INTRODUCTION TO
NUCLEAR
EFFECTIVE FIELD THEORIES

BIRA

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CNRS and US DOE
“Simplicity, like everything else, must be explained.”

S. Weinberg
Exercise #1

For each lecture, seminar, colloquium, and poster where the holy words “effective field theory” are invoked, are they being used in the original sense?

(e.g. Is “leading order” leading?)
References:

U. van Kolck,
**Effective Field Theories of Loosely Bound Nuclei,**
in *The Euroschool on Exotic Beams, Vol. IV*
C. Scheidenberger and M. Pfützer (eds.),
Springer, Berlin Heidelberg (2014)

U. van Kolck,
**Les Houches Lectures on Effective Field Theories for Nuclear and (some) Atomic Physics,**
in *Proceedings of the 2017 Les Houches Summer School on EFTs in Particle Physics and Cosmology,*
S. Davidson *et al.* (eds.),
Oxford University Press (to appear)
*arXiv:1902.03141 (nucl-th)*
Effective Field Theories
- Introduction
- What is effective
- Summary

EFT for Short-Range Forces
Two-Body System
Three-Body System
More-Body Systems
Long-range Forces
FORMULATION OF NUCLEAR PHYSICS CONSISTENT WITH STANDARD MODEL (SM) OF PARTICLE PHYSICS

Reward
understanding emergence of complexity at the most fundamental level:
nucleus made out of quarks and gluons interacting strongly (QCD), yet exhibiting many regularities

Beware
coupling constants not small: not an easy problem!
“There are few problems in nuclear theoretical physics which have attracted more attention than that of trying to determine the fundamental interaction between two nucleons. It is also true that scarcely ever has the world of physics owed so little to so many … … It is hard to believe that many of the authors are talking about the same problem or, in fact, that they know what the problem is.”

M. L. Goldberger

*Midwestern Conference on Theoretical Physics, Purdue University, 1960*
Nuclear Physics
The canons of tradition

I. Nuclei are essentially made out of non-relativistic nucleons in two isospin states (protons and neutrons).

II. The interaction potential is mostly two-body, but there is evidence for smaller three-body forces.

III. Isospin is a good symmetry, except for the Coulomb interaction, breaking in two-nucleon scattering lengths, and smaller effects.

IV. External probes (e.g. photons) interact mainly with each nucleon, but there is evidence for smaller two-nucleon currents.

but... WHY?
Quantum Chromodynamics
On the road to infrared slavery

1. Up, down quarks are relatively light, \( m_{u,d} \sim 5 \text{ MeV}, \) and thus relativistic.

2. The interaction is a multi-gluon, and thus a multi-quark, process.

3. Isospin symmetry is not obvious: \( \varepsilon = \frac{m_d - m_u}{m_d + m_u} \sim \frac{1}{3} \)

4. External probes can interact with collection of quarks.

**difficulty**

Quarks and gluons not the most convenient degrees of freedom at low energies.

How does nuclear structure emerge from QCD?
Strongly interacting particles (hadrons)

(many observed states not shown!)

Mesons

\( S = 1 \)

\( S = 0 \)

\( S = 1/2 \)

\( S = 3/2 \)

Baryons

\( \Sigma \)

\( \Delta \)

\( \Omega \)

\( \Xi \)

\( \Xi^* \)

\( \Sigma^* \)

\( \Lambda \)

\( N \)

\( K^* \)

\( \rho \)

\( \pi \)

\( K \)

\( \text{Budapest-Marseille-Wuppertal collaboration} \)

Exception: pion

\( m_\pi = 140 \text{ MeV}/c^2 \ll M_{\text{QCD}} \)

we’ll return to it!

\( M_{\text{QCD}} \sim 1000 \text{ MeV}/c^2 \)

= 1 GeV/c^2
Nuclear scales

\[ B \sim \frac{Q^2}{m_N} \sim \frac{M_{\text{nuc}}^2 c^2}{M_{\text{QCD}}} \sim 10 \text{ MeV} \]

\[ \frac{r}{A^{1/3}} \sim \frac{\hbar}{Q} \sim \frac{\hbar}{M_{\text{nuc}} c} \sim 1 \text{ fm} \]

\[ Q \sim M_{\text{nuc}} c \sim 100 \text{ MeV/c} \]
Multi-scale problems

\[ H = \left( \frac{p^2}{2m_e} - \frac{\alpha \hbar c}{r} \right) \left[ 1 + \mathcal{O} \left( \alpha; \frac{p^2}{m_e^2 c^2}; \frac{\hbar^2}{m_e^2 c^2 r^2} \right) \right] \]

\[ \alpha \equiv \frac{e^2}{4\pi \hbar c} \approx \frac{1}{137} \ll 1 \]

\[ r \sim R \]
\[ p \sim \frac{\hbar}{R} \]
\[ E(R) \sim \left( \frac{\hbar^2}{2m_e R^2} - \frac{\alpha \hbar c}{R} \right) \]

\[ \frac{dE(R)}{dR} = 0 \quad \Rightarrow \quad R = \frac{\hbar}{\alpha m_e c} \]

Three scales

\[ m_e c^2 = 0.5 \text{ MeV} \]
\[ pc \sim \alpha m_e c^2 = 3.6 \text{ keV} \]
\[ -E \sim \frac{p^2}{2m_e} \sim \frac{1}{2} \alpha^2 m_e c^2 = 13.6 \text{ eV} \]

(from now on, units such that \( \hbar = 1, c = 1 \) )
However...
no obvious small coupling
in nuclear forces.

QCD
"fine-structure"
constant

Needed:
method that does not
rely on small couplings

\[ \sim 1 \]

\[ \sim M_{\text{QCD}} \]

\[ \text{EFFECTIVE FIELD THEORY} \]
“I do not believe that scientific progress is always best advanced by keeping an altogether open mind. It is often necessary to forget one’s doubts and to follow the consequences of one’s assumptions wherever they may lead --- the great thing is not to be free of theoretical prejudices, but to have the right theoretical prejudices. And always, the test of any theoretical preconception is in where it leads.”

S. Weinberg

The First Three Minutes, 1972
Ingredients

➢ Relevant degrees of freedom
Ingredients

- Relevant degrees of freedom

  choose the coordinates that fit the problem

- All possible interactions
Example: Earth-moon-satellite system

\[ R_m \approx 1.7 \text{ Mm} \quad d \approx 384 \text{ Mm} \quad R_E \approx 6.4 \text{ Mm} \]

2-body forces \( \rightarrow \) 2+3-body forces

change in resolution

3-body force
Ingredients

- Relevant degrees of freedom
  
  *choose the coordinates that fit the problem*

- All possible interactions
  
  *what is not forbidden is compulsory*

- Symmetries
A farmer is having trouble with a cow whose milk has gone sour. He asks three scientists—a biologist, a chemist, and a physicist—to help him. The biologist figures the cow must be sick or have some kind of infection, but none of the antibiotics he gives the cow work. Then, the chemist supposes that there must be a chemical imbalance affecting the production of milk, but none of the solutions he proposes do any good either. Finally, the physicist comes in and says, “First, we assume a spherical cow...”

\[
\sum_{ij} \alpha_{ij} u_i v_j \rightarrow \vec{u} \cdot \vec{v} + \sum_{ij} \delta \alpha_{ij} u_i v_j
\]

no, say, \( u_1 v_2 \)

\[|\delta \alpha_{ij}| \ll 1\]

amenable to perturbation theory
Ingredients

- Relevant degrees of freedom

  choose the coordinates that fit the problem

- All possible interactions

  what is not forbidden is compulsory

- Symmetries

  not everything is allowed

- Naturalness
After scales have been identified, the remaining, dimensionless parameters are \( \mathcal{O}(1) \) unless suppressed by a symmetry.

Occam’s razor: simplest assumption, to be revised if necessary.

\( E \quad \text{energy of probe} \)
\( E_{\text{und}} \quad \text{energy scale of underlying theory} \)

Expansion in powers of...
A classical example: the flat Earth light object near surface of a large body

\[ E \sim mgh \ll E_{\text{und}} \equiv mgR \]

\[
V_{\text{eff}} (h) = m \sum_{i=0}^{\infty} g_i h^i = \text{const} + mg \left\{ h + \frac{g^2}{g} h^2 + \ldots \right\}
\]

\[
\text{naturalness: } \frac{mg_i h^{i+1}}{mg_i h^i} = \frac{E}{E_{\text{und}}} \times \mathcal{O}(1) = \frac{h}{R} \times \mathcal{O}(1) \quad \iff \quad g_{i+1} = \mathcal{O} \left( \frac{g}{R^i} \right)
\]

\[
V_{\text{und}} (h) = -GMm \frac{1}{R + h} = m \left( \frac{GM}{R^2} \right) \sum_{i=0}^{\infty} \left( -\frac{1}{R} \right)^{i-1} h^i \quad \iff \quad g_{i+1} = (-1)^i \frac{g}{R^i}
\]

\[ h \ll R \quad \equiv g \]

itself the first term in a low-energy EFT of general relativity...
Going a bit deeper...  

A short path to quantum mechanics

\[ P = |P_a = iA_2 + A_{23}|^2 + A_4|^2 \]

sum over all paths

Path Integral  
Feynman '48

\[ A = \int Dq \exp \left( i \int dt \mathcal{L}(q(t)) \right) \]

\[ \prod_i \int dq(t_i) \]

each path contributes a phase given by the classical* action

* Not strictly true when interactions involve time derivatives (Salam + Strahdee '68)
\[ \mathcal{L}(q(t_i)) \rightarrow \mathcal{L}(q(t_i)) + \frac{dq}{dt}\bigg|_{t_i} (t-t_i) + \frac{1}{2} \frac{d^2 q}{dt^2}\bigg|_{t_i} (t-t_i)^2 + \ldots \]
More generally,

\[ A = \int Dq \exp \left( i \int dt \mathcal{L}_{\text{und}}(q) \right) \]

\[ \times \int D\tilde{q} \delta(\tilde{q} - f_\Lambda(q)) \]

\[ \leftrightarrow \prod_i \int d\tilde{q}(t_i) \delta(\tilde{q}(t_i) - f(q(t_i))) \]

\[ = \int D\tilde{q} \exp \left( i \int dt \mathcal{L}_{\text{EFT}}(\tilde{q}) \right) \]

\[ \mathcal{L}_{\text{EFT}}(\tilde{q}) = \sum_{d,n=0}^{\infty} c_{d+n}(M, \Lambda) O_{d+n} \left( \tilde{q}, \left( \frac{d^d \tilde{q}}{dt^d} \right)^n \right) \]

Naturalness

\[ c_{d+n} \sim \frac{c_0}{M^{d+n}} \]

e.g.

\[ V_{\text{EFT}}(\tilde{q}) = c_0\tilde{q}^4 + c_2\tilde{q}^2 \left( \frac{d\tilde{q}}{dt} \right)^2 + \ldots \]

Observables \sim expansion in \( \frac{Q}{M} \)
All information is in the S matrix...

elastic scattering (for simplicity)

\[ |\vec{k}'| = |\vec{k}| \equiv k \] (conservation of energy)

\[ \theta : \text{given by certain probability amplitude - the "scattering amplitude" } \]

\[ T(k, \theta) = \sum_{l=0}^{\infty} T_l(k) P_l(\cos \theta) \]

Legendre polynomial

angular momentum

partial-wave amplitude

1. Parametrization by phase shifts:

\[ T_l(k) = -\frac{4\pi i}{\mu k} \left[ \exp(2i\delta_l(k)) - 1 \right] \]

effective range

shape parameter

2. Low-energy expansion:

\[ T_l^{-1}(k) = \frac{\mu}{2\pi k^{2l}} \left[ -\frac{1}{a_l} + \frac{r_l}{2} k^2 + \frac{P_l}{4} k^4 + \ldots - \frac{i k^{2l+1}}{2l+1} \right] \]

scattering length

3. Poles:

\[ T_l^{-1}(\kappa_r + i\kappa_i) = 0 \]

\[ \kappa_r = 0 \text{ bound states } \Rightarrow E = -B < 0 \]

\[ \kappa_i \leq 0 \text{ resonances } \Rightarrow E = E_R - i \Gamma_R / 2 \]
\[
T = T^{(\infty)}(Q) \sim N(M) \sum Q \sum M_i^{\nu = \nu_{\text{min}}} F_{\nu} \left( \frac{Q}{M}, \frac{Q}{\Lambda}; \{c_{\nu}(\Lambda)\} \right)
\]

\[dT \over d\Lambda = 0 \quad \text{“renormalization-group invariance”}
\]

\[\nu = \nu(d, n, \ldots) \quad \text{“power counting”}
\]

For \(k \sim m\), truncate consistently with RG invariance so as to allow systematic improvement (perturbation theory):

\[T = T^{(\nu)} \left[ 1 + \mathcal{O} \left( \frac{Q}{M}, \frac{Q}{\Lambda} \right) \right]
\]

\[\frac{\Lambda}{T^{(\nu)}} \frac{dT^{(\nu)}}{d\Lambda} = \mathcal{O} \left( \frac{Q}{\Lambda} \right)
\]

† Warning: renormalization **NOT** optional †

Otherwise, just a model since dependent on regularization choice
"second quantization":

\[ q(t) \rightarrow \psi(\vec{r}, t), \psi^*(\vec{r}, t) \]

+ Lorentz invariance

representation of \( \text{SO}(3,1) \)

\[ d\tau \rightarrow dt d^3r \]

\[ \frac{d}{dt} \rightarrow \frac{\partial}{\partial t}, \frac{\partial}{\partial \vec{r}} \]

\[ \equiv d^4x \]

\[ \rightarrow \frac{\partial}{\partial x^\mu} \]

**EFFECTIVE FIELD THEORIES**

Euler + Heisenberg '36
Weinberg '67 ... '79
Wilson, early 70s

...
\[ A = \int D\psi D\psi^* \exp \left( i \int d^4 x \left\{ \mathcal{L}_{\text{free}}(\psi) + \mathcal{L}_{\text{int}}(\psi) \right\} \right) \]
\[ = \int D\psi D\psi^* \left\{ 1 + i \int d^4 x \mathcal{L}_{\text{int}}(\psi) + \frac{1}{2} \left[ i \int d^4 x \mathcal{L}_{\text{int}}(\psi) \right]^2 + \ldots \right\} \exp \left( i \int d^4 x \mathcal{L}_{\text{free}}(\psi) \right) \]

momentum space

\[ \mathcal{L}_{\text{int}} = \frac{\lambda}{4} (\psi^* \psi)^2 \quad \times \quad = i \lambda \]

\[ = \int \frac{d^4 l}{(2\pi)^4} i \lambda \frac{i}{(p_1 + l)^2 - m^2 + i\varepsilon} \frac{i}{(p_2 - l)^2 - m^2 + i\varepsilon} \]

= \ldots but divergent from high-momentum region…

needs a cutoff to separate high and low momenta
Two possibilities:

- know **and** can solve underlying theory --
  get $c_i$'s in terms of parameters in $\mathcal{L}_{\text{und}}$

- know **but** cannot solve, or do **not** know, underlying theory --
  invoke Weinberg's "folk theorem":
  
  "The quantum field theory generated by the most general
  Lagrangian with some assumed symmetries will produce
  the most general $S$ matrix incorporating quantum mechanics,
  Lorentz invariance, unitarity, cluster decomposition and
  those symmetries, with no further physical content."

Note: proven only for scalar field with $Z_2$ symmetry in $E_4$, Ball + Thorne '94
but no known counterexamples
Bira's EFT Recipe

1. identify degrees of freedom and symmetries
2. construct most general Lagrangian
3. run the methods of field theory
   • compute Feynman diagrams with all momenta \( Q < \Lambda \)
     (“regularization”)
   • relate \( c_i(\Lambda), \Lambda \) to observables, which should be independent of \( \Lambda \)
     (“renormalization”)

controlled expansion in \( \frac{Q}{M} \times \mathcal{O}(1) \)

⇒ “naturalness”: what else?

contrast to models, which have fewer, but \textit{ad hoc}, interactions;
useful in the identification of relevant degrees of freedom
and symmetries, but plagued with uncontrolled errors

what is not forbidden
is mandatory!

not a model form factor!!!
“Nonrenormalizable theories are as renormalizable as renormalizable theories.”

S. Weinberg
A modern view of renormalization...

“A significant change in physicists’ attitude towards what should be taken as a guiding principle in theory construction is taking place in recent years in the context of the development of EFT. For many years (…) renormalizability has been taken as a necessary requirement. Now, considering the fact that experiments can probe only a limited range of energies, it seems natural to take EFT as a general framework for analyzing experimental results.”

T. Y. Cao

in Renormalization, From Lorentz to Landau (and Beyond), L.M. Brown (ed.), 1993
Time for a paradigm change, perhaps?

“Say... What's a mountain goat doing way up here in a cloud bank?”
The world as an onion

- General Relativity + higher-curvature terms
- Chiral EFT
- QCD-lite+NRQCD (2 or 3 flavors)
- QCD (6 flavors)
- Electroweak Th + higher-dim ops
- (SUSY)
- GUT?
- Pionless EFT
- Halo EFT

- atomic physics
- nuclear physics
- condensed-matter physics and beyond

$E(\text{GeV})$

$10^{-18}$ $10^{-15}$ $10^{-2}$ $0.1$ $0.05$

$r(\text{fm})$

$10^{-20}$ $10^{-16}$ $10^{-3}$ $10^{-1}$ $1$
Summary

❖ Nuclear systems involve multiple scales but no obvious small coupling constant

❖ EFT is a general framework to deal with a multi-scale problem using the small ratio of scales as an expansion parameter

❖ EFT is not just any field theory with “effective” degrees of freedom

❖ Symmetries and renormalization essential for model independence

Stay tuned:
next, how we can describe low-energy nuclear physics systematically
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Effective Field Theories
EFT for Short-Range Forces
- Scales
- Degrees of freedom
- Symmetries
- Action
Two-Body System
Three-Body System
More-Body Systems
Long-range Forces
Very Brief History of Nuclear EFTs

> 1990: Weinberg generalizes Chiral Perturbation Theory to nuclei --- “Chiral EFT”. He identifies infrared enhancements in multi-nucleon amplitudes and suggests:
  i) the potential be calculated in the usual ChPT expansion;
  ii) amplitudes be calculated from the exact solution of the Schrödinger equation with truncated potentials

> 1992: Others carry out this procedure obtaining good description of few-body data with only limited cutoff variation

> 1996: Weinberg’s prescription shown to fail to produce renormalizable amplitudes; problem ignored by most nuclear physicists

> 1998: Pionless EFT developed to resolve these issues; it additionally finds great success for light nuclei

> 1998: Pionless EFT extended with perturbative pions and shown to work more or less as well as Pionless EFT

> 2005: Weinberg’s power counting modified so that nonperturbative pions produce renormalizable amplitudes

today: Phenomenological problems with Weinberg’s prescription mounting in the case of heavier nuclei
\[ B_3 = 8.48 \text{ MeV} \]

<table>
<thead>
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<th>( 3B_A/(AB_3) )</th>
<th>nucleons</th>
<th>experiment</th>
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<tbody>
<tr>
<td>( A )</td>
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<tr>
<td>2</td>
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<td>3</td>
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<td>5</td>
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<td>( \vdots )</td>
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<tr>
<td>16</td>
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<td>2.82</td>
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<tr>
<td>( \vdots )</td>
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<tr>
<td>( \rightarrow \infty )</td>
<td></td>
<td>5.7</td>
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\[ = \kappa \left( 1 + \eta A^{-1/3} + \ldots \right) \]  

"liquid-drop formula" \hspace{1mm} v. Weizsäcker '36
Finite-range two-body potential

\[ V(r) = -\frac{\lambda}{2\mu R^2} \frac{f(r/R)}{(r/R)^n} + \ldots \]

nucleons: pion exchange

\[
\begin{align*}
2\mu &= m_N \approx 940 \text{ MeV} \\
&\quad \text{ } n = 3, \ldots \\
R &= 1/m_\pi \approx 1.4 \text{ fm} \\
\lambda &= m_\pi / M_{NN} \approx 1/2 \\
f(r/R) &= \exp(-r/R)
\end{align*}
\]

\[ 4\text{He atoms: two-photon exchange} \]

\[
\begin{align*}
2\mu &= m_{^4\text{He}} \\
n &= 6, \ldots \\
R &= l_{\text{vdW}} \approx 5.4 \text{ A} \\
\lambda &= 1 \\
f(r/R) &= 1
\end{align*}
\]
A measure of (inverse) size: the binding momentum

\[ Q_A \equiv \frac{2mB_A}{A} \]

\[ Q_2 = \sqrt{2\mu B_2} \]

right position of two-body pole

\[ B_{A \gg 1} \sim \sum_{i=1}^{4} \frac{Q_i^2}{2m} \]

all particles contribute equally to energy

nucleons:

\[ (Q_3 R)^{-1} \approx 2 \]

\[ (Q_2 R)^{-1} \approx 3 \]

\[ (Q_3 R)^{-1} \approx 2 \]

\[ (Q_2 R)^{-1} \approx 10 \]

\[ "unitarity\ limit" \ (Q_2 R)^{-1} \rightarrow \infty \]

\[ \psi (r \gg R) \propto e^{-\sqrt{2\mu B_2}r} \]

\[ V(r) = -\frac{\lambda}{2\mu R^2} \frac{f(r/R)}{(r/R)^n} + \ldots \]
\[ V(r) = C_0 \frac{\delta(r)}{4\pi r^2} + \text{derivatives of the delta function} \]
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- Fermi Th
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E(GeV) vs. r(fm)

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condensed-matter physics and beyond
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- EFT for Short-Range Forces
- Two-Body System
  - Amplitude
  - Connection to ERE, potentials
  - Potentials and fine tuning
  - Non-trivial fixed point, scale invariance
- Three-Body System
- More-Body Systems
- Long-range Forces
On blackboard
Nucleon-nucleon phase shifts
Pionless EFT

$^1S_0$

$^3S_1$

\[ C_{00}^{(0)} \rightarrow a_0 = -20.0 \text{ fm (exp)} \]
\[ C_{20}^{(1)} \rightarrow r_0 = 2.78 \text{ fm (exp)} \]

predict
\[ B_d = 1.91 \text{ MeV (NLO)} \]
\[ B_d = 2.22 \text{ MeV (exp)} \]

fit
\[ C_{01}^{(0)} \rightarrow a_0 = -20.0 \text{ fm (exp)} \]
\[ C_{21}^{(1)} \rightarrow r_0 = 2.78 \text{ fm (exp)} \]

predict
\[ B_d^* = 0.09 \text{ MeV (NLO)} \]
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Effective Field Theories

- EFT for Short-Range Forces
- Two-Body System
- Three-Body System
  - Auxiliary Field
  - Amplitude
  - Bound states and correlations
  - Limit cycle, discrete scale invariance

- More-Body Systems
- Long-range Forces
Issue: at what order a three-body force?

First, LO two-body force:

\[ \sim i \frac{(4\pi)^2}{m} D_0(\Lambda) \]

NDA: \[ D_0 \sim \frac{1}{M_{hi}^4} \]

but with fine tuning?

If there is a shallow two-body state

\[ \sim \left( \frac{4\pi C_0}{m} \right)^3 \frac{Q^3}{4\pi} \left( \frac{m}{Q^2} \right)^2 \]

\[ \sim \frac{4\pi C_0}{m} \frac{mQ}{4\pi} \sim \frac{Q}{M_{lo}} \]

need to resum LO two-body force
Auxiliary field

\[ T_2 = \text{full propagator of auxiliary "dimeron" } d \]

with mass \( 2m - \Delta \)

removed by choice of heavy field

\[ \mathcal{L} \rightarrow \mathcal{L}_d = \psi^+ \left( i \frac{\partial}{\partial t} + \frac{\nabla^2}{2m} + \ldots \right) \psi + d^+ \left[ \Delta + \sigma \left( i \frac{\partial}{\partial t} + \frac{\nabla^2}{4m} \right) + \ldots \right] d \]

\[ -\sqrt{\frac{2\pi}{m}} \left[ d^+ \psi \psi + \psi^+ \psi^+ d \right] - h_0 d^+ d \psi^+ \psi + \ldots \]
\[ \sim \frac{i}{\Delta} \]

\[ \sim -2 \frac{2\pi}{m} \left( \frac{i}{\Delta} \right)^2 \int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^0 + p^0 - (\vec{l} + \vec{p})^2 + i\varepsilon} \]

\[ \cdots = -\frac{i}{\Delta^2} \left( \theta_1 \Lambda + ik + \theta_{-1} \frac{k^2}{\Lambda} + \cdots \right) \]

**LO**

\[ = + + \cdots = \cdots = i(\Delta + \theta_1 \Lambda + ik)^{-1} + \mathcal{O} \left( \frac{1}{M_{hi}} \right) \equiv \Delta_R = 1/a_2 \]

**NLO**

\[ = \cdots = -i(\Delta_R + ik)^{-2} \left( \sigma + \theta_{-1} \frac{m}{\Lambda} \right) \frac{k^2}{m} \]

\[ \equiv \sigma_R = -mr_2/2 \]

renormalization simpler than before because momentum \( \rightarrow \) energy
$$\frac{k^2}{4m} - B_2 + E', \bar{p}'$$

$$\frac{k^2}{2m} - E', -\bar{p}'$$

LO

$$T_3$$

$$= \quad +$$

$$+ \ldots$$

$$\frac{k^2}{2m}, -\bar{p} \quad \frac{k^2}{4m} - B_2, \bar{p}$$

$$E = \frac{3k^2}{4m} - B_2$$

$$\sim \frac{4\pi}{Q^2}$$

$$\sim \frac{Q^3}{4\pi} \left( \frac{4\pi}{Q^2} \right)^2 \frac{1}{\Lambda} \sim \frac{4\pi}{Q^2} \frac{Q}{M_{lo}}$$

$$= \quad +$$

$$+ T_3$$

$$T_3 = K_{OPE} + \lambda \int_0^{\Lambda} \frac{d^3l}{(2\pi)^3} \frac{T_3 K_{OPE}}{D}$$

Skorniakov--Ter-Martirosian '57
Only “half-off-shell” amplitude enters:

\[ E' = \frac{k^2 - \not{p}'^2}{2m} \]

\[ T_3(\not{p}', \not{p}) = -\nu_3(\not{p}', \not{p}) - \lambda \int_0^\Lambda \frac{d^3l}{(2\pi)^3} \frac{T_3(\not{p}', \not{l}) \nu_3(\not{l}, \not{p})}{-1/a_2 + \sqrt{3l^2/4 - mE}} \]

\[ \nu_3(\not{p}', \not{p}) = \frac{8\pi}{mE - \not{p}'^2 - \not{p}^2 - \not{p}' \cdot \not{p}} \]

\[ \begin{align*}
\text{bosons} & \quad S = 0 \quad \lambda = 1 \\
\text{nucleons} & \quad \begin{cases} 
S = 3/2 & \lambda = -1/2 \\
S = 1/2 & \text{two equations that decouple in the UV} \quad \lambda = 1 \quad \& \quad \lambda = -1/2
\end{cases}
\end{align*} \]
Simplest case: \(S\) wave

\[
T_{3,0}(p', k) = -\nu_{3,0}(p', k) - \frac{\lambda}{2\pi^2} \int_0^\Lambda dl \frac{T_{3,0}(p', l) \nu_{3,0}(l, k)}{-1/a_2 + \sqrt{3l^2/4 - mE}}
\]

\[
= -\nu_{3,0}(p', k) - \frac{\lambda}{2\pi^2} \int_0^\Lambda dl \frac{\nu_{3,0}(p', l) T_{3,0}(l, k)}{-1/a_2 + \sqrt{3l^2/4 - mE}}
\]

\[
\nu_{3,0}(p', k) = \frac{4\pi}{p'k} \ln \left( \frac{p'^2 + k^2 - p'k - mE}{p'^2 + k^2 + p'k - mE} \right)
\]

As before, first look at UV properties: \(p' \gg k \geq 1/a_2\)

\[
T_{3,0}(p' \gg k) = \frac{4\lambda}{\sqrt{3}\pi} \int_0^\Lambda \frac{dl}{p'} \ln \left( \frac{p'^2 + p'l + l^2}{p'^2 - p'l + l^2} \right) T_{3,0}(l \gg k)
\]

Ansatz \(T_{3,0}(p' \gg k) \propto p'^{-(s+1)} \Rightarrow \frac{8\lambda}{\sqrt{3\pi}} \frac{\sin(s\pi/6)}{\sqrt{3s} \cos(s\pi/2)} = 1\) 

Danilov '63
nucleons \( S = \frac{3}{2} \) \( \lambda = -\frac{1}{2} \) \( \Rightarrow \ s \approx 2.17 \)

\[
a(x) \propto T_{Nd,3/2}(p'a_2,0)
\]

\[
\frac{\partial T_{Nd,3/2}(p'a_2 \sim 1,0)}{\partial \Lambda} \approx 0
\]

\[
T_{Nd,3/2}(p'a_2 \gg 1,0) \propto p'^{-3.17}
\]

...
nucleons $S = 3/2$

postdict

\[ a_{Nd,3/2} = 5.09 + 0.89 + 0.35 + \ldots \text{ fm} = 6.33 \pm 0.10 \text{ fm} \]

\[ a_{Nd,3/2} = 6.35 \pm 0.02 \text{ fm (exp)} \]

\[ \text{LO} \]

\[ \text{NLO} \]

\[ \text{v.Oers + Seagrave '67} \]

\[ \text{Dilg et al. '71} \]

\[ \text{Bedaque, Hammer + v.K. '98} \]

\[ \text{Bedaque + v.K. '97} \]
bosons \[ S = 0 \quad \lambda = 1 \quad \Rightarrow \quad s = \pm is_0 \quad s_0 \approx 1.00624 \]

\[ \frac{\partial T_{\psi d}(p'a_2 \sim 1,0)}{\partial \Lambda} \neq 0 \quad T_{\psi d}(p'a_2 \gg 1,0) \propto \cos \left( s_0 \ln \frac{p'}{\Lambda} + \delta \right) \]

Bedaque, Hammer + v.K. '98 '99
“Thomas collapse”

\[ B_T \]

\[ D_1 = 0, \text{ ground state} \]
\[ D_1 = 0, 1^{\text{st}} \text{ ex. state} \]

deuteron-neutron threshold

Kirscher, Barnea, Gazit, Pederiva + v.K. '15

Thomas '35
\[ T_3 = \frac{4\pi}{Q^2} + \int_0^\Lambda \frac{d^3l}{(2\pi)^3} \frac{T_3 K_{OPE}}{D} + \lambda \int_0^\Lambda \frac{d^3l}{(2\pi)^3} \frac{T_3 K_{TBF}}{D} \]

\( \sim \frac{4\pi}{M_{lo}^2} \)

Bedaque, Hammer + v.K. '98 '99
Repeat with
\[
\begin{align*}
\nu_3(\vec{p}', \vec{p}) &\rightarrow \nu_3(\vec{p}', \vec{p}) + h_0(\Lambda) \\
\nu_{3,0}(p', k) &\rightarrow \nu_{3,0}(p', k) + h_0(\Lambda)
\end{align*}
\]

\[h_0(\Lambda) \equiv 8\pi \frac{H(\Lambda)}{\Lambda^2}\]

UV properties: \(p' \gg k \geq 1/a_2\)

\[
T_{3,0}(p' \gg k) = \frac{4}{\sqrt{3}\pi} \int_0^\Lambda dl \left[ \ln \left( \frac{p'^2 + p'l + l^2}{p'^2 - p'l + l^2} \right) - 2 \frac{p'l}{\Lambda^2} H(\Lambda) \right] T_{3,0}(l \gg k)
\]

only important for \(p' \sim \Lambda\)

\[
T_{3,0}(\Lambda \gg p' \gg a_2^{-1}, 0) \propto \cos \left( s_0 \ln \frac{p'}{\Lambda} + \delta(H(\Lambda)) \right)
\]

\[\delta(H(\Lambda)) \equiv s_0 \ln \left( \frac{\Lambda}{\Lambda_*} \right)\]

physical, dimensionful parameter
Fitting to $a_3$

\[ \frac{\partial T_{\psi d}(p'a_2 \sim 1, 0)}{\partial \Lambda} \neq 0 \]

\[ T_{\psi d}(p'a_2 \gg 1, 0) \propto \cos \left( \frac{p + p'}{\Lambda \Lambda_*} \right) \]

Renormalized!
"Thomas collapse"
Fitting $\Lambda_*$ to $B_3$:

Thomas '35

outside EFT

physical triton

Kirscher, Barnea, Gazit, Pederiva + v.K. '15
Correlations

Phillips line

potential models (Phillips ’68)

varying \( \Lambda_* \)

(Bedaque, Hammer + v.K. ’00)

nucleon-deuteron scattering length

triton binding energy
nucleons \( S = 1/2 \)

\[ a_{Nd,1/2} = 0.65 \text{ fm (exp)} \]

\[ B_t = 8.0 + 0.8 + \ldots \text{ MeV} = 8.8 \pm 0.5 \text{ MeV} \]

\[ B_t = 8.48 \text{ MeV (expt)} \]
Studying UV properties in more detail:

\[ H(\Lambda) \equiv \frac{\Lambda^2 D_0(\Lambda)}{mC_0^2(\Lambda)} \]

\[ \approx \frac{\sin(\ln(\Lambda/\Lambda_*) + \arctan(1/s_0))}{\sin(\ln(\Lambda/\Lambda_*) - \arctan(1/s_0))} \]

Fitting \( \Lambda_* \) to \( B_3 \):

\[ \Lambda \to \Lambda e^{n\pi/s_0} \]

\[ D_0 \sim \frac{1}{M_{lo}^4} \]

not just the effective-range expansion!
Discrete Scale Invariance

\[ A = 2 \]

Unitarity limit

\[ C_0(\Lambda) = -\frac{1}{\theta_1 \Lambda} \quad \leftrightarrow \quad mB_2^{(0)} = 0 \]

regulator-dependent number

scale invariance

\[
\begin{align*}
    x & \rightarrow \alpha x \\
    t / m & \rightarrow \alpha^2 t / m \\
    \Lambda & \rightarrow \alpha^{-1} \Lambda \\
    \psi & \rightarrow \alpha^{-3/2} \psi
\end{align*}
\]

(includes Wigner spin-isospin symmetry)

\[ SU(4)_W \]

Mehen, Stewart + Wise '00

König, Grießhammer, Hammer + v.K. '16
König '16
v.K. '17

\[ \cdots \]
$$A = 3$$

$$D_{0}^{(0)}(\Lambda) \propto \frac{1}{\Lambda^{4}} \frac{\sin\left(s_{0} \ln\left(\Lambda/\Lambda_{*}\right) + \arctan s_{0}^{-1}\right)}{\sin\left(s_{0} \ln\left(\Lambda/\Lambda_{*}\right) - \arctan s_{0}^{-1}\right)}$$

**Discrete scale invariance**

$$x \rightarrow \alpha_{n} x \quad \Lambda \rightarrow \alpha_{n}^{-1} \Lambda$$

$$t \rightarrow \alpha_{n}^{2} \frac{t}{m} \quad \psi \rightarrow \alpha_{n}^{-3/2} \psi$$

\[ S_{EFT}^{(0)} \rightarrow S_{EFT}^{(0)} \]

$$\alpha_{n} = \exp\left(n\pi/s_{0}\right) = (22.7)^{n} \quad n = \ldots, -1, 0, 1, 2, \ldots$$

Ground state

$$mB_{3n}^{(0)} = k_{*}^{2} \exp\left(-2n\pi/s_{0}\right) \lesssim M_{hi}^{2}$$

$$\ln\left(\frac{k_{*}}{\Lambda_{*}}\right) = \ln \beta, \mod \pi/s_{0} \quad \beta \approx 0.383$$

(Efimov '71)

(Efimov states)

Bedaque, Hammer + v.K. '00

(includes Wigner spin-isospin symmetry)

SU(4)$_{w}$
Observation of the Efimov state of the helium trimer

Science 348 (2015) 551


\[ -\left( mB_2 / \Lambda_*^2 \right)^{1/4} \]

\[ = -\left( a_2 \Lambda_* \right)^{-1/2} \]

Borromean

bound 3 state

virtual 1+2 state

triton & helion

decreasing two-body attraction

Braaten + Hammer '03

Vaghani, Rupak, Higa + v.K. '18

4He atoms nucleons
INTRODUCTION TO NUCLEAR EFFECTIVE FIELD THEORIES

U. van Kolck
Institut de Physique Nucléaire d’Orsay
and University of Arizona

Supported in part by CNRS and US DOE
Effective Field Theories
EFT for Short-Range Forces
Two-Body System
Three-Body System
More-Body Systems
- Four bodies
- Beyond
Long-range Forces
$A = 4$

Unitary bosons

Hammer + Platter '07

No 4BF

ground state

first excited state

Efimov descendants
Correlations

nucleons

Tjon line

potential models
(Tjon '74)

exp

varying $\Lambda_*$

(Hammer, Meißner + Platter '04)

LO

$B_\alpha$ [MeV]

$B_1$ [MeV]
Nucleons at unitarity

additional expansion in \((Qa_2)^{-1}\)

Nucleons perturbatively close to unitarity
Bosons

$^4$He atoms

No more-body forces up to NLO

Bazak, Eliyahu + v.K. '16
Bazak, Kirscher, König, Pavón-Valderrama, Barnea + v.K. '18
Summary:

Expansion parameter \( \frac{Q}{M_{hi}} \)

- **LO**: no-derivative two- and three-body interactions
  
  \[ C_0, D_0 \]

  Around unitarity: a SINGLE physical parameter \( \Lambda_* \)

- **NLO**: two-derivative two- and no-derivative four-body interactions
  
  \[ C_2, E_0 \]

- **Etc.**
Unitary bosons

**discrete scale invariance**

Hammer + Platter ‘06
von Stecher ‘10’11
Gattobigio, Kievsky + Viviani ‘11’12

Ground states

single scale

\[ \frac{B_N^{(0)}(\Lambda_*)}{N} = \kappa_N \frac{B_3(\Lambda_*)}{3} \]

universal numbers

\[ \kappa_2 \equiv 0 \]
\[ \kappa_3 \equiv 1 \]
\[ \kappa_4 \approx 3.5 \]
\[ \kappa_{N \geq 5} \approx ? \]
\[ \kappa_N \approx \frac{3}{N} (N - 2)^2 \]

Bazak, Eliyahu + v.K. '16

\[ \kappa_N = \kappa_\infty \left[ 1 + \eta N^{-1/3} + \mathcal{O}(N^{-2/3}) \right] \]

\[ \kappa_\infty = 90 \pm 10 \quad \eta = -1.7 \pm 0.3 \]
Nucleons around unitarity

\[ A \geq 4 \]

**discrete scale invariance**

single scale

\[ \text{LO} \]

\[ \frac{B^{(0)}_A(\Lambda^*_A)}{A} = \kappa_A \frac{B_3(\Lambda^*_A)}{3} \]

\[ \begin{align*}
\kappa_2 &\equiv 0 \\
\kappa_3 &\equiv 1 \\
\kappa_4 &\approx 3.5 \\
\kappa_{A \geq 5} &\approx ?
\end{align*} \]

\text{Ground states}

same as for bosons should grow slower than bosons

same saturation mechanism as for bosons?
Larger nuclei?

adapt \textit{ab initio} many-body methods

- $^4\text{He}$ excited state + $^6\text{Li}$ at LO with no-core shell model
  Stetcu, Barrett + v.K. '07

- $^{16}\text{O}$ at LO with auxiliary-field diffusion Monte Carlo
  Contessi et al. '17

- $^4\text{He}$ at LO + resummed NLO with resonating group
  and stochastic variational
  Kirscher \textit{et al.} '09; Lensky, Birse + Wallet '15

- $^4\text{He}$, $^{16}\text{O}$, $^{40}\text{Ca}$ at LO + resummed NLO
  Bansal \textit{et al.} '17

+ many applications to exotic hadronic states, cold atoms, ...
\[ B_3 = 8.48 \text{ MeV} \quad B_3 = 2.28 \cdot 10^{-5} \text{ eV} \]

<table>
<thead>
<tr>
<th>( \frac{3B_A}{(AB_3)} )</th>
<th>nucleons</th>
<th></th>
<th></th>
<th>( ^4 \text{He atoms} )</th>
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<tr>
<td>( A )</td>
<td>experiment</td>
<td>LO EFT</td>
<td>unitarity</td>
<td>potential</td>
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<tr>
<td>3*</td>
<td>(0.17)</td>
<td>(0.19)</td>
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<td>0.0180</td>
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<td>?</td>
<td>6.2</td>
<td>5.7</td>
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<td>6</td>
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<td>?</td>
<td>9.2</td>
<td>8.2</td>
<td>8.9</td>
</tr>
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<td>( \vdots )</td>
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<td>?</td>
<td>27.4</td>
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<tr>
<td>( \rightarrow \infty )</td>
<td>5.7</td>
<td>?</td>
<td>?</td>
<td>180</td>
<td>?</td>
<td>90</td>
</tr>
</tbody>
</table>

\[ = \kappa \left( 1 + \eta A^{-1/3} + \ldots \right) \quad \text{“liquid-drop formula”} \quad \text{v. Weizsäcker '36} \]
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Long-range Forces
Next time…
Conclusion

- Systems near unitarity can be described model-independently by Pionless EFT properties given by essentially one parameter $\Lambda$.
- Details obtained in perturbation theory.
- Cf. parallel results from a model perspective: Kievsky et al.

- How far can we go for nuclei?
  - More nucleons: Gezerlis et al., in progress
  - Nuclear matter: Lyu et al., in progress
  - Higher orders: Bazak et al., in progress