' 2^{-2h}\Gamma^2}\,. \tag{19}$$

*In the above formulas, m stands for the number of vertices in the graph and M is an estimate for the coupling constants.*

*A completely analogous result holds for exchange graphs by substituting* $\Gamma$ *by* $\Delta$.

The theorem shows, for sufficiently small $h$ (sufficiently small is here a $\Gamma$-dependent concept) and for some external-momenta configurations, *i.e.* $\Gamma \neq 0$, an improvement with respect to power counting for some 4-legged graphs already sufficient to transform a divergent bound into convergent. If $\Gamma = 0$, we get no improvement at all and are left only with the power counting estimate.

Notice that $\Gamma = 0$ means a configuration of equal and opposite external momenta 1 and 2 and, by total momentum conservation, also equal and opposite momenta 3 and 4; in other words a scattering process in which two fermions of equal and opposite momenta, *i.e.* a Cooper pair, are annihilated and another Cooper pair is created. We call $\Gamma = 0$ a *Cooper pairs* configuration.

A completely analogous result holds for exchange graphs with $\Delta$ in the place of $\Gamma$. Notice that $\Delta = 0$ means a *forward scattering* configuration.

**Sketch of proof for theorem 1** :

We will be referring to the simplest direct graph in Fig. 1 as an example, but at the end the reader can convince himself that the same reasoning will extend trivially to each loop of the other graphs, being it possible to gain one extra improvement per loop.

We begin from the definition (12) of the form factor of the graph. If $G_h$ represents either $g_h$ or $\overline{g}_h$, then it can be easily seen from their definitions that $|G_h(k)| < Ce^{-\gamma[(k^0)^2+\frac{1}{4}\beta^2(k^1)^2]}$, where $C$ and $\gamma$ are positive constants independent of $h$. Using this bound for both $g$ functions in (12), bounding each of the running couplings by a constant $M$, rescaling $k_5' = 2^{-h}k_5$ and integrating $k_5^0$, we arrive at

$$|f_h(k_1,\ldots,k_4,\vec{\omega}_1,\ldots,\vec{\omega}_4)| \leq C^2M^2e^{-\gamma 2^{-h}|\Gamma_0|}\int dk_5^1 e^{-\frac{1}{4}\gamma\beta^2(k_5^1)^2}I(k_5^1)\ , \tag{20}$$

where

$$I(k_5^1) = \int d\vec{\omega}_5\, e^{-\frac{1}{4}\gamma\beta^2 2^{-2h}(|\vec{\Gamma}-(2^h k_5^1+p_F)\vec{\omega}_5|-p_F)^2}\ . \tag{21}$$

In the last formula, integration over $\vec{\omega}_5$ means integration over the $d-1$ angles that localize the unitary vector $\vec{\omega}_5$. But the integrand depends only on the angle $\theta$ between $\vec{\omega}_5$ and $\vec{\Gamma}$. We can then consider without loss of generality only the case $d = 2$, the difference with other dimensions being only a numerical factor, which we can incorporate into the constant $C$ and a factor $\sin^{d-2}\theta$ that we bound by 1.

If we define

$$N = \frac{1}{4}\gamma\beta^2 2^{-2h}|\vec{\Gamma}|^2 \tag{22}$$

$$p(\theta) = \frac{1}{|\vec{\Gamma}|^2}(|\vec{\Gamma} - r\vec{\omega}_5| - p_F)^2 \tag{23}$$

$$r = 2^h k_5^1 + p_F\ , \tag{24}$$

then

$$I(k_5^1) = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, e^{-Np(\theta)}\ . \tag{25}$$

Asymptotic values when $N \to \infty$, *i.e.* $h \to -\infty$, of integrals of this kind can be calculated by Laplace's method[21]. There are three different cases, corresponding to the three cases in theorem 1: *1.* If $|\vec{\Gamma}| < p_F + r$, then $p$ has two points of minimum and $p(\theta)$ is zero in these minima; *2.* If $|\vec{\Gamma}| = p_F + r$, then the unique point of minimum is $\theta = 0$ and $p$ is still zero at this minimum; *3.* If $|\vec{\Gamma}| > p_F + r$, then again $\theta = 0$ is the unique point of minimum and now the value of $p$ at this minimum is greater than zero.

Let us define $\theta_0$ to be the position of the minimum of $p(\theta)$ (if there are two points of minimum we choose $\theta_0$ to be the one with positive sine). If we apply Laplace's method in the first case we get to

$$I(k_5^1) \overset{N\to\infty}{\sim} \frac{2\pi^{1/2}}{\left[\frac{1}{2}Np''(\theta_0)\right]^{1/2}} = \frac{4\pi^{1/2}p_F}{\gamma^{1/2}\beta r\,|\sin\,\theta_0|}\,\frac{2^h}{|\vec{\Gamma}|}\,. \tag{26}$$

By definition of asymptoticity, this implies that there exists $N_0$ such that for $2^{-h}|\vec{\Gamma}| > N_0$ we have

$$I(k_5^1) < 2\,\frac{4\pi^{1/2}p_F}{\gamma^{1/2}\beta r\,|\sin\,\theta_0|}\,\frac{2^h}{|\vec{\Gamma}|}\,.$$

If we could use this result in (20), then the proof of the first assertion in the theorem would follow easily. The problem is that in using "normal" Laplace's method to obtain the last formula, we have tacitly assumed that $|\vec{\Gamma}|$ and $r$ are fixed, so that $N_0$ in principle depends on them. As we still have to integrate on $k_5^1$, which is related to $r$ by (24) and we want $N_0$ to be independent of $|\vec{\Gamma}|$, we need a sort of Laplace's method *uniform* in $|\vec{\Gamma}|$ and $r$.

We will proceed proving the first statement in theorem 1 and in the end comment on the differences in applying the same method to the other two statements. Imposing

$$0 < |\vec{\Gamma}| < 2p_F - \rho \quad \text{and} \quad p_F - \frac{1}{2}|\vec{\Gamma}| < r < p_F + \frac{1}{2}|\vec{\Gamma}| \tag{27}$$

we guarantee that $p(\theta)$ has two minima $\pm\theta_0$ with $\theta_0$ bounded away from zero. This is the key feature in the first part of the theorem.

The proof goes as follows. Under conditions (27) we prove in lemma 2 that $p(\theta)$ has some uniformity properties in $|\vec{\Gamma}|$ and $r$. Then we use these properties in lemma 3 to show that $I(k_5^1)$ has a good bound if $|k_5^1| < \frac{1}{2}2^{-h}|\vec{\Gamma}|$. Finally, lemma 4 shows that these values of $k_5^1$ really give the dominant contribution and resumes the proof of the first part of the theorem for the special direct graph in Fig. 1.

**Lemma 2 (Uniformity lemma)** *Suppose conditions (27). Under these, given any $\epsilon > 0$, there exists $\delta > 0$, $\delta$ independent of $|\vec{\Gamma}|$ and $r$, such that for $|\theta - \theta_0| < \delta$ we have*

$$|p(\theta) - \frac{1}{2}p''(\theta_0)\,(\theta-\theta_0)^2| < \epsilon\,\frac{1}{2}p''(\theta_0)\,(\theta-\theta_0)^2\,. \tag{28}$$

**Proof:** Approximate $p(\theta)$ by its 2nd. order Taylor polynomial and use the differential formula for the remainder to estimate it. Conditions (27) are used to guarantee that each derivative of $p(\theta)$, in particular the third derivative, is bounded.

**Lemma 3 (Uniform Laplace's method)** *Under conditions (27) (which imply $|k_5^1| < \frac{1}{2}2^{-h}|\vec{\Gamma}|$), for any given $\epsilon > 0$ there exists a number $N_0$ independent of $|\vec{\Gamma}|$ and $r$ such that for $N > N_0$ we have*

$$\left|I(k_5^1) - \frac{2\pi^{1/2}}{\left[\frac{1}{2}Np''(\theta_0)\right]^{1/2}}\right| < \epsilon\,\frac{2\pi^{1/2}}{\left[\frac{1}{2}Np''(\theta_0)\right]^{1/2}}\,.$$

*In other words, $I(k_5^1)$ is asymptotic to $2\pi^{1/2}/\left[\frac{1}{2}Np''(\theta_0)\right]^{1/2}$, asymptoticity being uniform in $|\vec{\Gamma}|$ and $r$.*

**Proof:** Just imitate the proof of the ordinary Laplace's method given in [21]. The only difference is that lemma 2 guarantees for any $\epsilon > 0$ existence of $\theta_1$ independent of $|\vec{\Gamma}|$ and $r$ such that for all $\theta \in [\theta_0, \theta_1]$ we have $|p(\theta) - \frac{1}{2}p''(\theta_0)\,(\theta-\theta_0)^2| < \epsilon\,\frac{1}{2}p''(\theta_0)\,(\theta-\theta_0)^2$. This will be used to estimate

$$\int_{\theta_0}^{\theta_1} d\theta\,\left(e^{-Np(\theta)} - e^{-\frac{1}{2}Np''(\theta_0)(\theta-\theta_0)^2}\right)$$

somewhere in the proof, guaranteeing uniformity.

**Lemma 4** *If $0 < \Gamma < 2p_F - \rho$, where $\rho$ is any positive number given, there exists a number $N_0$ independent of $h$ and $\Gamma$ such that the form factor of the graph in Fig. 1 satisfies*

$$|f_h| < C^2 M^2 \frac{2^h}{\Gamma} . \tag{29}$$

**Proof:** Use lemma 3 to show that there exists $N_0$ such that for $2^{-h}|\vec{\Gamma}| > N_0$, $0 < |\vec{\Gamma}| < 2p_F - \rho$ and $|k_5^1| < \frac{1}{2} 2^{-h}|\vec{\Gamma}|$ we have

$$I(k_5^1) < 2 \frac{4\pi^{1/2} p_F}{\gamma^{1/2}\beta r |\sin\theta_0|} \frac{2^h}{|\vec{\Gamma}|} < C'' \frac{2^h}{|\vec{\Gamma}|} ,$$

with $C''$ independent of $h$, $|\vec{\Gamma}|$ and $k_5^1$.

In order to finish the proof of the lemma, and thus of the first part of the theorem we have to control the large values $|k_5^1| > \frac{1}{2} 2^{-h}|\vec{\Gamma}|$ in the integration (20). This is easy because we can use the easy power counting bound $I(k_5^1) < 1$ and exploit the smallness of $e^{-\frac{1}{4}\gamma\beta^2(k_5^1)^2}$ for the large values. This ends the proof of the first part of theorem 1.

The proof of the remaining two parts of theorem 1 is now a mere repetition of the ideas used to prove the first part. We explain the differences.

The hypothesis $|\vec{\Gamma}| > 2p_F + \rho$ of the third part together with $r < p_F + \rho/2$ implies $\theta_0 = 0$ and $p(0)$ and $p''(0)$ bounded away from zero, so that we can prove an analog of lemma 2 approximating $p(\theta)$ near $\theta = 0$ by $p(0) + \frac{1}{2}p''(0)\theta^2$ *uniformly* in $|\vec{\Gamma}|$ and $r$. An analog of lemma 3, shows that $I(k_5^1)$ is uniformly asymptotic to $C\, 2^h\, e^{-C'.2^{-2h}|\vec{\Gamma}|^2}$. Finally, the large values of $k_5^1$ such that $r \geq p_F + \rho/2$ are dealt with by an analog of lemma 4.

The proof of the second part of the theorem is a bit more subtle, for when $|\vec{\Gamma}|$ is in a neighborhood of $2p_F$ and $r$ is close to the dominant value $p_F$, then both $p(\theta_0)$ and $p''(\theta_0)$ are small. There exists $\sigma > 0$, $\sigma$ dependent on $\rho$ but independent of $|\vec{\Gamma}|$, such that for $|r - p_F| < \sigma$ both are also non-negative and the fourth derivative $p^{(IV)}(\theta_0)$ is positive and bounded away from zero. We prove an analog of lemma 2 asserting that for any $\epsilon > 0$ there exists $\delta > 0$, $\delta$ dependent on $\rho$ but independent of $|\vec{\Gamma}|$ and $r$ such that if $2p_F - \rho < |\vec{\Gamma}| < 2p_F + \rho$, $r < p_F + \rho/2$ and $|\theta - \theta_0| < \delta$, then $p(\theta)$ is *bounded below* by $(1-\epsilon)\,\frac{1}{4!}\,p^{(IV)}(\theta_0)(\theta-\theta_0)^4$. We can use then an analog of lemma 3 to show that $I(k_5^1)$ has an *upper bound* uniformly asymptotic to $C/N^{1/4} = O(2^{\frac{1}{4}h})$. Again, as in the other cases, an analog of lemma 4 controls the values of $k_5^1$ such that $|r - p_F| \geq \sigma$.

We will finish this section with one more improvement theorem, corollary of theorem 1, for some classes of graphs, whose prototype is the graph in Fig. 5, the simplest 4-legged one which is neither direct, nor exchange.

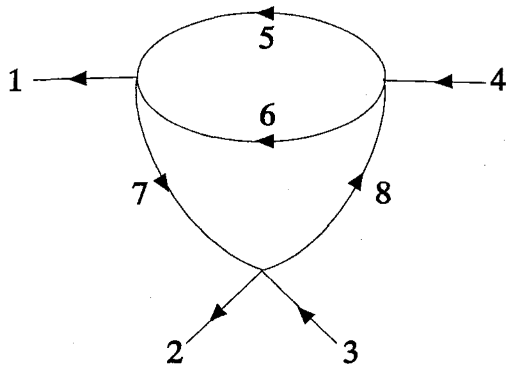


Figure 5. The simplest graph which is neither direct, nor exchange.

**Theorem 5 (Loop improvements)** *Let $G$ be a graph in which all lines have the same frequency $h$ and such that there is at least one internal line being shared by two loops. If $\delta$ is the dimension of the graph, $M$ is an upper bound for the coupling constants, $m$ is the number of vertices and $0 < \epsilon < 1$, then there exists a constant $C$ independent of $G$ such that the form factor $f_h$ obeys*

$$|f_h| < C^m M^m\, 2^{-\delta h}\, 2^{\frac{1}{2}(1-\epsilon)h} . \tag{30}$$

*In other words, for these graphs power counting can be improved by a factor $2^{\frac{1}{2}(1-\epsilon)h}$.*

**Sketch of proof:** Using the notation of Fig. 5, the idea is to integrate internal lines 7 and 8 only in the end. While they are fixed, integration of 5 and 6 leads to the form factor of a direct graph with $\Gamma_0 = k_1^0 + k_7^0$, $\vec{\Gamma} = (k_1^1 + p_F)\vec{\omega}_1 + (k_7^1 + p_F)\vec{\omega}_7$, and $\Gamma = |\Gamma_0| + |\vec{\Gamma}|$, to which we can apply theorem 1, remembering that $\Gamma$ depends on the integration variables $k_7, \vec{\omega}_7$. As the measure of the region in $k_7, \vec{\omega}_7$ where $\Gamma \leq 2^h N_0$ (no improvement in theorem 1) is $O(2^h)$, we can have an improvement independent of the external momenta configuration. The final result is an improvement of only

$2^{\frac{1}{2}(1-\epsilon)h}$ because the second case in theorem 1 gives only $2^{\frac{1}{2}h}$ and a logarithmic correction due to integration of $1/\Gamma$ in the region where $\Gamma > 2^h N_0$ but $\Gamma$ is still $O(2^h)$ small accounts for the $2^{-\epsilon h}$.

## IV. Renormalization

In the usual position-space version of the tree expansion[22,4] renormalization is accomplished by separating the local (relevant) and non-local (irrelevant) parts in the fields of each contribution to the effective potential. In our momentum-space formalism, the same separation into irrelevant and relevant parts is achieved by splitting each form factor $f_h$ in parts $f_h^R$ (irrelevant) and $f_h^L$ (relevant) such that $f_h^R + f_h^L = f_h$, but local will not be anymore a synonym of relevant. The separation scheme should be such that summing on frequency attributions the $R$ parts, one has no more divergences, these being "hidden" in the relevant parts $f_h^L$.

As in any scale $h$, $\mathcal{L}V^{(h)}$, the relevant part of $V^{(h)}$, has to be a term with the same form as $V^{(0)}$, we may write it explicitly as

$$\begin{aligned}\mathcal{L}V^{(h)} &= \int dp_1 dp_2\, 2^h \left[1-(1-e^{-2^{-h}|k|})^2\right] \nu_h(k_1)\,\delta(p_1-p_2)\, :\psi^+_{k_1,\vec{\omega}_1}\psi^-_{k_2,\vec{\omega}_2}: \\ &+ \int dp_1\ldots dp_4 \left[e^{-2^{-h}\Gamma}\lambda_h(k_1,\ldots,k_4,\vec{\omega}_1,\ldots,\vec{\omega}_4) + e^{-2^{-h}\Delta}\mu_h(k_1,\ldots,k_4,\vec{\omega}_1,\ldots,\vec{\omega}_4)\right] \\ &\quad \delta(p_1+p_2-p_3-p_4) : \psi^+_{k_1,\vec{\omega}_1}\psi^+_{k_2,\vec{\omega}_2}\psi^-_{k_3,\vec{\omega}_3}\psi^-_{k_4,\vec{\omega}_4} :\,, \end{aligned} \tag{31}$$

where we are using the same $p_i$ convention used in (7).

The functions $\nu_h(k)$, $\lambda_h(k_1,\ldots,k_4,\vec{\omega}_1,\ldots,\vec{\omega}_4)$ and $\mu_h(k_1,\ldots,k_4,\vec{\omega}_1,\ldots,\vec{\omega}_4)$ appearing at the formula above as coefficients of the terms with 2 and 4 fields are the *running coupling functions.* In usual problems, these functions do not depend on $k$ and $\vec{\omega}$ and are called running coupling *constants.* We will sometimes indicate collectively all running couplings at scale $h$ by $v_h$. In the process of constructing the $\mathcal{R}$ and $\mathcal{L}$ operations separating the irrelevant and relevant parts of the effective potentials it will become clear why we have running functions instead of running constants, why we appended some strange factors to these functions and why we used 2 running coupling functions $\lambda$ and $\mu$ to describe the 4-field interaction.

We begin now the task of defining the $\mathcal{L}$ and $\mathcal{R}$ operations. A very easy definition of $f^L$ and $f^R$ is the one for a cluster with 6 or more external legs. As these clusters are already convergent by power counting, then we can define for them $f_h^L = 0$ and $f_h^R = f_h$.

Let us now define the $\mathcal{L}$ and $\mathcal{R}$ operations for a direct graph with all internal lines at frequency $h$. If we use the pure power-counting prescription of Benfatto and Gallavotti[4] of taking as relevant only the local part in position space, then we know it will work in the sense $\sum_{h=h'+1}^{0} f_h^R$ is not diverging when $h' \to -\infty$, but we also know from theorem 1 that power counting may be treating as divergent some convergent configurations.

A good prescription should use theorem 1 and contain the fact that only configurations with $\Gamma = 0$ cause divergence, but also the information that each $\Gamma \neq 0$ configuration for a fixed $\Gamma$ behaves as if it were relevant while $h$ is not small enough, *i.e.* $2^{-h}\Gamma < N_0$, and irrelevant when $2^{-h}\Gamma > N_0$. Physically, a configuration with $\Gamma \neq 0$ behaves as a Cooper pair, being thus relevant, if the scale $2^h$ of momenta considered is much greater than $\Gamma$, passing smoothly to an irrelevant non-Cooper-pairs configuration when $2^h \ll \Gamma$. We propose then to define

$$f_h^R = (1 - e^{-2^{-(h-1)}\Gamma})\, f_h\,. \tag{32}$$

In fact, we have $f_h^R \approx 0$ and $f_h^L \approx f_h$ if $2^{-h}\Gamma \ll N_0$ and $f_h^R \approx f_h$ and $f_h^L \approx 0$ if $2^{-h}\Gamma \gg N_0$. We show now that this prescription does really exclude from $f_h^R$ all divergences:

**Proposition 6** *If $f_h^R$ is the renormalized form factor of a direct graph with m vertices, all internal lines at frequency h and root h', then, if all running couplings are bounded by M,*

$$|\sum_{h=h'+1}^{0} f_h^R| < C^m M^m ,$$

*where C is a constant independent of h', Γ and of the graph.*

**Sketch of proof:** In order to simplify things, take $\rho = p_F$ in theorem 1, so for $0 < \Gamma < p_F$ we have an improvement in the bound for $f_h$ of the form $\frac{2^h}{\Gamma}$, if $2^{-h}\Gamma > N_0$. If $\Gamma > p_F$, then, take $\epsilon = \frac{1}{2}$ and get an improvement of at least $2^{\frac{1}{4}h}$ proving the proposition in this case. Let us then concentrate on the less easy case $0 < \Gamma < p_F$.

Given Γ, let $H$ be defined as the smallest integer such that $2^{-H}\Gamma \leq N_0$. If $h < H$, then the factor $1 - e^{-2^{-(h-1)}\Gamma}$ is almost equal to 1, but we can use the $\frac{2^h}{\Gamma}$ improvement; summing it from $h = -\infty$ up to $h = H$, we get $\frac{2^H}{\Gamma}$, which is $O(1)$. If $h \geq H$, we can use that the factor $1 - e^{-2^{-(h-1)}\Gamma}$ is bounded by $2^{-(h-1)}\Gamma$; summing from $h = H$ up to $h = 0$ we get again something of order $2^{-H}\Gamma$, which is again $O(1)$. This ends the proof.

Analogously, if $f_h$ is the form factor of an *exchange* graph in which all lines are at the same frequency $h$, then we define its irrelevant part as

$$f_h^R = (1 - e^{-2^{-(h-1)}\Delta}) f_h . \qquad (33)$$

We are now able to explain some strange features of (31). First of all, we define two running couplings $\lambda_h$ and $\mu_h$, because we want to keep separated the contributions to the relevant part $\mathcal{L}V^{(h)}$ coming from direct and exchange graphs. So, $\lambda_h$ is by definition formed from contributions of direct graphs and $\mu_h$ from contributions of exchange graphs. This separation means that instead of having only one type of 4-line vertex in the renormalized graphs, we now have two. Every time we draw a graph we must also specify for each 4-line vertex if that vertex is a $\lambda$ or a $\mu$.

Now the factors accompanying the vertices. As the relevant part of the form factor $f_{h+1}$ of a direct graph is $f_{h+1}^L = e^{-2^{-h}\Gamma} f_{h+1}$, then it is natural to define the running coupling $\lambda_h$ so that the sum of all these contributions is not $\lambda_h$, but $e^{-2^{-h}\Gamma}\lambda_h$. The same for exchange graphs.

But as we are talking of two different types of 4-line vertices, we must now correct our renormalization prescription for direct and exchange graphs to take into account not only the two different types of vertices but also the factors accompanying them. All our preceding estimates on direct and exchange graphs remain correct, because the factors accompanying the $\lambda$ and $\mu$ functions are both less than 1, but we can improve them in some cases.

Suppose for example that in a direct graph like the one in Fig. 1 the vertex at the left is a $\lambda$ and the other a $\mu$. Accompanying the $\mu_h$ function of the right vertex we have a factor $\exp[-2^{-(h-1)}\Delta]$. As we have to integrate momentum of line 5, on which $\Delta = |k_3^0 - k_5^0| + |(k_3^1 + p_F)\vec{\omega}_3 - (k_5^1 + p_F)\vec{\omega}_5|$ depends, and as the factor containing $\Delta$ decreases very quickly when $\Delta$ is greater than $O(2^h)$, then the effective size of the integration region is reduced to $O(2^h)$ around $k_5^0 = k_3^0$ and $(k_5^1 + p_F)\vec{\omega}_5 = (k_3^1 + p_F)\vec{\omega}_3$. We can then gain an extra, momentum-independent, $2^h$ factor with respect to power counting and the graph can be considered as irrelevant and renormalized as $f_h^R = f_h$. An analogous result holds for exchange graphs with some $\lambda$-vertex.

If a graph has 4 external lines, all internal lines of the same frequency and is neither direct, nor exchange, then, by theorem 5, it is already convergent and we define $f_h^R = f_h$ and $f_h^L = 0$. Of course these contribute neither to $\lambda$-, nor to $\mu$-couplings.

With all these observations we can now conclude that the only graphs with 4 external lines and all internal lines with the same frequency that need to be non-trivially renormalized are the *direct* ones in which *all vertices are* $\lambda$ and the *exchange* ones in which *all vertices are* $\mu$. For these graphs we use respectively prescriptions (32) and (33).

This means that we have already shown how to renormalize any graph with all internal lines of the same frequency and 4 or more legs. We must still learn how to do the same for 2-legged graphs with all lines at the same frequency and then how to deal with graphs having lines of different frequencies.

By (11), the dimension of a cluster with 2 external legs in which there is no 2-legged vertex inside (*i.e.* $m_{2,v} = 0$) is -1. It is usual in renormalization group to define all running couplings to be *dimensionless.* This means that the running coupling $\nu_h$ associated to the contribution of 2-lined graphs should be defined so that each graph contributes to $2^h \nu_h$. As the dimension of 4-legged vertices is 0, their running couplings $\lambda_h$ and $\mu_h$ are already dimensionless.

Of course, when we are using power counting to bound renormalized graphs, in which vertices should be thought of as representing not the bare constants $v_0$ but the running couplings $v_h$, then it is necessary to take into account the extra $2^h$ factor accompanying all the 2-legged vertices. That can be easily done and the result is that the dimensions (11) of the clusters are changed to

$$\delta_v = -2 + \frac{1}{2} n_v^e , \tag{34}$$

now dependent only on the number of external legs of the cluster.

Notice however that this argument is up to now pure power counting. We showed for 4-legged clusters some improvements beyond power counting which proved that many of the configurations we thought were relevant are indeed irrelevant. In the case of 2-legged clusters, the (power counting) dimension -1 seems to imply that these clusters are relevant for any momentum configuration. We now show easily that 2-legged clusters are relevant only when their external momentum is $O(2^h)$ or smaller. More exactly, if $k = (k^0, k^1)$ is the external momentum variable of a 2-legged cluster, we define $|k| = |k^0| + |k^1|$ and the irrelevant part $f_h^R$ of the form factor $f_h$ of a 2-legged cluster is defined as

$$f_h^R(k) = \left(1 - e^{-2^{-(h-1)}|k|}\right)^2 f_h(k) . \tag{35}$$

A calculation analogous to the one in the proof of proposition 6 now shows that prescription (35) does really work. If we define $H$ as the smallest integer such that $2^{-H}|k| \leq 1$, then

$$\begin{aligned} |\sum_{h=h'+1}^{0} f_h^R| &< C^m M^m C' 2^H = C^m M^m C' 2^{h'} 2^{-h'} |k| \frac{1}{2^{-H}|k|} \\ &< C^m M^m 2C' 2^{h'} 2^{-h'} |k| . \end{aligned}$$

In this formula $2^{h'}$ is the correct dimensional factor for the sum, as it is a contribution to the 2-field part in $V^{(h')}$, and the dimensionless factor $2^{-h'}|k|$ offers no problem, as it can be absorbed in the propagator of lines at scale $h'$, defining an effective propagator $g_{h'}^{eff}(2^{-h'}k) = 2^{-h'}|k| g_{h'}(2^{-h'}k)$ at that scale which decays exponentially when $k$ is $O(2^{h'})$, just as the plain propagator.

As a consequence of (35), the relevant part of a graph with 2 external lines, root frequency $h-1$ and first non-trivial cluster at frequency $h$ is then $f_h^L(k) = \chi(2^{-(h-1)}|k|) f_h$, where $\chi(k) = \left[1 - \left(1 - e^{-|k|}\right)^2\right]$. It is then natural to define the running couplings $\nu_{h-1}$ such that the sum of the contributions coming from all graphs like the one considered is $2^{h-1}\chi(2^{-(h-1)}|k|)\nu_{h-1}(k)$. This explains the strange form of the coefficient of the 2-field term in (31). As it happened to the running couplings related to 4-field interactions, $\nu_h$ is also a function, not a constant. It can be shown, by rotational invariance, that the $\nu$ are functions only of $k$ and not of quasimomenta $\vec{\omega}$.

In order to complete the renormalization prescription, we now have to say how to renormalize graphs in which there are internal lines of different frequencies. The prescription is to begin by renormalizing the innermost clusters in the graphs, in which all lines have the same frequency. Erase then these clusters and substitute them for "effective" vertices with the same number of lines; in the resulting graph renormalize the innermost clusters again and continue the procedure until you arrive at the outermost cluster. With this idea it can be shown that all renormalized graphs are convergent.

As an example take the graph in Fig. 6, a contribution to $\mathcal{R}V^{(h)}$, generated by the tree in the same figure. We renormalize it as follows.

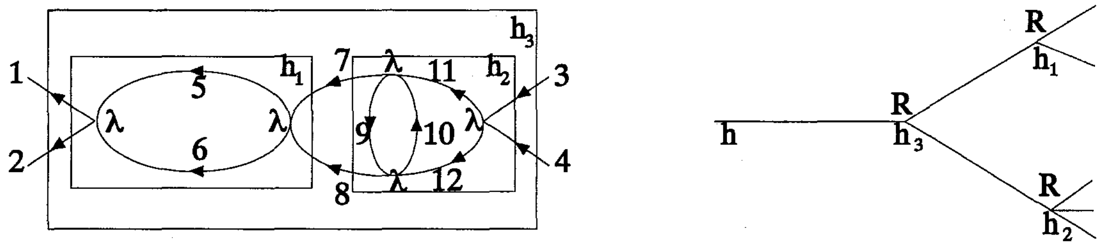


Figure 6. An example.

If we erase the clusters of frequencies $h_1$ and $h_2$ and substitute them for effective vertices we get the graph in Fig. 7, which we call *the graph of Fig. 6 seen at frequency* $h_3$.

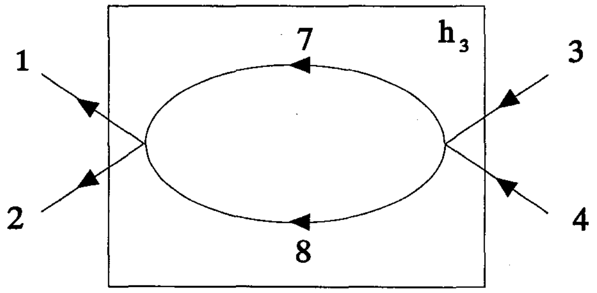


Figure 7. The graph of the previous figure seen at frequency $h_3$.

As it is a direct graph, we may renormalize it at frequency $h_3$ as $f^R_{h_3} = (1 - e^{-2^{-(h_3-1)}\Gamma}) f_{h_3}$, where $f_{h_3}$ was obtained renormalizing clusters $h_1$ and $h_2$, *i.e.*

$$f_{h_3} = \int dk_7 d\vec{\omega}_7 \, f^R_{h_1}(1,2,7,8)\, f^R_{h_2}(7,8,3,4)\, 2^{-2h_3}\, g_{h_3}(2^{-h_3}k_7)\, \overline{g}_{h_3}(2^{-h_3}k_8) \,, \tag{36}$$

with $f^R_{h_1} = (1 - e^{-2^{-(h_1-1)}\Gamma}) f_{h_1}$ and $f^R_{h_2} = f_{h_2}$.

## V. The beta functional and properties of the Fermi surface

Having renormalized our theory, now the (renormalized) tree expansion gives us the running couplings at each scale $h$ as a perturbative series in the running couplings at every preceding scale $l > h$, [22]. These evolution equations for the running couplings are called the beta functional.

As we said at the beginning of this work, our main concern is about normality or anomaly of the Fermi surface. The answer to this question depends on the behaviour of the running couplings predicted by the beta functional, as we now explain.

From the relation between the effective potentials and Schwinger functions, see [4] or [22], we know that the 2-point Schwinger function is given as

$$S_{(\geq h)}(p) = \frac{1}{ip_0 - e(\vec{p})} + \left(\frac{1}{ip_0 - e(\vec{p})}\right)^2 \left[2^h \nu_h(k)\, \chi(2^{-(h-1)}|k|) + w_h(k)\right] \,, \tag{37}$$

where $S_{(\geq h)}$ is the Schwinger function evaluated with the infrared cut-off at frequency $h$, $p$ is the momentum, related to the $k, \vec{\omega}$ variables by the convention explained after equation (8) and $e(\vec{p}) = (\vec{p}^2 - p_F^2)/(2m)$ . The first term in the right-hand side is the *free* 2-point Schwinger function, whereas the second term is the contribution of the interaction. In this second term there are also two parts: the $\nu$ term is the relevant part of the 2-field term in $V^{(h)}$ and the $w$ term, which we do not write explicitly, is the irrelevant part of the same 2-field term.

In order to have in the interacting case a Fermi surface with the same thermodynamical properties as the free case, *i.e.* a normal Fermi liquid, it is necessary that the 2-point Schwinger function maintains the $[ip_0 - e(\vec{p})]^{-1}$ singularity of the free case. So, it seems reasonable to *define* a normal Fermi liquid as one in which the term with singularity proportional to $[ip_0 - e(\vec{p})]^{-2}$ in (37) vanishes. As $w_h(k)$ is the irrelevant part of the 2-field term in $V^{(h)}$, then it is proportional to $\left(1 - e^{-2^{-(h-1)}|k|}\right)^2$, which goes quadratically to 0 when $k \to 0$, causing then no singularity. The necessary condition for a Fermi liquid to be normal is then that the function $Z(k)$ defined by

$$Z(k) = \lim_{h \to -\infty} \left[2^h \nu_h(k)\, \chi(2^{-(h-1)}|k|)\right] \qquad (38)$$

goes to zero when $k \to 0$ as $[ip_0 - e(\vec{p})]^n$, with $n \geq 1$. In particular, when $k = 0$ this implies that we must have

$$2^h \nu_h(0) \stackrel{h \to -\infty}{\longrightarrow} 0\ . \qquad (39)$$

Observe that this definition accounts for a possible wave-function renormalization if it happens that $n = 1$ above.

The counterterm $\nu_0$ should be thought of as a constant function $\nu_0(k)$ fixed from the beginning so that the physical value of the Fermi momentum is $p_F$. But the natural definition of physical value of $p_F$ is the position of the singularity in the 2-point Schwinger function[3], so that normality of the Fermi liquid depends on the possibility of fine-tuning $\nu_0$ such that (39) is satisfied.

By standard renormalization group arguments, versions of the stable manifold theorem of Dynamical Systems, one can show existence of a unique $\lambda_0$-dependent value of $\nu_0$ such that $\nu_h(0) \stackrel{h \to -\infty}{\longrightarrow} 0$. This fulfills condition (39), which is part of the complete condition for normality of the Fermi surface. The rest of the normality definition, including the possible wave function renormalization, should then be checked *a posteriori*.

If we want to believe in our perturbative results, then the other condition to be checked, necessary for normality of the Fermi surface, is that the running couplings do not grow unboundedly when $h \to -\infty$. This is what is called *consistence* of perturbation theory, to which we now turn.

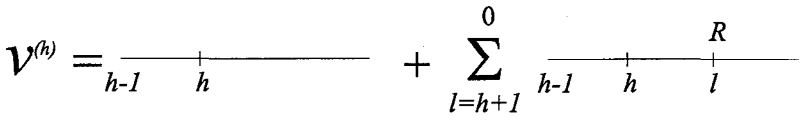


Figure 8. Tree expansion for $V^{(h)}$ to first order.

In Fig. 8 we show the renormalized tree expansion for $V^{(h-1)}$ to first order in perturbation theory. As usual, see[22], an end line of a tree attached to a frequency label $l$ means $\mathcal{L}V^{(l)}$. In the formalism of Benfatto and Gallavotti, where a $\mathcal{R}$ applied over a $\mathcal{L}$ gives a zero result, all the trees other than the first are null. That does not happen in our formalism. As the sum over $l$ in Fig. 8 is already convergent, as required for $R$ vertices in trees, it is natural to define the $\mathcal{L}$ operation at scale $h$ for all the trees within that sum as 0; thus only the first tree in the figure will contribute to the

[3]This is analogous to the definition of physical mass in Quantum Field Theory as the position of the pole of the 2-point Schwinger function.

running couplings $v_{h-1}$.[4] As a consequence, we have as the first-order beta functional

$$\begin{aligned} \nu_{h-1}(k) &= 2\chi(2^{-h}|k|)\,\nu_h(k) \\ \lambda_{h-1}(k_1,\ldots,k_4,\vec{\omega}_1,\ldots,\vec{\omega}_4) &= e^{-2^{-h}\Gamma}\,\lambda_h(k_1,\ldots,k_4,\vec{\omega}_1,\ldots,\vec{\omega}_4) \\ \mu_{h-1}(k_1,\ldots,k_4,\vec{\omega}_1,\ldots,\vec{\omega}_4) &= e^{-2^{-h}\Delta}\,\mu_h(k_1,\ldots,k_4,\vec{\omega}_1,\ldots,\vec{\omega}_4) \end{aligned} \tag{40}$$

From these, we see to first order that running couplings in configurations with non-zero $k,\Gamma,\Delta$, respectively, are strongly depleted to zero as $h \to -\infty$. As other terms depleting configurations with non-zero $k,\Gamma,\Delta$ are present also at higher orders, it seems natural to *conjecture* that $\nu,\lambda,\mu$ for non-zero $k,\Gamma,\Delta$, respectively, quickly converge to zero as $h \to -\infty$. If this conjecture above is true, then we are left only with $\nu_h(0)$, $\lambda_h$ at Cooper pairs configurations and $\mu_h$ at forward scattering configurations to be concerned about. Of these three couplings, we already know that it is possible to fine tune the initial value $\nu_0$ such that $\nu_h(0) \to 0$. We are then left with $\lambda_h$ and $\mu_h$ at Cooper pairs and forward scattering, respectively.

At second order, the only contribution to the beta functional for the $\lambda_{h-1}$ is given by the graph in Fig. 1, where each vertex may contribute either as $\lambda_h$, or as a renormalized cluster of frequency $l > h$, *i.e.* one of the trees within the sum in Fig. 8. Accordingly, taking into account the multiplicities of the graphs and the possible soft lines, see page 48, we have

$$\begin{aligned} \lambda_{h-1}(1,\ldots,4) &= e^{-2^{-h}\Gamma}\lambda_h(1,\ldots,4) \\ &- e^{-2\cdot 2^{-h}\Gamma}\int dk_5\, d\vec{\omega}_5\, G_h(k_5,\vec{\omega}_5)\,\Lambda_h(1,2,5,6)\,\Lambda_h(5,6,3,4)\,, \end{aligned} \tag{41}$$

where

$$G_h(k_5,\vec{\omega}_5) = \frac{4}{(2\pi)^{d+1}}\,2^{-2h}\,\overline{g}_h(2^{-h}k_6)\,\left[g_h(2^{-h}k_5) + 2g_h^s(2^{-h}k_5)\right] \tag{42}$$

and

$$\Lambda_h = \lambda_h + \sum_{l=h+1}^{0}\left[(1-e^{-2^{-(l-1)}\Gamma})e^{-2^{-l}\Gamma}\lambda_l + (1-e^{-2^{-(l-1)}\Delta})e^{-2^{-l}\Delta}\mu_l\right]\,. \tag{43}$$

In (41) and (42), $k_6$ and $\vec{\omega}_6$ are functions of $k_5,\vec{\omega}_5$ given by (13).

Equation (41) is a complicated evolution, but it is remarkably simplified for Cooper pairs configurations. In order to simplify the notation, let us define a symbol for the $\lambda$ in Cooper pairs configurations: $\lambda_h(1,3) = \lambda_h(1,-1,3,-3)$, where $-i$ means $k_i,-\vec{\omega}_i$.

As a first simplification, the contributions to $\Lambda_h$ coming from renormalized $\lambda$-vertices are null, if $\Gamma = 0$. We may also neglect the contributions from renormalized $\mu$-vertices, because the $\mu$ will be sizeable only if $\Delta = 0$ and, in this case, they will be null because of the factor $1 - e^{-2^{-(l-1)}\Delta}$.

Another simplification is that as $\lambda_0$ does not depend on the zero components $k_i^0$ of the $k$ variables, then, for $\Gamma = 0$, it can be seen from (41) that $\lambda_{-1}(1,3)$ does not depend on $k_1^0$ and $k_3^0$. Continuing the inductive procedure, we have that to second order, *i.e.* evolution given by (41), the $\lambda_h$ in Cooper pairs do not depend on the $k^0$ variables.

Another fact is that the imaginary part of the func-

[4] Of course that does not mean that the contribution of the remaining trees in Fig. 8 is zero. They contribute directly to the irrelevant part of the effective potential and, at higher orders, they will also contribute to the relevant part.

tion $G_h(k_5, \vec{\omega}_5)$ appearing in (41) is an odd function of the variable $k_5^0$. As the $\lambda_h$ in Cooper pairs do not depend on $k_5^0$, then the imaginary part of $G_h(k_5, \vec{\omega}_5)$ integrates to zero. As the $\lambda_0$ are real numbers, then this argument shows that the $\lambda_{-1}$ are also real and, by induction, the $\lambda_h$ in Cooper pairs are real for any value of $h$. Of course the argument does not hold for configurations which are not Cooper pairs. It can be seen that this property is also easily extensible to any finite order in perturbation theory.

We continue by noticing that for $d = 2$ we can expand the $\lambda_h$ in Cooper pairs in a Fourier series

$$\lambda_h(1,3) = \sum_{l=-\infty}^{l=\infty} \lambda_h^l(k_1^1, k_3^1) \frac{e^{il(\theta_1 - \theta_3)}}{\sqrt{2\pi}} , \tag{44}$$

where $\lambda_h^l(k_1^1, k_3^1)$ are the Fourier coefficients and, by rotational invariance, $\lambda_h(1,3)$ depends only on the difference of the angles $\theta_1, \theta_3$ which the $\vec{\omega}$'s make with some axis. There is an analogous expansion in Legendre polynomials in the case $d = 3$, see[4], section 13, and all the results we describe from now on remain valid also in this case.

By Fourier expanding we can evaluate the angular integral in (41), obtaining then that the evolution equations for the coefficients $\lambda_h^l$ are diagonal in $l$:

$$\lambda_{h-1}^l(k_1^1, k_3^1) = \lambda_h^l(k_1^1, k_3^1) - \sqrt{2\pi} \int dk_5\, G_h(k_5)\, \lambda_h^l(k_1^1, k_5^1)\, \lambda_h^l(k_5^1, k_3^1) . \tag{45}$$

In order to arrive at this result, it has been used that $k_6$ becomes independent of $\vec{\omega}_5$, for Cooper pairs, and then $G_h$ becomes independent of it, too, and can be taken out of the angular integral.

This trick of Fourier expanding is very interesting, because the complicated dependence of (41) on the 3 angles $\theta_1$, $\theta_3$ and $\theta_5$ disappears and we have the same evolution for any $l$ without ever mixing different values.

But there is more we can say. As a consequence of the real-valuedness of $\lambda_h(1,3)$ it follows that $\overline{\lambda_h^l} = \lambda_h^{-l}$, where the bar means here complex conjugation. As $\lambda_0(1,3)$ is an even function of $\theta$, then $\lambda_0^l = \lambda_0^{-l}$ and, by the evolution (45), $\lambda_h^l = \lambda_h^{-l}$ for all $l$. This shows in turn that $\lambda_h^l(k_1^1, k_3^1)$ is real for all $l$.

As the real part of $G_h(k_5)$ is positive, the last result shows, by the evolution (45), that for each set of values $l, k_1^1, k_3^1$, we have $\lambda_{h-1}^l(k_1^1, k_3^1) < \lambda_h^l(k_1^1, k_3^1)$ if $\lambda_h^l(k_1^1, k_3^1)$ has the same sign for all values of $k_1^1, k_3^1$.

This means that if we begin with an attractive potential $\lambda_0(\vec{x} - \vec{y})$, then all $\lambda_h^l(k_1^1, k_3^1)$ are going to be negative and explode in the limit $h \to -\infty$. Thus, attractive interactions cause an inconsistent perturbation theory already at second order and our methods can do no more in this case. This failure is in agreement with the BCS theory of superconductivity, which expects formation of *bounded* Cooper pairs even at arbitrarily weak attraction. We could not hope that our perturbative theory around the free Fermi gas could describe a superconductor, an object too different of the Fermi gas.

On the other hand, if the interaction $\lambda_0(\vec{x} - \vec{y})$ is repulsive, then, if we manage to prove that the $\lambda_h^l(k_1^1, k_3^1)$ never become negative, then they tend to zero as $h \to -\infty$. In this case we have not only consistence to second order, but also asymptotic freedom in the infrared.

Stating it all again, we cannot solve the evolution equations (41) for the $\lambda$-couplings, yet, not even to second order, but it can be seen that the equations for the couplings at Cooper pairs configurations imply that they are always real numbers and that their evolution at second order can be simplified by an angular momentum expansion. They are also *compatible* with a failure of the perturbative expansion even for arbitrarily weak attractive potentials and asymptotic freedom for repulsive potentials, which by its turn is compatible with a normal Fermi liquid. Furthermore, it seems that the $\lambda$-couplings for all non-zero $\Gamma$, *i.e* non-Cooper pairs, go rapidly to zero when $h \to -\infty$, as well as all $\mu$ with non-zero $\Delta$ and all $\nu$ with non-zero $k$.

The situation for the $\mu$-couplings with $\Delta = 0$ is similar. We can show exactly the same real-valuedness properties and diagonalization of the second order evolution equations by an angular momentum expansion. The only difference is that the function $\overline{G}_h$ analogous to the $G_h$ appearing in (45), given by

$$\overline{G}_h(k_5,\vec{\omega}_5) = \frac{16}{(2\pi)^{d+1}} 2^{-2h} \overline{g}_h(2^{-h}k_6) \left[g_h(2^{-h}k_5) + 2g_h^s(2^{-h}k_5)\right] , \quad (46)$$

has not any more a positive real part at forward scattering configurations. This follows from the fact that at these configurations we have $k_6 = k_5$, instead of $k_6^1 = k_5^1$ and $k_6^0 = -k_5^0$ of the Cooper pairs. As a consequence, the evolution equations for the angular momentum coefficients do not have any more a clear behavior related to the sign of the interaction potential, but that does not rule out a bounded evolution (thus normality of the Fermi liquid) of the $\mu_h$ at forward scattering.

Another important consequence of our different renormalization procedure is that even to third and higher orders in perturbation theory the $\lambda_h$ and $\mu_h$ evolutions are given respectively by direct and exchange graphs. The simple form of these graphs implies, by use of symmetry properties of the form factors, see[15], section 5.4, that the diagonalization of the evolution at Cooper pairs or forward scattering holds also to any perturbative order. We think that this property will be of fundamental importance when trying to analyze further orders of the perturbative series.

At this point we can resume the discussion by noting that, as all running couplings seem to flow to zero, then it is *possible* that, to second order in perturbation theory, Fermi liquids at dimensions $d > 1$ are *normal* if the interaction potential is *repulsive*. This conjecture means absence of the Kohn-Luttinger effect in any dimension $d > 1$. It is confirmed to second order and refuted at third in $d = 2$ in [18] for one particular model and in the Bethe-Salpeter approximation. We know yet of no conclusive evidence either for the Kohn-Luttinger effect, or to its absence, so the question to us is still open.

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## References

[1] D. Pines and P. Nozières: *The Theory of Quantum Liquids*, vol. I: *Normal Fermi Liquids*, (W. A. Benjamin, New York, 1966).

[2] P. W. Anderson: "Luttinger-liquid" behavior of the normal metallic state of the 2D Hubbard model, Phys. Rev. Lett. **64**: 1839(1990).

[3] D. Mattis and E. Lieb: Exact solution of a many fermion system and its associated boson field, *J. Math. Phys.* **6**: 304 (1965). Reprinted in E. Lieb, D. Mattis, *Mathematical Physics in One Dimension* (Academic Press, New York, 1966).

[4] G. Benfatto, G. Gallavotti: Perturbation theory of the Fermi surface in a quantum liquid. A general quasiparticle formalism and one-dimensional systems, J. Stat. Phys. **59**: 541 (1990).

[5] G. Benfatto, G. Gallavotti, A. Procacci, B. Scoppola: Beta function and Schwinger functions for a many fermion system in one dimension. Anomaly of the Fermi surface, Comm. Math. Phys. **160**: 93(1994).

[6] D. Pines: *The Many-Body Problem* (W. A. Benjamin, Reading, 1962).

[7] A. A. Abrikosov, L. P. Gorkov, I. E. Dzyaloshinski: *Methods of Quantum Field Theory in Statistical Physics* (Dover Publications, New York, 1963).

[8] J. Engelbrecht, M. Randeria: New collective mode and corrections to Fermi-liquid theory in two dimensions, Phys. Rev. Lett. **65**, 1032 (1990).

[9] H. Fukuyama, Y. Hasegawa, O. Narikiyo: J. Phys. Soc. Japan **60**, 2013 (1991).

[10] W. Kohn and J. Luttinger, Phys. Rev. Lett. **15**, 524 (1965).

[11] M. A. Baranov, A. V. Chubukov, M. Yu. Kagan: Superconductivity and superfluidity in Fermi systems with repulsive interactions, Int. J. Mod. Phys. B **6**, 2471 (1992).

[12] A. V. Chubukov: Kohn-Luttinger effect and the instability of a two-dimensional repulsive Fermi liquid at $T = 0$, Phys. Rev. B **48**, 1097 (1993).

[13] J. Feldman, E. Trubowitz: Perturbation theory for many fermion systems, Helv. Phys. Acta **63** (1990).

[14] R. Shankar: Renormalization group approach to interacting fermions, Rev. Mod. Phys. **66**, no. 1 (1994).

[15] A. G. M. Neves: *Teoria perturbativa dei liquidi di Fermi in* $d > 1$, Doctorate thesis, Università di Roma "La Sapienza", 1993.

[16] A. G. M. Neves: Renormalization group around a sphere for interacting fermion systems in $d > 1$, preprint.

[17] G. Benfatto, G. Gallavotti: Renormalization group approach to the theory of the Fermi surface, Phys. Rev. B **42**: 9967 (1990).

[18] J. Feldman, H. Knörrer, R. Sinclair, E. Trubowitz: Superconductivity in a repulsive model, *preprint*.

[19] J. Feldman, M. Salmhofer, E. Trubowitz: Perturbation theory around non-nested Fermi surfaces I. Keeping the Fermi surface fixed, to appear in the September 1996 issue of J. Stat. Phys. .

[20] G. Gallavotti: Renormalization theory and ultraviolet stability for scalar fields via renormalization group methods, *Rev. Mod. Phys.* **57**: 471 (1985).

[21] F. W. Olver: *Asymptotics and Special Functions* (Academic Press, New York, 1970).

[22] G. Gallavotti: Renormalization group, *Troisième cycle de la Physique*, Lausanne (1990).