Sliding Block on a Semicircular Track with Friction

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This work presents the dynamics of a block as it slides down a semicircular track with both Coulomb's and viscous frictions. Analytical solutions and graphic forms of the equations for the velocity, acceleration and energy as functions of angular position are displayed and discussed for several friction coefficients. Following these solutions the plots as function of time for these damped motions are also presented.

É apresentado neste trabalho a dinâmica de um bloco quando este desliza sobre um trajeto semicircular na presença de ambos os atritos, entre superfícies e viscoso. Soluções analíticas e formas gráficas das equações para a velocidade, aceleração e energia como funções da posição angular são exibidas e discutidas para diversos coeficientes de atrito. Seguindo estes soluções, os gráficos em função do tempo para estes movimentos amortecidos são também apresentados.

I Introduction

In courses of intermediary mechanics it is helpful to introduce oscillatory motions where no approximations are made. The purpose is to familiarize the students with an ellaborated mathematical apparatus that describes these experimentally observed motions, although little theoretical discussions is found, mainly when friction forces are involved. In fact, these forces are known to act opposing the motion (direction contrary to that of velocity), changing in direction for each half cycle executed. Franklin and Kimmel[1], made a study where a block starts its motion from the top of a semicircular track in the presence of a Coulomb's frictional force. The parameters velocity, acceleration and work performed by the frictional force in the first quadrant were found relative to a horizontal line passing through the center of the circunference constituing the track, varying thus its position from 0 until $+\pi/2$. Lapidus[2], extended the former work, considering now motion in the second quadrant and where the block started from any initial position relative to a vertical line varying from $-\pi/2$ until $+\pi/2$[2].

We propose here to study this kind of frictional motion including now a viscous force generally considered as resulting from the atmosphere. We assume that a block of mass m starts its downward sliding motion at rest on a semicircular track of radius R from a point in some initial angular position θ_0 relative to a horizontal line passing through the center of the circunference. We also assume that the forces of resistance to the motion are the Coulomb's frictional force F_d, proportional to the surface reaction and a viscous force with magnitude proportional to v^2, where v is the tangential velocity of the block relative to the surface, Fig. 1. The force of reaction N due to the contact with the surface is given by $N = mgsen\theta + F_c$, where F_c is the centripetal force given by mv^2/K and K is the radius of the trajectory described by the center of mass of the block. Initially and in each return point it is necessary for the motion to continue that the tangential component of the gravitational force relative to the surface be greater than the frictional force. This condition is given by $\theta > \theta_c$, where $\theta_c = \tan^{-1}\mu_s$ being μ_s the static friction coefficient. In fact, it is still possible to find analytical solutions for the dynamic parameters if this quadratic approximation in the velocity is considered. Applying Newton's second law of motion for the block that slides down yields the following equation for the tangential acceleration:

$$a_t = gcos\theta \pm \mu(gsin\theta + \frac{v^2}{K}) \pm \frac{\beta}{m}v^2 \qquad (1)$$

In each return point there is a change in the direction of motion. In this way, if more than the first half cycle is to be found, substitution in the initial angular position is required and the negative signal is to be applied when θ is increasing, while the positive signal is to be employed when θ is decreasing. As before, it is assumed that the viscous force F_v is proportional to v^2 and directed contrary to the velocity of the block, $F_v = -\beta v^2$. It is thus possible to extend the results obtained by Franklin and Kimmel[1]. The proportionality

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constant β with dimension [mass/length] is an intrinsic parameter that depends on the area of contact A of the block with the frictional medium, as well as on the viscosity coefficient κ where the block is immersed. It can then be written as $\beta = \kappa A$.

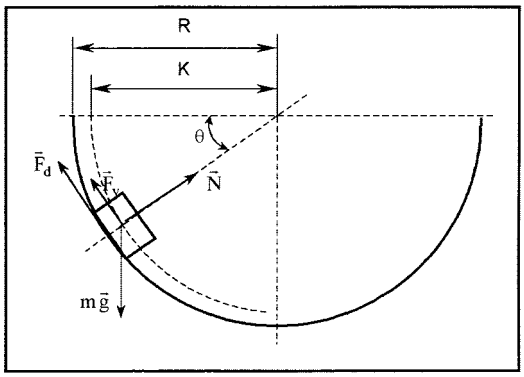

Figure 1. Block sliding down from the top of a vertical semicircular track of radius R. The center of mass executes a motion following the radius K, where $K = R - h$ and h is the half-height of the block. For simplifications in the calculations were considered $R = 1.1m$ and $h = 0.1m$ resulting in a trajectory of radius $K = 1.0m$.

II Results

For the first half cycle writing Eq. (1) as a function of the angular position θ and rearranging terms yields:

$$\frac{dv^2}{d\theta} + \alpha v^2 = 2gK(\cos\theta - \mu\sin\theta) \tag{2}$$

For simplification we use the following definition:

$$\alpha = 2(\mu + \frac{\beta K}{m}) \tag{3}$$

Eq. (2) is a standard differential equation which can also be analitically solved by multiplying both sides by the integrating factor $exp(\alpha\theta)$, as indicated by Franklin and Kimmel[1]

$$e^{\alpha\theta}(\frac{dv^2}{d\theta} + \alpha v^2) = \frac{d}{d\theta}(v^2 e^{\alpha\theta}) = 2gK(\cos\theta - \mu\sin\theta)e^{\alpha\theta} \tag{4}$$

Integrating from an initial angular position θ_0 to a final position θ results in the following expression for the quadractic velocity:

$$v^2 = \frac{2gK}{1+\alpha^2}\{(\mu+\alpha)\cos\theta + (1-\mu\alpha)\sin\theta - [(\mu+\alpha)\cos\theta_0 + (1-\mu\alpha)\sin\theta_0]e^{-\alpha(\theta-\theta_0)}\} \tag{5}$$

In Table I are shown the special values of the dynamic parameters as numerically obtained. In the limiting case of no viscous force ($\beta = 0$) and with the special initial conditions $\theta_0 = 0$ we get:

$$v^2 = \frac{2gK}{1+4\mu^2}(3\mu\cos\theta + (1-2\mu^2)\sin\theta - 3\mu e^{-2\mu\theta}) \tag{6}$$

This is in agreement with the previous work[1]. The process can be continued by reaching the next return point, changing now the initial angular position and the signal for the frictional forces in the Eq. (1). In Fig. 2 are shown curves of v as a function of θ after several cycles of oscillation. It must be noted that the maximum value reached by the velocity is not situated at $\pi/2$, except when there is no friction ($\mu = \beta = 0$). Mathematically this can also be shown by finding the maximal velocity as a function of θ. Using the same relations given by Eqs. (1-3) for the first half cycle and substituing the expression for v^2, Eq. (5), into Eq. (2), differentiating and rearranging the terms yields:

$$a_t = \frac{g}{1+\alpha^2}\{(1-\mu\alpha)\cos\theta - (\mu+\alpha)\sin\theta + \alpha[(1-\mu\alpha)\sin\theta_0 + (\mu+\alpha)\cos\theta_0]e^{-\alpha(\theta-\theta_0)}\} \tag{7}$$

In Fig. 3 are displayed the curves for the tangential and centripetal acceleration (v^2/K) as functions of θ in several cases. The energy spent by the friction when the block travels a length K$d\theta$ is given by

$$dW_f = -(F_d + F_v)Kd\theta = -(\mu N + \beta v^2)Kd\theta \tag{8}$$

The total energy W_f spent in the first half cycle can be found integrating the former expression or taking the negative of the diference between potential and kinetic energies. This results in

$$W_f = mgK\sin\theta_0 - mgK\frac{\mu+\alpha}{1+\alpha^2}[(\alpha\sin\theta - \cos\theta) - (\frac{1-\mu}{\mu+\alpha}\sin\theta_0 + \cos\theta_0)e^{-\alpha(\theta-\theta_0)}] \tag{9}$$

The curves are also displayed in Fig. 3. Eq. (7) can be used for obtaining the angular velocity $\dot{\theta}(\theta)$ in the first half cycle. Using the fact that initially $\dot{\theta}_0 = 0$ (this also happens in each return point), yields:

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{K(1+\alpha^2)}\{(1-\mu\alpha)\sin\theta + (\mu+\alpha)\cos\theta - [(1-\mu\alpha)\sin\theta_0 + (\mu+\alpha)\cos\theta_0]e^{-\alpha(\theta-\theta_0)}\}} \tag{10}$$

In Fig. 4 are shown curves of oscillation for several friction coefficients according to these solutions.

	μ=0.10 β=0.0	μ=0.20 β=0.0	μ=0.50 β=0.0	μ=0.0 β=0.05	μ=0.0 β=0.10	μ=0.0 β=0.15	μ=0.10 β=0.05	μ=0.20 β=0.10	μ=0.50 β=0.15
v_{max}	3.8688	3.4562	2.6768	3.7535	3.3217	3.0126	3.3836	2.8226	2.2076
θ	1.3190	1.1324	0.7745	1.2797	1.1044	0.9825	1.1155	0.8755	0.5966
$v(\theta{=}\pi/2)$	3.7888*	3.1856*	1.3595*	3.6476	3.0399	2.5646	3.1028	2.0985	stop
	2.7803	1.4909		2.7641	1.9916	1.5426	1.7650		
	1.9511			2.2306	1.4845	1.1052	0.8991		
	1.2232			1.8717	1.1839	0.8617			
	0.5219			1.6129	0.9851	0.7063			
				1.4175	0.8437	0.5984			
				1.2644	0.7381	0.5190			
				1.1412	0.6561	0.4583			
				0.9555	0.5903				
				0.8839	0.5365				
				0.8224					
R. Points	2.6722	2.3486	1.7220	2.6307	2.3731	2.2114	2.3557	1.9871	1.4966
	0.7937	1.2673	0.7569	1.0162	1.1523	1.1362	1.5819		
	2.0910			2.2347	1.9965	1.8830	1.7662		
	1.2727			1.0090	1.2246	1.3213	1.5781		
	1.6644			2.0581	1.8627	1.7785			
				1.1401	1.3182	1.3927			
				1.9565	1.7934	1.7266			
				1.2212	1.3718	1.4321			
				1.8903	1.7507				
				1.2763	1.4066				
				1.8439					
				1.3163					

Table I - It is shown here the maximal velocities (in [m/s]) reached by the block and its corresponding angular positions (in radians), values for the velocities when it crosses the lowest point and at the return points (where $v = 0$) for each half cycle of oscillation. For these values initially we use $\theta_0 = 0$. For $\mu = 0.50$, $\beta = 0.15$, the block stops before reaching the lowest position. $(*)$ values marked in agreement with Ref. [1].

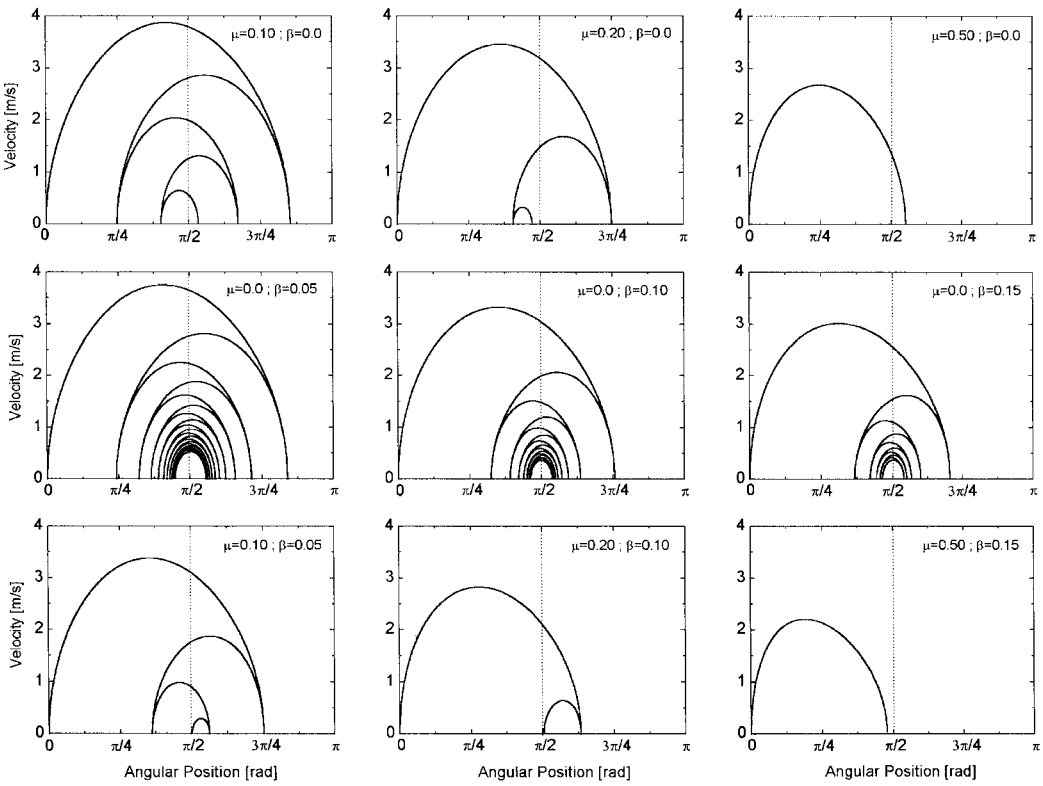

Figure 2. Comparative curves for the velocity as a function of θ. For $\mu = 0.0$ the curves were truncated before reaching the rest point.

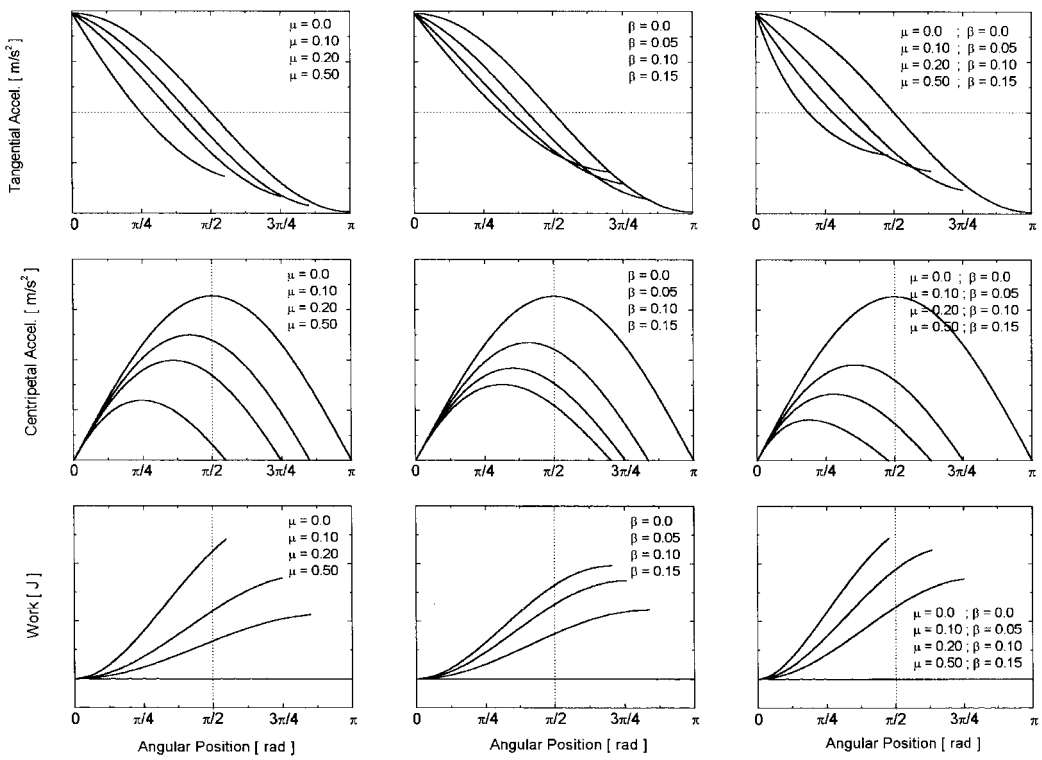

Figure 3. Curves for the tangential and centripetal accelerations, as well as for the energy spent by the frictional forces.

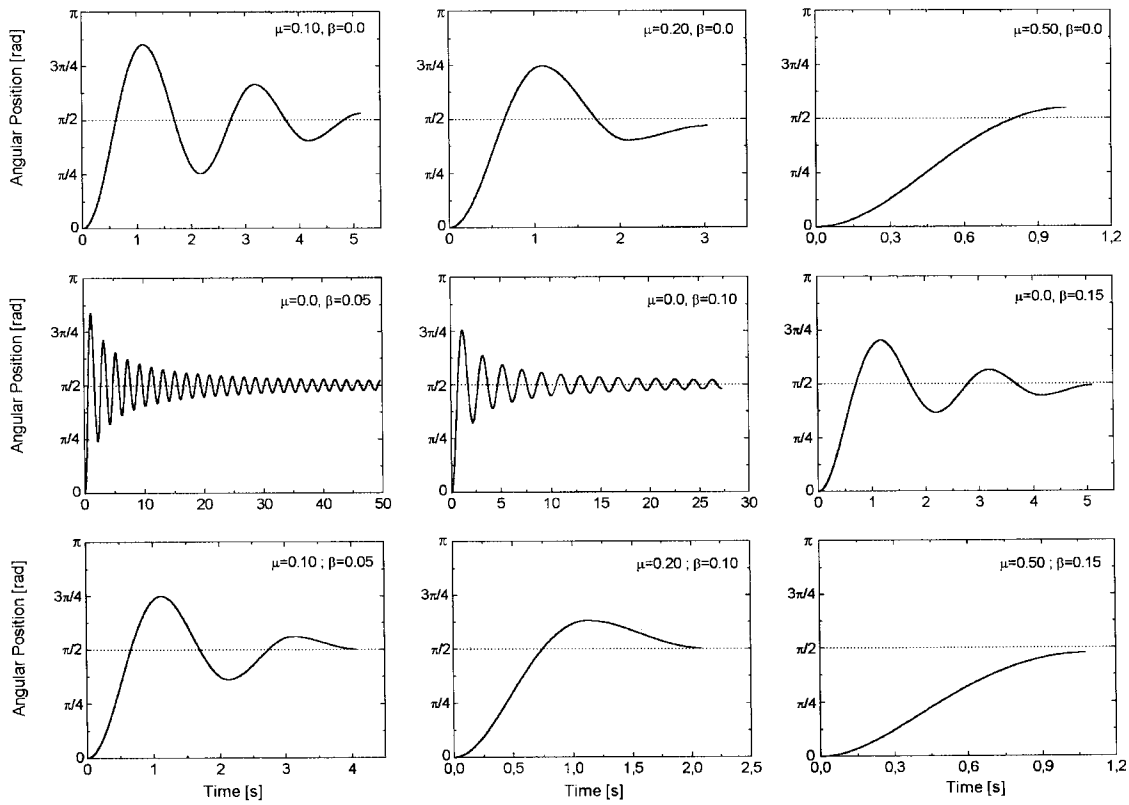

Figure 4. Damped motion as a function of time for several Coulomb's and viscous friction coefficients.

III Discussion

As seen in the acceleration curves with friction, the maximum velocity occurs before $\pi/2$ (making Eq. (2) equal to zero). The reason is that the block is under the action of a direction dependent resistive force while it executes its semicircular motion. The values given by Franklin and Kimmel[1], (3.7881, 3.1849, 1.3585) for the velocities in $\pi/2$ when $\beta = 0$ (for $\mu = 0.1$, 0.2 and 0.5 respectively) are also numerically confirmed. The return points can be checked by making use of Lapidus's method[2], for the same conditions as given above. In the absence of both frictional forces we return to the ideal case of an oscillating pendulum without approximation for small angles. In this case the period can also be analitically found by means of elliptic functions[3], which results in a value of 2.367 seconds for a length equal to 1 m. We present here a suggestion for determining the amplitude and period of motion: to mark the oscillatory trajectory covering the track with paper and fixing a pen to the block, although it can be difficult to determine the friction coefficient between these surfaces.

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