Some Quantum Aspects of D=3 Space-Time Massive Gravity

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We discuss some features of Einstein-Proca gravity in $D = 3$ and 4 space-times. Our study includes a discussion on the tree-level unitarity and on the issue of light deflection in $3D$ gravity in the presence of a mass term.

I. Introduction

It is a well-known fact that the gauge-field approach to pure Einstein theory of gravitation in D=4 exhibits unitarity at the tree-level, although renormalizability is not attained. Pure gravity is on-shell finite only at the 1-loop level[1].

Recently, some effort has been done to analyse Einstein-Chern-Simons models in D=3 space-time[2], which provides not only renormalizability, but finiteness and unitarity[3]. Essentially, the physics of this model consists in giving the graviton a mass without explicitly or spontaneously breaking gauge symmetry.

Our actual purpose in this paper is to discuss possible consequences of adding a Proca-like mass term to the Einstein-Chern-Simons model. We start our study by analysing the simpler case of Einstein-Proca gravity in three and four-dimensional space-times (section II). Then, in section III, we concentrate on the Einstein-Chern-Simons-Proca gravity model in D=3, drawing our attention to the behaviour of the graviton propagator, in order to infer about unitarity at the tree-level. The problem of light-ray deflection in a gravitational field is also contemplated. Finally, in section IV, our concluding remarks are presented.

II. The Einstein-Proca Model

Our starting point is the Einstein-Hilbert action for gravity in D dimensions:

$$\pounds_{H.E} = -\frac{1}{2\kappa^2}\sqrt{-g}\ R\ , \tag{1}$$

where R is understood to be the scalar curvature. In the weak-field approach, the metric can be decomposed around the background flat-space metric as

$$g^{\mu\nu}(x) = \eta^{\mu\nu} - \kappa h^{\mu\nu}\ . \tag{2}$$

So, by replacing (2) into (1), the non-interacting part of the Lagrangian turns out to be given by the bilinear terms on the fluctuation $h_{\mu\nu}$ – field as below:

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$$\pounds^{(2)}_{H.E} = \frac{1}{4}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} - \frac{1}{4}\partial_\lambda h^\mu{}_\mu \partial^\lambda h^\nu{}_\nu + \frac{1}{2}\partial_\lambda h^\lambda{}_\mu \partial^\mu h^\nu{}_\nu - \frac{1}{2}\partial_\lambda h^\lambda{}_\mu \partial_\nu h^{\nu\mu} . \tag{3}$$

If we add by hand a Proca mass term for the graviton,

$$\pounds_{mass} = -\frac{1}{4}\ m^2(h^{\mu\nu}h_{\mu\nu} - h^\mu{}_\mu h^\nu{}_\nu) , \tag{4}$$

the complete bilinear Lagrangian can be cast as follows:

$$\pounds = \frac{1}{2}\ h^{\mu\nu}\mathcal{O}_{\mu\nu,k\lambda}h^{k\lambda} , \tag{5}$$

where $\mathcal{O}_{\mu\nu,k\lambda}$ can be decomposed on it's Lorentz basis of spin-operators for symmetric rank-2 tensors [4]:

$$\mathcal{O}_{\mu\nu,k\lambda} = \left\{ -\frac{1}{2}(\Box + m^2)P^{(2)} - \frac{m^2}{2}P^{(1)}_m + (\Box + m^2)P^{(0)}_s + \frac{\sqrt{3}}{2}m^2\left(P^{(0)}_{sw} + P^{(0)}_{ws}\right) \right\}_{\mu\nu,k\lambda} . \tag{6}$$

We stress that, since this theory is not gauge invariant, due to the Proca-like mass term, we do not need to introduce any gauge-fixing term into (6).

The graviton propagator, in configuration space, is given by:

$$\langle T[h_{\mu\nu}(x)h_{k\lambda}(y)]\rangle = i\mathcal{O}^{-1}_{\mu\nu,k\lambda}\delta^D(x-y) , \tag{7}$$

where the inverse operator $\mathcal{O}^{-1}$ can be determined with the help of the multiplicative table of the Barnes and Rivers spin-operators [4,5]. Basically, one writes the inverse operator as a linear combination of the spin-operators as below :

$$\mathcal{O}^{-1}_{\mu\nu,k\lambda} = \left\{ XP^{(2)} + YP^{(1)}_m + ZP^{(0)}_s + WP^{(0)}_w + RP^{(0)}_{sw} + SP^{(0)}_{ws} \right\}_{\mu\nu,k\lambda} . \tag{8}$$

Then, with the help of the tensorial identity for the rank-2 symmetric tensor,

$$\mathcal{O}^{\rho\sigma}_{\mu\nu}\mathcal{O}^{-1}_{\rho\sigma,k\lambda} = \left(P^{(2)} + P^{(1)}_m + P^{(0)}_s + P^{(0)}_w\right)_{\mu\nu,k\lambda} , \tag{9}$$

the propagators are uniquely determined. The explicit forms of the operators appearing above can all be found in Ref. [4,5].

In our case,

$$\mathcal{O}_{\mu\nu,k\lambda} = \left\{ AP^{(2)} + BP^{(1)}_m + CP^{(0)}_s + DP^{(0)}_w + EP^{(0)}_{sw} + FP^{(0)}_{ws} \right\} , \tag{10}$$

where

$$\begin{aligned} A &= -\frac{1}{2}(\Box + m^2) ; \\ B &= -\frac{m^2}{2} ; \\ C &= (\Box + m^2) ; \\ D &= 0 ; \\ E &= \frac{\sqrt{3}}{2}m^2 ; \\ F &= \frac{\sqrt{3}}{2}m^2 . \end{aligned} \tag{11}$$

Then, one gets the solution:

$$X = -\frac{2}{(\Box + m^2)} \quad ; \quad W = -\frac{2(-2\Box + \Box D - 2m^2 + m^2 D)}{m^4(D-1)};$$
$$Y = -\frac{2}{m^2} \quad ; \quad R = \frac{2\sqrt{3}}{m^2(D-1)};$$
$$Z = -\frac{2(D-4)}{(D-1)(\Box + m^2)} \quad ; \quad S = \frac{2\sqrt{3}}{m^2(D-1)}. \tag{12}$$

For $D = 3$, the Einstein-Proca propagator reads as below :

$$\begin{aligned} \mathcal{O}^{-1}_{\mu\nu,k\lambda} = & -\frac{1}{(\Box + m^2)}(\eta_{\mu k}\eta_{\nu\lambda} + \eta_{\mu\lambda}\eta_{\nu k}) + \frac{2}{3(\Box + m^2)}\eta_{\mu\nu}\eta_{k\lambda} + \\ & + \frac{1}{3}\frac{1}{(\Box + m^2)}\eta_{\mu\nu}\eta_{k\lambda} + \omega_{\mu\nu}\omega_{k\lambda}\left[\frac{-2}{\Box + m^2} + \frac{2}{3(\Box + m^2)} + \frac{4}{m^2} + \right. \\ & \left. + \frac{1}{3}\frac{1}{(\Box + m^2)} - \frac{(\Box + m^2)}{m^4} - \frac{2}{m^2}\right] + \eta_{\mu k}\omega_{\nu\lambda}\left[\frac{1}{(\Box + m^2)} - \frac{1}{m^2}\right] + \\ & + \eta_{\mu\lambda}\omega_{\nu k}\left[\frac{1}{(\Box + m^2)} - \frac{1}{m^2}\right] + \omega_{\mu k}\eta_{\nu\lambda}\left[\frac{1}{(\Box + m^2)} - \frac{1}{m^2}\right] + \\ & + \omega_{\mu\lambda}\eta_{\nu k}\left[\frac{1}{(\Box + m^2)} - \frac{1}{m^2}\right] + \eta_{\mu\nu}\omega_{k\lambda}\left[\frac{-2}{3(\Box + m^2)} - \frac{1}{3(\Box + m^2)} + \frac{1}{m^2}\right] + \\ & + \omega_{\mu\nu}\eta_{k\lambda}\left[\frac{-2}{3(\Box + m^2)} - \frac{1}{3(\Box + m^2)} + \frac{1}{m^2}\right] . \end{aligned} \tag{13}$$

Proceeding along the same lines, it can be found that the propagators for Einstein gravitation [5] in $D = 4$, $m = 0$; $D = 3$, $m = 0$; and for Einstein-Proca [5], $D = 3$, $m \neq 0$ and $D = 4$, $m \neq 0$ (Einstein-Proca) are respectively given by:

$$\langle T[h_{\mu\nu}(-k)h_{k\lambda}(k)]\rangle = \frac{i}{k^2}(\eta_{\mu k}\eta_{\nu\lambda} + \eta_{\mu\lambda}\eta_{\nu k} - \eta_{\mu\nu}\eta_{k\lambda}); \tag{14}$$

$$\langle T[h_{\mu\nu}(-k)h_{k\lambda}(k)]\rangle = \frac{i}{k^2}(\eta_{\mu k}\eta_{\nu\lambda} + \eta_{\mu\lambda}\eta_{\nu k} - 2\eta_{\mu\nu}\eta_{k\lambda}); \tag{15}$$

$$\langle T[h_{\mu\nu}(-k)h_{k\lambda}(k)]\rangle = \frac{i}{(k^2 - m^2)}(\eta_{\mu k}\eta_{\nu\lambda} + \eta_{\mu\lambda}\eta_{\nu k} - \eta_{\mu\nu}\eta_{k\lambda}); \tag{16}$$

$$\langle T[h_{\mu\nu}(-k)h_{k\lambda}(k)]\rangle = \frac{i}{(k^2 - m^2)}(\eta_{\mu k}\eta_{\nu\lambda} + \eta_{\mu\lambda}\eta_{\nu k} - \frac{2}{3}\eta_{\mu\nu}\eta_{k\lambda}). \tag{17}$$

Coupling these propagators to conserved external sources (energy-momentum tensors), $T^{\mu\nu}$, one writes the following current-current amplitudes:

$$A_{m=0} = T^{\mu\nu}(\Delta_{\mu\nu,k\lambda})T^{'k\lambda} = g^2 T^{\mu\nu}\langle h_{\mu\nu}h_{k\lambda}\rangle_{m=0}\, T^{'k\lambda} \tag{18}$$

and

$$A_{m\neq 0} = T^{\mu\nu}(\Delta_{\mu\nu,k\lambda})T^{'k\lambda} = \tilde{g}^2 T^{\mu\nu}\langle h_{\mu\nu}h_{k\lambda}\rangle_{m\neq 0}\, T^{'k\lambda} . \tag{19}$$

Now taking static sources, where only the T^{00} – component contributes, the explicit forms of the amplitudes for massless graviton exchange gives us the relation below for their effective coupling constants, g and $\tilde{g}$:

$$g^2 = \frac{4}{3}\tilde{g}^2 . \tag{20}$$

We now wish to consider the problem of photons propagating in the gravitational field of some static matter distribution. In this situation, the propagator describes graviton exchanges between the photons and the matter that generates the gravitational field. In the metric formalism for General Relativity, light follows geodesic lines. In a field-theoretical treatment, it is the photon-graviton interaction that replaces this geometrical fact. Now, in (18), $T'_{k\lambda}$ is the usual energy-momentum tensor for electromagnetism:

$$T'_{k\lambda} = -F_{k\alpha}F^{\alpha}{}_{\lambda} + \frac{1}{4}\eta_{k\lambda}F_{\mu\nu}F^{\mu\nu} \ . \tag{21}$$

So, in principle, the transition amplitudes for the deflection of light induced by massless and massive gravitons are respectively given by:

$$A_{m=0} = \frac{i}{k^2}2g^2T^{00}T'^{00}, \qquad A_{m\neq 0} = \frac{i}{(k^2-m^2)}\frac{8}{3}\tilde{g}^2T^{00}T'^{00}. \tag{22}$$

However, since in $D = 3$ pure Einstein gravity there is no graviton propagation [5], one does not talk about light-ray deflection. As a consequence, in $3D$-gravity, light deflection may be discussed only if the graviton becomes massive, the mass being introduced by a Proca-like or a topological Chern-Simons term.

We conclude this section by studying the necessary condition to get tree-level unitarity for the Einstein-Proca gravity in $D = 3$. To do so, we must take the current-current amplitude and analyse the positivity condition on the imaginary part of the residues at the pole $k^2 = m^2$. Otherwise, if positivity is not attained, there will be ghosts or non-dynamical gravitons in the model.

The transition amplitude is

$$A = \tau^{\mu\nu *}(\vec{\mathbf{k}})\langle T[h_{\mu\nu}(-(\vec{\mathbf{k}})h_{k\lambda}(\vec{\mathbf{k}})]\rangle\tau^{k\lambda}(\mathbf{k}) \ ; \tag{23}$$

where, after some algebraic manipulation, we obtain

$$A = \frac{i}{(k^2-m^2)}\left[2|\tau^{\mu\nu}(\vec{k})|^2 - |\tau^{\mu}_{\mu}(\vec{k})|^2\right] \ . \tag{24}$$

Choosing a suitable basis of vectors in momentum space,

$$k^{\mu} = (k^0;\vec{k});\ \tilde{k}^{\mu} = (k^0;-\vec{k});\ \varepsilon^{\mu}_{i} = (0;\vec{\varepsilon}_i);\ i = 1,\cdots,D-2\ , \tag{25}$$

where

$$\begin{aligned} \tilde{k}^{\mu}k_{\mu} &= (k^0)^2 + (\vec{k})^2 > 0 \\ \varepsilon^{\mu}_{i}\varepsilon_{j\mu} &= -\delta_{ij} \\ k^{\mu}\varepsilon^{i}_{\mu} &= \tilde{k}^{\mu}\varepsilon^{i}_{\mu} = 0\ , \end{aligned} \tag{26}$$

and expanding the tensorial current on this basis, we get:

$$\tau_{\mu\nu} = a_{(k)}k_{\mu}k_{\nu} + b_{(k)}k_{(\mu}\tilde{k}_{\nu)} + c_{i(k)}k_{(\mu}\varepsilon^{i}_{\nu)} + d_{(k)}\tilde{k}_{\mu}\tilde{k}_{\nu} + e_{i(k)}\tilde{k}_{(\mu}\varepsilon^{i}_{\nu)} + f_{ij}\varepsilon^{i}_{(\mu}\varepsilon^{j}_{\nu)} \ . \tag{27}$$

The current conservation, $k^{\mu}\tau_{\mu\nu} = 0$, at the pole, dictates the number of degrees of freedom among the coefficients a, b, c, d, e_i and f_{ij}.

It is shown that if $\left(|c_{i(k)}|^2 + |e_{i(k)}|^2\right) < 0$, the necessary condition for tree-level unitarity, $Res\ Im\ A > 0$ at the pole, is automatically ensured.

III. The Einstein-Chern-Simons-Proca (E.C.S.P.) Gravity

Let us consider topological gravity in $(2+1)$ and discuss some of its features whenever a Proca-like mass term is present [6]. The generalized (E.C.S.P.) Lagrangian reads as below:

$$\pounds = -\frac{1}{2\kappa^2}\sqrt{-g}R + \frac{1}{\mu}\varepsilon^{\lambda\mu\nu}\Gamma^{\rho}{}_{\lambda\sigma}(\partial_{\mu}\Gamma_{\rho}{}^{\sigma}{}_{\nu} + \frac{2}{3}\Gamma_{\mu}{}^{\sigma}{}_{\omega}\Gamma_{\nu}{}^{\omega}{}_{\rho}) - \frac{1}{2}\ m^2(h^{\mu\nu}h_{\mu\nu} - \xi h^{\mu}{}_{\mu}h^{\nu}{}_{\nu}) \ . \tag{28}$$

Notice that the massive Einstein-Proca Lagrangian can be recovered by taking $\mu \to \infty$. Also, just like before (E.P.), there is no gauge invariance. In this sense, again, a gauge-fixing term is not adjoined. In order to obtain the propagators for this model, we just take the bilinear terms on the quantum field $h_{\mu\nu}$, in (28); this yields the following operator to be inverted:

$$\begin{aligned} \mathcal{O}_{\mu\nu k\lambda} &= -\frac{1}{2}(\Box+m^2)P^{(2)}-\frac{m^2}{2}P^{(1)}+\left[\Box-\frac{m^2}{2}(1-3\xi)\right]P_s^{(0)}-\frac{m^2}{2}(1-\xi)P_w^{(0)} \\ &+\frac{\sqrt{3}}{2}m^2\left(P_{sw}^{(0)}+P_{ws}^{(0)}\right)+\frac{1}{2M}(S_1+S_2)\ , \end{aligned} \quad (29)$$

where S_1 and S_2 are new spin-operators coming from the Chern-Simons term [5]. The M parameter is the topological Chern-Simons mass and is just a redefinition of μ $(M=\frac{\mu}{8\kappa^2})$. To get the Einstein-Proca operator (6), we must simply take $\xi=1$ and M going to infinity in (29).

The method to invert (29) is the same as the one followed in the previous section, and so we just need to obtain the coefficients presented in the linear combination of the whole set of spin operators:

$$\mathcal{O}^{-1}_{\mu\nu,k\lambda}=X_1P^{(2)}+X_2P^{(1)}+X_3P_s^{(0)}+X_4P_w^{(0)}+X_5P_{sw}^{(0)}+X_6P_{ws}^{(0)}+X_7S_1+X_8S_2\ . \quad (30)$$

It can be found that

$$\begin{aligned} X_1 &= \frac{-2M^2(\Box+m^2)}{(\Box^2M^2+2\Box m^2M^2+m^4M^2+\Box^3)}; \quad (31) \\ X_2 &= \frac{-2}{(m^2)}; \\ X_3 &= A/B; \\ X_4 &= 2\left[\frac{2m^2\xi+\Box-m^2}{m^2[(\xi-1)\Box-3m^2\xi+m^2]}\right]; \\ X_5 &= \frac{-2\sqrt{3\xi}}{(\Box\xi-3m^2\xi-\Box+m^2)}; \\ X_6 &= \frac{-2\sqrt{3\xi}}{(\Box\xi-3m^2\xi-\Box+m^2)}; \\ X_7 &= \frac{-2M}{(\Box M^2+2\Box m^2M^2+m^4M^2+\Box^3)}; \\ X_8 &= \frac{-2M}{(\Box M^2+2\Box m^2M^2+m^4M^2+\Box^3)}. \end{aligned}$$

where

$$A=[(-3+3\xi)\Box^3+(-4M^2+4\xi M^2)\Box^2+(4m^2\xi M^2-6m^2M^2)\Box-2m^4M^2]$$

and

$$\begin{aligned} B &= [(-1+\xi)\Box^4+(m^2-3m^2\xi-M^2+\xi M^2)\Box^3+(-m^2M^2-m^2\xi M^2)\Box^2+ \\ &+ (m^4M^2-5m^4M^2\xi)\Box-3m^6M^2\xi+m^6M^2]\ . \end{aligned}$$

Next point, we comment on the dynamical possibility for the graviton in (2+1) to appear as the mediator of the gravitational interaction. As before, one couple the propagators to conserved external currents $\tau^{\mu\nu}(-\vec{k})$ and $\tau^{k\lambda}(\vec{k})$ and observe that only the X_1 and X_3 coefficients survive the transversallity of these currents. So, in the same way as analysed in (23), the dynamics at tree-level for the E.C.S.P. theory can be discussed. We need first to analyse poles to get information on the physicallity of the masses. For simplicity, take $\xi=1$, where X_1 and X_3 have the same pole structure:

$$\Box^2 M^2 + 2\Box m^2 M^2 + m^4 M^2 + \Box^3 = 0 \quad (32)$$

or, in the momentum space,

$$(k^2)^3 - M^2(k^2)^2 + 2m^2M^2(k^2) - m^4M^2 = 0\ , \quad (33)$$

which is a cubic equation in k^2. It is checked that tachyons and ghosts are excluded from the spectrum whenever $|\frac{M}{m}|\rangle\frac{3\sqrt{3}}{2}$. For example, with $|\frac{M}{m}| = 3$, three massive poles can be obtained:

$$\begin{aligned} k_1^2 &= 6,4113m^2\ , \\ k_2^2 &= 0,7737m^2, \\ k_3^2 &= 1,8154m^2\ . \end{aligned} \quad (34)$$

Therefore, in principle, there is the possibility of having three massive excitations mediating the gravitational interaction in $(2+1)$ (E.C.S.P.). Also, one can verify that these three massive gravitons satisfies the condition of positive-definite residue of their amplitudes [6] that guarantees tree-level unitarity. Finally, we could say that, by power-counting (just by observing the X_1 and X_3 coefficients for $D = 3$), the propagator is proportional to $\frac{1}{k^4}$, which for assymptotic values of k, leads to a renormalizable theory.

IV. Concluding Remarks

We concentrated our efforts in this paper to understand some features of $3D$ massive gravity. Massive graviton propagators are derived and conditions for the absence of ghosts and tachyons are found out. The analysis of propators is also used to conclude that light-ray deflection in $D = 3$ is possible only in the context of massive gravity. In the general case of Proca and Chern-Simons mass terms, the spectrum of propagating gravitons is discussed and 3 massive excitations are identified in a special case of parameters.

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