(In)stability in Classical Mechanics

((In)estabilidade em mecânica clássica)

César R. de Oliveira
Departamento de Matemática, UFSCar
C.P. 676, São Carlos, SP, 13560-970 Brazil
e-mail: oliveira@dm.ufscar.br

Recebido em 2 de Setembro, 1998

Different aspects of the stability and instability concepts in Classical Mechanics are reviewed. It is also commented upon some specific points not usually considered in textbooks on Mechanics. The background motivation is the solar system stability question and it is also included a discussion of the KAM theorem. It is stressed the concepts instead of applications.

Através de um pequeno passeio pela Mecânica Clássica são discutidos vários aspectos dos conceitos gerais de estabilidade e instabilidade. Aproveita-se a oportunidade para tecer comentários sobre pontos específicos normalmente não abordados em livros-texto de Mecânica. Toma-se como pano de fundo a questão da estabilidade do sistema solar e finaliza-se abordando o teorema KAM. Privilegia-se a discussão de conceitos (com uma roupagem matemática) em detrimento às aplicações diretas.

I Introduction

From a very broad point of view stability is the conservation of a given type of behavior when there is only moderate changes in the underlying conditions. Here we aim at reviewing some important stability concepts in Classical Mechanics of particles under a force field generated by a potential $U(q)$, where q denotes the N generalized position coordinates $q = (q_1, \cdots, q_N)$; N is the number of degrees of freedom (d.f.) of the system. We think of stability for time going to infinity and general references for what follows are [1, 2, 3, 8, 9, 10, 12, 14, 17, 19].

Some concepts like orbits and vector fields will be freely used; we think their intuitive meanings are enough on first reading this note, which should be considered just an introduction and (hopefully) motivation to further studies. With confidence, at present it has become important a modern approach to this veteran subject.

We shall account for some modern aspects of the subject that are relevant for both Physics and Mathematics. We begin with Newton's Second Law and some pertinent comments. Then we discuss two notions of stability, Lyapunov and Structural Stabilities, that became key ingredients in the theory of Dynamical Systems and Differential Equations. In order to motivate the consideration of geometric aspects of Mechanics we briefly discuss the introduction of the concept of manifold; however our subsequent discussion will be restricted to the more common spaces $\mathbb{R}^n$.

After reviewing the Hamiltonian formulation of Classical Mechanics we discuss a theorem by Poincaré about return times and call it Poincaré's Stability. Finally, we introduce the notion of integrable systems and discuss their perturbations via the KAM theorem. We finish with a series of remarks and also mention some open problems.

II Newton's Second Law

A basic quantitative step in Mechanics was the formulation of Newton's Second Law

$$F = \frac{dp}{dt},$$

where p denotes the momentum vector in cartesian coordinates, i.e., it is just the product of mass and velocity, $p = mv$. Also at this point Newton has revealed his talent since this formulation (in contrast to force=mass$\times$acceleration) is adequate for applications in relativity, for the case of mass variation and for the inclusion of magnetic fields.

This fundamental relation brings out a rather delicate conceptual question: What is force? From Newton's Second Law force is the ratio of momentum variation, but one could argue that we are just renaming things and instead of calling it *momentum variation* we say *force*. A satisfactory answer to this question is that we know explicit expressions of forces in important cases, such as Hooke's harmonic law and gravitational attraction.

Newton's Second Law is a differential equation, so every tool developed in the theory of Differential Equations could, in principle, be used to investigate the motion, particularly the stability questions; perhaps most of these tools were developed due to Newton's Second Law...

The ancient Greeks thought that it was needed the presence of a force to keep a particle in motion (I admit it is not at all clear for me what they called *force*); but assuming Newton's Laws of motion, if $F = 0$ it follows that the one-particle acceleration (in one-dimension) is null and denoting the particle position at time t by $x(t)$ we have

$$\frac{d^2x(t)}{dt^2} = 0 \Longrightarrow x(t) = vt + x(0),$$

where v is particle's constant velocity (since $dv/dt = F/m = 0$). Therefore, since v can be different from zero, one concludes that it is possible to have motion with no force at all.

However, we have just skipped one of the main arguments that have led Newton to formulate his famous Laws, the inertia that had already been verified by Galileo. Working mainly with slopes, Galileo realized that when there is little frictional resistance a moving body travels farther and farther before stopping, and so he concluded that if there was no friction at all there would be no change in the body velocity (including speed and direction). This became Newton's First Law also known as Law of Inertia. It is worth mentioning that in his first experiments Galileo used his own pulse to measure time.

Besides presenting his Laws of motion Newton used them to study important systems, with special emphasis on planetary motion. Notice that he has also developed the Differential and Integral Calculus as a tool for his Second Law! Here we have one of the first instances of unification in Physics, the celestial bodies and the motion of objects on Earth are governed by the same rules.

Newton not only found that the planets are attracted by the sun according to the inverse square of the distance r between them, but has also deduced that the elliptic orbits follows from this law, and this was not at all clear at that time. Since it is very simple, we reproduce an argument for the inverse square law for circular orbits with radius r and period of motion $T = 2\pi r/v$, where v denotes the constant speed of the particle of mass m. In this case the gravitational force F_G equals the centripetal force $F_c = mv^2/r$, and combining these two relations with Kepler's Third Law $r^3/T^2 = K_1$ one concludes that $F_G = K_2/r^2$ (K_1 and K_2 are numeric constants). This inverse square law rules the planetary motion, which has been the prototype of order for centuries; however, as we shall discuss later on, it still holds some surprises.

After Newton's celebrated work a period of development of quantitative tools for Classical Mechanics has followed; for instance, some specific methods and adaptation of Newton's Laws to problems in fluids, waves (linear and nonlinear), different kinds of structures and also new formulations of Newtonian Mechanics, particularly Lagrangian and Hamiltonian Mechanics, with major contribution by Euler, Jacobi, Hamilton, Lagrange and Laplace.

The so-called Lagrangian and Hamiltonian equations can be deduced from variational principles which have led some people to speculate on metaphysical and philosophical interpretation of them. One of the main problems that have arisen was the stability problem, with particular emphasis on the stability of the solar system: will our solar system behave as it does today forever?

On purpose we have been vague in formulating this question since we have not discussed details of what we mean by stability and instability. In fact, there are different notions of stability and one should be aware of this when formulating his/her questions and conclusions. The main goal of this work is to discuss some of these notions with perspective to apply them to the solar system.

III Lyapunov and Structural Stabilities

The Lyapunov stability refers to stability of a single known orbit of a system; it is widely used in the theory of Differential Equations and although it is a very natural notion its importance has been overcome by more recent concepts. Loosely speaking, an orbit $x(t)$ is Lyapunov stable if given an arbitrary small error any other orbit that starts close enough of $x(t)$ keeps its distance from $x(t)$ less than that error forever. Let's present its definition in a rather general setting of differential equations in $\mathbb{R}^n$.

Definition 1 *Let $x(t)$ be a solution (for $t \in \mathbb{R}$) of the differential equation*

$$\frac{dx(t)}{dt} = f(x)$$

with f being continuous in $\mathbb{R}^n$. This solution is Lyapunov stable if for all $\varepsilon > 0$ there exists $\delta > 0$ such that any other solution $y(t)$ of this equation such that $|y(0) - x(0)| < \delta \Longrightarrow |y(t) - x(t)| < \varepsilon$ for any $t > 0$.

A simple example. The null solution $x(t) = 0$ of the differential equation in $\mathbb{R}$, $dx(t)/dt = -2x(t)$ is Lyapunov stable, since all solutions $y(t)$ of this equation converges (uniformly for initial conditions in compact neighborhoods of zero) to $x(t)$ for $t \to \infty$.

Some comments are in order. (i) the continuity assumptions for f is just to guarantee the existence of solutions of the differential equation.

(ii) ε plays the role of the *error* in the informal discussion above.

(iii) we can get a useful picture of a Lyapunov stable solution $x(t)$ by considering a tube of radius ε with $x(t)$ at its center, so that any other solution that starts from a distance less than δ from $x(0)$ continues inside this tube, as depicted in Figure 1.

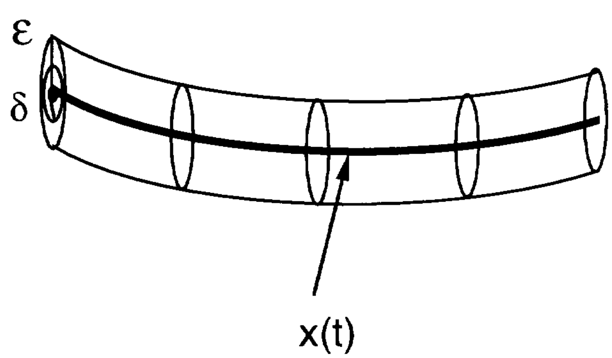

Figure 1. An illustration of a Lyapunov stable solution $x(t)$.

(iv) if the condition $\lim_{t\to\infty} |y(t)-x(t)| = 0$ is added to the above definition, then $x(t)$ is said to be asymptotically Lyapunov stable; notice the null solution in the above example is asymptotically stable. As we shall discuss later on, the Lyapunov asymptotically stable does not take place in conservative Classical Mechanics.

(v) in general it is not a simple task to verify whether a given solution is Lyapunov stable or not, particularly in the case of our solar system: If one were able to give a very small perturbation to the position of one planet, say, would the orbits of all planets, comets, satellites and asteroids keep very close to their original orbits?

(vi) we are in position to give a naive definition of "chaotic motion." If the solutions of a system of differential equations live in a finite (compact) region of $\mathbb{R}^n$ in which no orbit is Lyapunov stable, then we say this is a chaotic region.

It is common to study the stability of orbits via approximations and series expansions and some researchers (erroneously) claimed to have proved the stability of the solar system. However, other important contributions, mainly due to Poincaré, suggested that such series were divergent; in 1889 Poincaré won a prize from the King of Sweden for his discoveries. At the end of this article we will comment upon the main roots of such divergences while discussing the KAM theorem.

A broad discussion about periodic orbits had also taken place at that time. Even today many works in the physical and mathematical literature are concerned with the problem of existence and stability of periodic orbits of differential equations. Although we are not going to discuss these problems here, we would like to let a question to the reader: Why is it important to study periodic orbits?

With the seemingly failure of series convergence in Celestial Mechanics, Poincaré introduced the qualitative study of differential equations with new mathematical ideas. These ideas became a turning point in Mathematics and led to the development of Differential Topology and Geometry, and can be compared to Newton's development of Calculus. Many methods applied nowadays in Classical Mechanics, and differential equations in general, consist of refinements of Poincaré's ideas.

The notion of manifold is a generalization of two-dimensional surfaces, like a sphere and a torus, and is an important concept in Mathematics. There are several reasons to consider manifolds in the study of Mechanics:

(j) the differentiation depends only on local properties of the objects under consideration.

(jj) a constant of motion, say energy, in general restricts the particle motion to a surface.

(jjj) constraints also reduce the motion to surfaces; think of a pendulum in three-dimensional space whose mass position is restricted to the surface of a two-dimensional sphere. Another interesting example is a planar pendulum whose rod is pivot to another rod, as indicated in Figure 2; the positions of this mass are restricted to the surface of a two-dimensional torus. If you pivot n rods we get a systems with motion on the surface of a n-torus.

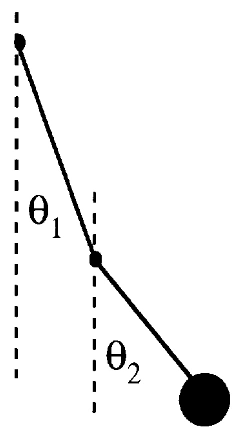

Figure 2. A pendulum with two pivoted rods; the positions of the mass are restricted to the surface of the Torus (θ_1, θ_2).

(jv) in many opportunities it is important to distinguish the local motion from the global one; for example, for short period of times the motion of a pendulum rotating with constant angular velocity is similar to the motion of a particle on a line with constant velocity, but the pendulum will return to its initial position, while on the line the particle *goes to infinity.*

Poincaré's geometric approach to differential equations originated another notion of stability, which gives the flavor of "global stability." Nowadays it is called structural stability and was proposed by Andronov e Pontriaguin in 1937. Denote by X a C^r-vector field on an open set M of $\mathbb{R}^n$ (think of a vector field as a field of forces); it is possible to introduce a precise notion of distance between two C^r-vector fields, but here we will restrict ourselves to the intuitive notion that two vector fields are C^r-close if they, and their respective derivatives up to order r, are close.

Definition 2 *The flow of the vector field X is the application $\varphi_t^X : M \to M$ such that $\varphi_t^X(x)$ is the unique solution at time t of the problem (also called orbits)*

$$\begin{cases} \frac{d\varphi(t)}{dt} = X(\varphi(t)) \\ \varphi(0) = x \end{cases}$$

Definition 3 *Two vector fields X and Y on M are topologicaly equivalent if there exists a homeomorphism $h : M \hookleftarrow$ such that*

$$h \circ \varphi_t^X = \varphi_t^Y \circ h,$$

for all t.

Recall that a homeomorphism is a continuous bijective application with continuous inverse. Notice that we can talk about vector fields or solutions of the corresponding differential equations. In case X and Y are topologicaly equivalent the above relation means that h can be thought of a change of coordinates, since it continuously maps orbits of X onto orbits of Y and vice-versa: $h(\varphi_t^X(x)) = \varphi_t^Y(h(x))$. Sometimes it is convenient to allow a change in the time scales, but we just ignore this possibility here.

Definition 4 *A vector field X on M is structurally stable if there exists a neighborhood V of X such that all vector field $Y \in V$ is topologicaly equivalent to X.*

We see that a vector field X is structurally stable if one adds a small perturbation and the structure of orbits is just slightly deformed. This gives a better idea of "global stability" and it becomes important to give conditions for a mechanical system to be structurally stable; notice that in this case the perturbations must be restricted to those admissible in Classical Mechanics, for example, they must be volume preserving. Is the solar system structurally stable?

IV Hamilton's Equations and Poincaré Stability

In the previous section we have stressed the importance of the concept of manifold in theoretical Mechanics, but here we consider only the particular case of $\mathbb{R}^n$. As already anticipated, there are alternative formulations to Newton's equations of Mechanics, and now we briefly present the Hamiltonian equations.

Given a differentiable function $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $H(q,p)$, the so-called Hamiltonian function (as usual in Mechanics $q = (q_1, \cdots, q_N)$ represents the positions of the particles and $p = (p_1, \cdots, p_N)$ the corresponding momenta; the set of all admissible (q,p) is called *phase space*), the Hamiltonian equations are given by

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}. \tag{4.1}$$

In case q and p are vectors in $\mathbb{R}^N$ one should think of (4.1) as a system of equations, one for each pair (q_j, p_j), $j = 1, \cdots, N$. The presence of the negative sign in just one of the equations has important consequences and has led to the development of Sympletic Geometry; see below for some consequences. The Hamiltonian function generates a vector field and we shall denote the corresponding flow by φ_t^H. Thoroughly we suppose the solutions of (4.1) are defined for all $t \in \mathbb{R}$.

One can recover Newton's equation for particles in $\mathbb{R}^N$, with masses m_j under a potential $U(q)$, by taking the particular form of the Hamilton's function

$$H(q,p) = \sum_j \frac{p_j^2}{2m_j} + U(q). \tag{4.2}$$

Just write out the Hamiltonian equations for this choice of H, recall that in cartesian coordinates $p = mv = mdq/dt$, the force field is given by $F(q) = -\nabla U(q)$, and you will get Newton's Second Law. From some points of view the Hamiltonian formulation generalizes Newton's formulation of Mechanics, since the function H does not need to be necessarily related to mechanical systems. This permits that some general results in Mechanics can been applied to other research fields, as they do indeed.

The mechanical systems are related to particular cases of differential equations and so they are expected to present special properties. The Hamiltonian formulation is very adequate to show up important particularities of the differential equations that model mechanical systems, and now we underline some of them.

P1. The very Hamiltonian function H is a constant of motion, in fact it is regarded the total mechanical energy of the system, as it is evident from (4.2). This is readily seen by differentiation along orbits and using (4.1)

$$\frac{dH(q,p)}{dt} = \frac{\partial H(q,p)}{\partial q}\dot{q} + \frac{\partial H(q,p)}{\partial p}\dot{p} = \frac{\partial H(q,p)}{\partial q}\frac{\partial H(q,p)}{\partial p} - \frac{\partial H(q,p)}{\partial p}\frac{\partial H(q,p)}{\partial q} = 0.$$

This holds because we have assumed the Hamiltonian function does not depend explicitly on time (i.e., conservative systems), otherwise we had got $dH/dt = \partial H/\partial t$.

P2. A result of Liouville says that the Hamiltonian flow φ_t^H generated by (4.1) preserves volumes in phase space: if one starts with a (measurable) region Ω in the space (q,p) with volume (Lebesgue measure) $V(\Omega)$ at initial time t_0, then this volume is maintained under time evolution, i.e.,

$$V(\varphi_t^H(\Omega)) = V(\Omega), \text{ for any } t.$$

P3. We have at our disposal a very important way to change coordinates in phase space (q,p), the so-called canonical transformations. Given a differentiable function, say $F_2(q,P)$, where P denotes the new momenta, one gets the new position coordinates Q and old momenta from the relations

$$Q = \frac{\partial F_2}{\partial P} \text{ and } p = \frac{\partial F_2}{\partial q}.$$

For functions $F_1(q,Q)$, $F_3(p,Q)$ and $F_4(p,P)$, all of them called generating functions of the canonical transformations, one obtains similar (not identical!) relations. The main advantages of considering canonical transformations are that they preserve the Hamiltonian form of the equations and the Hamiltonian in the new variables $K(Q,P)$ is readily obtained by direct substitution $K(Q,P) = H(q(Q,P), p(Q,P))$. Summing up

$$\dot{Q} = \frac{\partial K}{\partial P}, \quad \dot{P} = -\frac{\partial K}{\partial Q}.$$

As one example consider $F_2(q,P) = qP$, so that $Q = q$ and $p = P$, i.e., this is the generating function of the identity seen as a canonical transformation. Another simple and interesting example is provided by the generating function $F_1(q,Q) = qQ$; in this case one obtains $p = Q$ and $P = -q$ and, up to a sign, the roles of position and momentum are interchanged.

P4. This property is, in fact, a corollary of P2. Since the Hamiltonian temporal evolution does not change volumes in phase space, its equilibrium points are neither nodes nor repellors. Therefore the unique kind of stable equilibrium point for Hamiltonian systems are elliptic points (see Figure 3; like the phase space of the harmonic oscillator near the equilibrium). Another consequence of P2 is that no solution of Hamiltonian systems can be Lyapunov asymptotically stable, since this property requires shrinking of volumes in phase space.

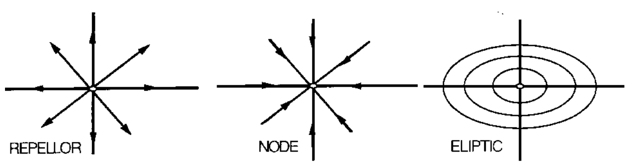

Figure 3. Three kinds of equilibrium points in the plane: each one is denoted by o.

If one combines the above mentioned result of Liouville with another one by Poincaré one gets a weak notion of stability. Although this is not usually referred to as stability we shall call it Poincaré stability. It is a typical result of ergodic theory and, in our case, can be stated as follows.

Theorem 1 (Poincaré's Recurrence) *Suppose the motion of a mechanical system is restricted to a region Ω in phase space, with finite volume $V(\Omega)$. For each (measurable) set $A \subset \Omega$, with volume $V(A)$, the return set*

$$R_A = \{(q,p) \in A | \exists (t_j)_{j=1}^{\infty}$$

with

$$t_j \to \infty$$

such that

$$\varphi_{t_j}^H(q,p) \in A\}$$

(is measurable and) is such that $V(A) = V(R_A)$.

In other words, the set of points that starts in A and does not return to A an infinite number of times under the Hamiltonian time evolution has null volume! It says, in some sense, that most points in phase space generalize the behavior of a periodic orbit that actually returns to its initial position. That is the Poincaré stability. Unfortunately it is very difficult to control the return times and it is known that in some cases, for example in systems described by Statistical Mechanics (large number of particles), this time can be of the order of the age of the universe.

V Integrability

As the very term indicates, integrability refers to sufficient conditions to obtain rather explicit solutions of some mechanical systems. Let's start with an one d.f. system (a system that can be described by just one position coordinate q) in cartesian coordinates. Since the energy value E is a constant of motion the system can be restricted to the energy surface $H^{-1}(E)$. At this point it is enlightening to see geometrically the rather simplicity of this system; think of the Harmonic oscillator, for example, or any other system whose phase space can be described by an elliptic equilibrium point (see the third picture in Figure 3); each curve is determined by the energy value and an angle determines a unique point on this curve, so the state of the system could be described by the energy and an angle. It is possible to get an analytical version of this geometric description by calculating the energy E from the initial conditions and taking into account that $p = m dq/dt$, so that

$$E = H(q,p) = \frac{p^2}{2m} + U(q) = \frac{m}{2}(\frac{dq(t)}{dt})^2 + U(q),$$

and we have got a separable differential equation for the solution $q(t)$, which, in principle, can be integrated by elementary methods. For example, it is enough to simplify this equation to get

$$t - t_0 = \pm\sqrt{\frac{m}{2}} \int_{q_0}^{q} \frac{d\xi}{\sqrt{E - U(\xi)}}$$

with q_0 being the initial condition at time t_0 and take the appropriate choice of sign. It is just a matter of inverting a function to get the solution $q(t)$, and one says the mechanical problem has been integrated by quadratures and it will be called "integrable."

For systems with $1 < N$ d.f. the natural generalization of the above procedure is obtained by requiring that there are at least N independent constants of motion $\{K_j = K_j(q,p)\}_{j=1}^N$. The relevant result here is due to Liouville that requires the N (differentiable) constants of motion are in involution, i.e., their mutual Poisson brackets $\{\cdot,\cdot\}$ vanish

$$\{K_i, K_j\} \equiv \sum_{l=1}^{N} \left(\frac{\partial K_i}{\partial q_l}\frac{\partial K_j}{\partial p_l} - \frac{\partial K_i}{\partial p_l}\frac{\partial K_j}{\partial q_l}\right) = 0,$$

for any pair (i,j).

Theorem 2 *Let $H(q,p)$ be the Hamiltonian function of an N d.f. system with N differentiable constants of motion $\{K_j\}_{j=1}^N$ whose Poisson brackets vanish $\{K_i, K_j\} = 0$ for any pair (i,j) and are independent, i.e., at each point of phase space the set of N gradient vector fields ∇K_j is linearly independent. Then, this system can be integrated by quadratures.*

For Hamiltonian systems with just 1 d.f. the phase space has dimension 2 and the conservation of energy reduces the motion to curves ("surfaces of dimension 1"). Notice that each independent constant of motion reduces by one unit the dimension of the surface in which the motion takes place, but for $N \geq 2$ the presence of N independent constants of motion restricts the orbits to a surface of dimension N (recall that the phase space has dimension $2N$), and the involution condition is required to "separate" individual d.f. and get explicitly the solution curve.

Definition 5 *A Hamiltonian systems that satisfies the hypotheses of Theorem 2 is called completely integrable system (for brevity, integrable system).*

Theorem 3 *Consider an integrable Hamiltonian system with N d.f. and let $\{K_j\}_{j=1}^N$ be the relevant constants of motion. Then, there is a canonical transformation such that the new momenta are given by $P_j = K_j$, $j = 1, \cdots, N$, and the new Hamiltonian function has the form*

$$H = H(K_1, \cdots, K_N)$$

(i.e., H does not depend on the new positions Q_j).

Therefore, for an integrable Hamiltonian system it is possible to find canonical coordinates $(Q_j, P_j)_{j=1}^N$ such that

$$P_j = const. \quad \text{and} \quad \dot{Q}_j = \omega_j \equiv \frac{\partial H}{\partial K_j} = const. \Longrightarrow Q_j(t) = \omega_j t + Q_j(0), \quad j = 1, \cdots, N.$$

By using topological arguments, Arnold and Jost have revealed the structure of phase space of integrable systems since they showed, in case the surface defined by $K_1, \cdots, K_N$ is compact and connected (this happens very often), such surfaces are N-dimensional tori and the above momenta P_j select a particular torus while the coordinates Q_j are the angles describing the points on such torus. In this case the phase space can be decomposed in layers of a "toroidal onion" and each torus is invariant by the Hamiltonian flow (see Figure 4). This is an important instance of restriction of motion to manifolds in phase space. Think again of the particular case of the harmonic oscillator in phase space, the energy selects a unique elliptic curve (a one-dimensional torus) and the angle coordinate describes the oscillations around this curve. The integrable motion on each torus in phase space consists of just compositions of rotations (i.e., linear flows on the torus) and is, therefore, to be considered synonymous of very simple motion.

There are several examples of completely integrable Hamiltonian systems, including the noticeable 1 d.f. systems discussed above, the problem of two bodies under mutual gravitational attraction and the Toda lattice. In fact, most traditional textbooks on Classical

Mechanics are concerned exclusively to integrable systems since their motions are relatively simple as the above theorem shows; however, such approach may give the *false* suspicion that most Hamiltonian systems are integrable and it is just a matter of technical skills to find the constants of motion (maybe from symmetries) and carry out the integration of Hamiltonian equations.

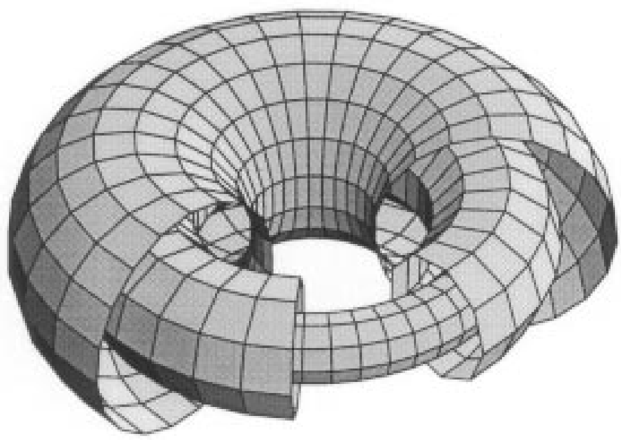

Figure 4. Scheme of the tori in an integrable 2 d.f. phase space.

The primary importance of integrable systems is that they are, in principle, soluble; fortunately there are important integrable Hamiltonian systems. Before the contributions of Poincaré it was usual to think that each problem in mechanics would be solved by transforming it into an integrable system, and the well-known Hamilton-Jacobi equation was developed for this purpose. An interesting physical oriented discussion and examples can be found in Chapter 3 of [10].

Is the set of completely integrable Hamiltonian systems structurally stable (in the world of Hamiltonian systems)? In other words, are sufficiently small perturbations of a completely integrable system also completely integrable?

Is the solar system completely integrable? This basic question has attracted the attention of many important scientists and even though no exact answer is known yet, there are many evidences supporting a negative answer. For example, even the general case of just three gravitational bodies is known to be nonintegrable.

Until the first half of the XX century there were two opposite expectations about the answers of these questions. Taking into account that the problem of two gravitational bodies is integrable (even Newton had got a complete solution) and the long distances among bodies in the solar system, it was expected that small perturbations of integrable systems should continue to be integrable, and therefore the solar system would be "stable." On the other hand, investigations on the foundations of Statistical Mechanics had led to the expectation that an arbitrary small perturbation of integrable systems would bring their orbits "covering all energy surface" so that only the energy would remain a constant of motion and the systems could not be completely integrable; this was the well-known ergodic hypothesis and would justify the use of the microcanonical ensemble. In this respect an important numerical work (one of the first numerical works in Physics) became famous as the Fermi-Pasta-Ulam experiment [6]; it consisted of the numerical integration of the equations of motion for a classical one-dimensional harmonic chain slightly perturbed by nonlinear forces, and the ergodicity was not found.

An "almost complete" answer to the question of perturbation of completely integrable systems appeared in the fifties with the celebrated KAM theorem.

VI KAM Stability

To begin with we summarize, in just one phrase, the main conclusion of the KAM theorem we are going to discuss:

— *most tori are just deformed and survive under the perturbation of integrable systems.*

It has become so common the citation of KAM theorem in texts on "chaos theory" that many students have been misled by supposing it is a result of instability, so we have stressed the stability meaning of this important set of results.

Consider an integrable system with N d.f. (e.g., two heavy bodies under gravitational interaction) described by appropriate coordinates (see last section) $K = (K_1, \cdots, K_N)$ with the Hamiltonian function $H_0 = H_0(K)$ and perturb it by $\varepsilon H_1(Q, K)$, where $0 < \varepsilon \ll 1$, so that the perturbed system is described by

$$H(Q, K) = H_0(K) + \varepsilon H_1(Q, K).$$

The main problem is to find a canonical transformation to new coordinates $(\tilde{Q}, \tilde{K})$ such that the new Hamiltonian function $\tilde{H}$ depends only on the new momenta $\tilde{K}$

$$H(Q(\tilde{Q}, \tilde{K}), K(\tilde{Q}, \tilde{K})) = \tilde{H}(\tilde{K}),$$

so that the perturbed system is also completely integrable. For simplicity we assume the unperturbed motion is confined to compacted and connected surfaces (so on tori, as we already know).

Usually one looks for a generating function $S(Q, \tilde{K}) = F_2(Q, \tilde{K})$ in power series of ε

$$S(Q, \tilde{K}) = Q\tilde{K} + \varepsilon S_1(Q, \tilde{K}) + \varepsilon^2 S_2(Q, \tilde{K}) + \cdots \quad (6.3)$$

Notice that the zeroth order term is the generating function of the identity canonical transformation, since for $\varepsilon = 0$ the unperturbed variables (Q, K) works. Taking into account that the variables Q are angles on the tori, one can use Fourier series

$$H_1(Q, K) = \sum_m H_{1m}(K) e^{imQ}, \quad S_1(Q, \tilde{K}) = \sum_m S_{1m}(\tilde{K}) e^{imQ}$$

so that (after some algebra)

$$S(Q, \tilde{K}) = Q\tilde{K} + i\varepsilon \sum_{m \neq 0} \frac{H_{1m}(\tilde{K})}{m.\omega_0(\tilde{K})} e^{imQ} + \cdots$$

where $m = (m_1, \cdots, m_N)$ are vectors of integer numbers, $\omega_0(\tilde{K}) \equiv \nabla_{\tilde{K}} H_0(\tilde{K}) = (\omega_{01}, \cdots, \omega_{0N})$ being the frequencies of the unperturbed tori.

The problem of integrability of the perturbed system has been reduced to the convergence of the series (6.3). For concreteness assume that we have no special symmetry so that all $H_{1m} \neq 0$. Besides the convergence of the series in ε one must control the scalar products $m.\omega_0 = m_1\omega_{01} + \cdots + m_N\omega_{0N}$ that appear in the denominators. Here, again, we emphasize some important details:

(I) if the frequencies $\omega_0 = (\omega_{01}, \cdots, \omega_{0N})$ are (rationally) commensurable then $m.\omega_0 = 0$ for an infinite number of values of the vector m and the series does not converge (this situation is usually called *resonant*).

(II) even if $(\omega_{01}, \cdots, \omega_{0N})$ is incommensurable the quantities $m.\omega_0$ never vanish but are arbitrary close to zero, since vectors of rational numbers are dense in $\mathbb{R}^N$. This property questions the convergence of (6.3) also for incommensurable frequencies: this is the well-known problem of "small divisors" that abounds Celestial Mechanics and recently has also permeated Quantum Mechanics, Spectral Theory, Partial Differential Equations, Nash-Moser implicit function theorems, etc.

(III) there is no hope of proving the convergence of (6.3) without connecting the speed $m.\omega_0 \to 0$ with the speed $H_{1m} \to 0$ for $m \to \infty$, a property related to differential properties of the Hamiltonian function. The coefficients H_{1m} are expected to control $m.\omega_0$ for $m \to \infty$.

In the fifties A. N. Kolmogorov (see the Appendix of [1]) proposed a method to proving the convergence of (6.3) for some values of the unperturbed frequencies ω_0. This method was based on Newton's method to find the root of nonlinear functions and the details was implemented independently by Arnold and Moser. In the first proofs Arnold supposed the Hamiltonian functions were analytic and Moser asked for class C^{333} for systems with two d.f.! Currently it is know that this theorem holds for class C^3 and there are counterexamples in class $C^{3-\alpha}$ due to Herman [11]. This result and its ramifications are know as KAM technique. Let's enunciate the KAM theorem for analytic Hamiltonians with two d.f. without some technical details in order to avoid cumbersome statements; e.g., it should be required that the frequencies of the unperturbed system are not degenerate $\det(\partial\omega_0/\partial K_j) = \det(\partial^2 H_0/\partial K_j \partial K_i) \neq 0$, because in the proof the tori are specified by their frequencies, a condition not satisfied for the harmonic oscillator! Let's mention that in [19, 4] the authors tried a pedagogical approach to the proof of KAM theorem.

Theorem 4 (KAM Theorem) *Let $\omega_0 = (\omega_{01}, \omega_{02})$ be incommensurate (i.e., $\alpha \equiv \omega_{01}/\omega_{02}$ is an irrational number). Then, for $0 < \varepsilon$ sufficiently small there exists $C(\varepsilon) > 0$ with $\lim_{\varepsilon \to 0} C(\varepsilon) = 0$ such that if*

$$|\frac{\omega_{01}}{\omega_{02}} - \frac{r}{s}| > \frac{C(\varepsilon)}{s^{2.5}} \text{ for all } r, s \in \mathbb{N}, s \neq 0, \qquad (6.4)$$

the series (6.3) converges. Furthermore, the Lebesgue measure of the numbers α (in bounded sets) that do not satisfy (6.4) goes to zero as $\varepsilon \to 0$.

Condition (6.4) is a particular specification of the irrational character of α and it says that the unperturbed tori whose ratio of frequencies are sufficiently far from rational numbers are preserved after the perturbation of an integrable system. It is a stability result! Notice that the KAM theorem deals with the behavior of orbits for all times; there are interesting estimates for finite times due to Nekhoroshev [16] for some analytic Hamiltonians.

Unfortunately the KAM theory says nothing for the remaining tori and there is room for speculations about chaotic behavior and pure numerical works. In fact, the numerical works have revealed many features not direct accessible to theorems yet. Notice that the set of numbers α that satisfies (6.4) and the one that doesn't are both dense in $\mathbb{R}$; this combination of results has shown that Hamiltonian systems in general present a complicated mixture of simple motion (represented by the reminiscent tori) with complicated motions in the regions not embraced by KAM theory. Moser has compared this mixture in phase space with a sponge; its solid part consisting of the reminiscent tori with bounded motions while the solutions in the holes of the sponge being capable of complicated behavior.

What about the solar system? We know that although the problem of just two gravitational bodies is integrable the perturbation by other bodies has important effects; for example, the Earth elliptic trajectory rotates at the ratio of $0.3°$ each century and certainly the present trajectories of the satellites are very different from their trajectories in the past.

In 1866 Kirkwood observed gaps in the asteroids distribution between Mars and Jupiter as function of their distance to the Sun. In the last decades such gaps have been directly related to the KAM theory in some approximations (they reside in the region of frequencies the KAM theorem does not hold).

Numerical simulations of the solar system have been useful but not very helpful (yet!) as far as the stability problem is concerned. For example, by using a specifically designed machine called Digital Orrery, a group from Massachusetts Institute of Technology (in Boston, U.S.A.) has numerically integrated the orbits of the planets of the solar system around 10^7 years (about 20 percent of the age of the Solar System), and just small variations in the energy of the outer planets have been verified.

VII Final Remarks

An interesting phenomenon may occur for systems with $3 \leq N$ d.f., which dramatizes yet again the nature of Classical Mechanics, the so-called Arnold diffusion. It is a consequence of different dimensions; in case of 2 d.f. a two-dimensional torus separates the three-dimensional energy surface into two connected components, while for $N \geq 3$ an N-dimensional torus does not separate the $(2N-1)-$dimensional energy surface into two distinct components (it is like curves in three-dimensional spaces). Therefore, in case of three or more d.f. an orbit that starts in a region where the tori were destroyed may wander through other regions of the energy surface. This possibility is still not well investigated but it is expected to occur very slowly in time. For examples exhibiting Arnold diffusion see [5].

For hundreds of years the solar system, governed by the inverse square law of force, has been considered the paradigm of order and predictability, which was reinforced by Newton's solutions and recovery of Kepler's laws, but today the best answer for the question of its stability, in a very general sense, is "we do not know." Maybe the appropriate formulation of this stability question should include the possibility of the trajectories of planets in the solar system be capable of Arnold diffusion.

Now we point up some important research topics in theoretical Classical Mechanics:

- What kind of motion does occur when a torus is destroyed? Significant contributions have been given by Aubry and Mather [7].

- Finding estimates for the perturbation intensity assuring all tori have been destroyed.
- A deeper understanding of Arnold diffusion.
- Building a semiclassical theory of nonintegrable Hamiltonian systems. This is part of a subject called Quantum Chaos. The inappropriate quantization rules of "old Quantum Mechanics" for nonintegrable systems was realized by Einstein in 1917.
- Existence of periodic orbits in Hamiltonian systems and presenting lower bounds for the number of such orbits [20]. This point has also relations to Quantum Chaos.
- Looking for weak conditions for the validity of KAM theorem; here we cite the contributions of Rüsmann [18] and Gallavotti [9].
- It was numerically found that the stabilization of Earth's obliquity [13], i.e., there is a large chaotic zone in obliquity avoided by the Earth, is due to the presence of the Moon. Its obliquity exhibits only small variations of $\pm 1.3°$ around the mean value of $23.3°$, while Mars's obliquity was found to follow a chaotic orbit. This is also related to the ice ages and asks for a rigorous treatment and a precise theorem.

To finish I would like to stress that we would hardly know so much about gravitation and Classical Mechanics if our solar system were composed by a binary star (with comparable masses)! Think of that...

Acknowledgments

This work was partially supported by CNPq (Brazil).

References

[1] Abraham, R. and Marsden, J. E., *Foundations of Mechanics*, 1978 (Benjamin, Reading, Massachusetts).

[2] Arnold, V. I., *Mathematical Methods of Classical Mechanics*, 1978 (Springer-Verlag, New York).

[3] Arnold, V. I. and A. Avez, *Ergodic Problems of Classical Mechanics*, 1968 (Benjamin, Reading, Massachusetts).

[4] Benettin, G., L. Galgani, A. Giorgilli and J.-M. Strelcyn, Nuovo Cimento **B79**, (1984) 201–223.

[5] Douady, R. and P. Le Calvez, C. R. Acad. Sc. Paris Ser. I **296**, (1983) 895–989.

[6] Ford, J., Phys. Rep. **213**, (1992) 271–310.

[7] Forni, G. and J. N. Mather, *Action minimizing orbits in Hamiltonian Systems*, in Lect. Notes Math. **1589**, 1994, 92–186 (Springer-Verlag, New York).

[8] Gallavotti, G. *the Elements of Mechanics*, 1983 (Springer-Verlag, New York).

[9] Gallavotti, G., Rev. Math. Phys. **6**, (1994) 343–411.

[10] Gutzwiller, M. C., *Chaos in Classical and Quantum Mechanics*, 1990 (Springer-Verlag, New York).

[11] Hermann, M.-R., Astérisque **144**, (1986) 1–248.

[12] Kozlov, K. K., Russ. Math. Survey **38**, (1983) 1–76.

[13] Laskar, J., F. Joutel and P. Robutel, Nature **361**, (1993) 615–617.

[14] Moser, J., *Stable and Random Motion in Dynamical Systems*, 1973 (Princeton Univ. Press, Princeton NJ).

[15] Moser, J., Math. Intell. **1**, (1978) 65–71.

[16] Nekhoroshev, N. N., Russ. Math. Survey **32**, (1977) 1–67.

[17] Ozorio de Almeida, A. M., *Hamiltonian Systems, Chaos and Quantization*, 1988 (Cambridge Univ. Press, Cambridge).

[18] Rüssmann, H., Math. Appl. **59**, (1990) 211–223.

[19] Wreszinski, W. F., *Mecânica Clássica Moderna*, 1997 (Edusp, São Paulo).

[20] Zehnder, E., *Periodic solutions of Hamiltonian equations*, Lect. Notes Math. **1031**, 1983, 172–213 (Springer-Verlag, New York).