Learning About Classical Limit from an Integrable Two-Spin System

G. Q. Pellegrino*
Division de Physique Théorique†
Institut de Physique Nucléaire
91406 Orsay Cedex, France

K. Furuya
Instituto de Física "Gleb Wataghin"
Universidade Estadual de Campinas
13083-970 Campinas - SP, Brasil

M. C. Nemes
Departamento de Física - ICEx
Universidade Federal de Minas Gerais
31270-901 Belo Horizonte - MG, Brasil

Recebido 20 de agosto, 1997

It is shown how the simple two-spin system, viewed as a possible paradigm to study the properties of generic classical integrable systems with two degrees of freedom, has quantum characteristics that make it suitable to illustrate those properties also in the quantum domain. We concentrate on quantum mechanical results and explore these characteristics to solve the problem, to point out how this system permits a controllable approximation to the classical limit, and to clarify some misleading associations concerning classical limit, quantum numbers, and energy levels.

1. Introduction

In the last decades, the field of Hamiltonian systems has known a burst of research concentrated mainly on the possibility of studying chaotic dynamics in the classical as well as quantum contexts. Nowadays, much of the physical properties of those systems is known thanks to the mathematical and numerical apparatus developed in the course of those years and, despite of their sophistication and complexity, the concepts, results, language and techniques related to this search become of common use in many branches of physics [1].

On more basic grounds, however, it is still difficult to illustrate the physical properties of Hamiltonian systems related either to classical phase space or to quantum spectra and eigenfunctions. In the classical context this is so because, in general, the phase space of any non-trivial systems has too many dimensions and the related mathematical expressions are so complicated functions of the phase space coordinates that not one of them can be eliminated to allow the generation of informative pictures. Gade and Gade III, [2] called attention to this difficulty and claimed that the integrable two-spin system could be viewed as a paradigm to visualize and understand the behaviour and properties of generic classical integrable systems, which are, by their turn, the point of departure for studying chaos. Listing the good features of that spin system for such a classical study, they show how one can observe its winding numbers, the foliation of phase space by rational and

*Present address: Departamento de Matemática, UFSCar, C.P. 676, 13560-905 São Carlos, SP, Brasil
†Unité de recherche des Universités de Paris XI et Paris VI associée au CNRS.

irrational tori, and the dense coverage of irrational tori by quasiperiodic orbits, all this coming up from uncomplicated mathematical expressions and easily applicable graphical procedures. Before that, spin system had already been extensively studied, special attention then being paid to the presentation of results of original research instead of their pedagogical qualities [3] (see, however, the series of articles referenced in [4]). Nevertheless, we can see in these studies the suggestion that much can be learned from spin systems, not only about integrable but also about chaotic classical dynamics.

Some of the difficulties found in the quantum mechanical context are related to the approximation to the classical limit. On the one hand, we deal in most of the cases with Hamiltonians characterized by infinite matrices and the (numerical) resolution of the eigenvalue problem is attained only after the truncation of the Hamiltonian matrix in question. This truncation poses problems related to the convergence of the numerical methods and also to the choice of what part of the matrix should be neglected and what should not. On the other hand, we generally take for granted that we should look at the highest quantum numbers to get the closest to the classical limit. In terms of the spectra, the higher the energy levels are considered the better should be our approximation.

Here we would like to show that the very two-spin system proposed by Gade and Gade III, in its quantum version, has the necessary quantum-mechanical features which make it suitable for elucidating also those points. In our case, we concentrate on quantum-mechanical results and explore the system's features to show that they allow for a controlled approximation to the classical limit - without any truncation of the Hamiltonian matrix - and also to call attention to the fact that some care must be taken in the use of the highest-energy-level prescription.

To accomplish our purposes, we present in section II the quantum version of the two-spin system, together with some of its symmetry properties which allow for a simple and transparent connection with the classical limit, explored in section III. Some concluding remarks are presented in the last section IV.

It should be emphasized that, while being the main object in this work, the two-spin system is not in itself the ultimate objective we are here interested in. It is taken more as a convenient model to learn about general properties of the classical limit. Besides the intrinsic characteristics presented above, this convenience is revealed by the possibility of its implementation as a pedagogical tool. As we will see in sections II and III, the properties of the model can be seen in practice through a numerical study that can be pursued by students with no unreasonable effort.

II. The quantum two-spin model

The quantum version of the integrable two-spin system, widely known as the anisotropic Heisenberg model, is given by the Hamiltonian

$$H = \mathbf{S}_1 \cdot \mathbf{S}_2 + \sigma S_1^z S_2^z, \tag{1}$$

where the anisotropy parameter σ is a real number and S_i^x, S_i^y, $S_i^z (i = 1, 2)$ correspond to the usual SU(2) spin operators. One can easily show that, besides the energy, this system has the z-component T^z of the total spin operator

$$\mathbf{T} = \mathbf{S}_1 + \mathbf{S}_2 \tag{2}$$

as a constant of motion:

$$[H, T^z] = 0. \tag{3}$$

We consider here the case $|\mathbf{S}_1| = |\mathbf{S}_2| = \mathbf{S}$ and diagonalize H using as basis states the (anti)symmetrized tensor product of spin states, defined as

$$|m_1 m_2 > \equiv \frac{1}{2!}[|S, m_1; S, m_2 > \pm |S, m_2; S, m_1 >] \tag{4}$$

where the +(-) sign corresponds to bosonic (fermionic) cases. Throughout the paper we take integer values for S (+ sign in eq. (4)), and work within the subspace where $< T^z > = 0$ which implies the relation $m_1 + m_2 = 0$ for m_1 and m_2, the quantum numbers corresponding to the z-projection of the spin operators $\mathbf{S}_1$ and $\mathbf{S}_2$ respectively [5]. The eigenvalue problem of the Hamiltonian matrix is then straightforwardly solved by any of the many numerical routines available, and one can easily obtain the system's energy spectra, examples of which are shown in Fig. 1 for spin size $S = 200$ and some values of the anisotropy parameter σ.

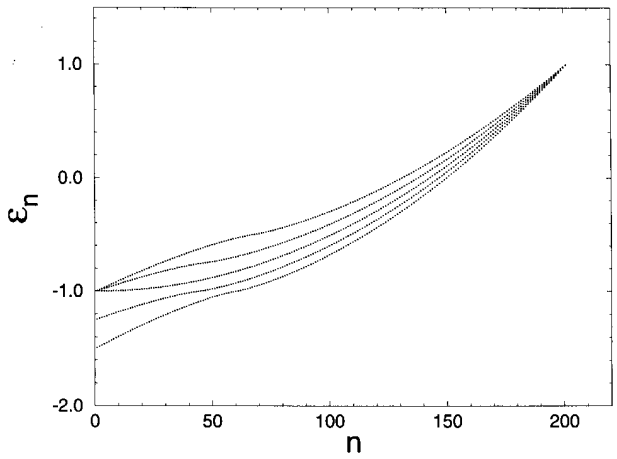

Figure 1. Energy spectra of the two-spin system for spin size $S = 200$ and some values of the anisotropy parameter: σ = - 0.5, -0.25,0.0,0.25,0.5 (the lowest spectra correspond to the greatest values of σ). We note here that all figures in the paper present the energies scaled by $S(S+1)$ (see item iii) in section II).

The spectra for this two-spin system have the following features to be noticed:

i) The Hilbert space of the system is finite. This feature comes from the SU(2) Lie algebra of spin operators and guarantees that the spectra are themselves finite. In fact, for the subspace given by $< T^z >= 0$ there are $S+1$ states $|m, -m >$, $m = 0, 1, ..., S$. Thus, we are here not concerned with truncation of the Hamiltonian matrices in order to suppress the upper and lower energy parts of the spectrum as is usually done, the ultimate constraint being nevertheless imposed by the computational system used, since the number of levels increases as $S + 1$. In addition to its finiteness, the spectra for this system are conveniently non-degenerate.

ii) System (1) and the Heisenberg models in general have a well defined classical limit, which is realized by taking $\hbar \to 0$ and $S \to \infty$ with $\hbar\sqrt{S(S+1)} = 1$ [6]. In practice, this limiting process can be observed through the large S scaling behaviour, characteristic of these systems and shown in Fig. 2: as the spin size tends to infinity, the quantum spectrum tends to its classical limit in the form of a limiting curve. It can be shown [7] that this curve is indeed the one obtained by evaluating the actions of the periodic orbits associated with the classical Hamiltonian for the model.

iii) The large S scaling behaviour and the prescription $\hbar\sqrt{S(S+1)} = 1$ for the classical limit permit us to set $\hbar = 1$ and scale the whole spectrum by $S(S+1)$, as we do throughout the paper. This scaling fixes the lower and upper extrema of the spectra at values independent of S, meaning that we can globally explore the entire spectrum without suppressing their low-energy part. By the same token we will see that, since the classical limit for this spin system is attained as S increases, controlling S allows for a control of the approximation to the classical limit for the spectrum as a whole. This could be qualitatively anticipated from Fig. 2 where we can see that, for a given value of σ, the number of levels increases with S, diminishing in this way the energy difference between neighbouring levels, the scaled energy range being now fixed. In the same spirit, it will be seen that the scaling of the spectra provides also a way to investigate and control the density of levels in a given region of the spectrum.

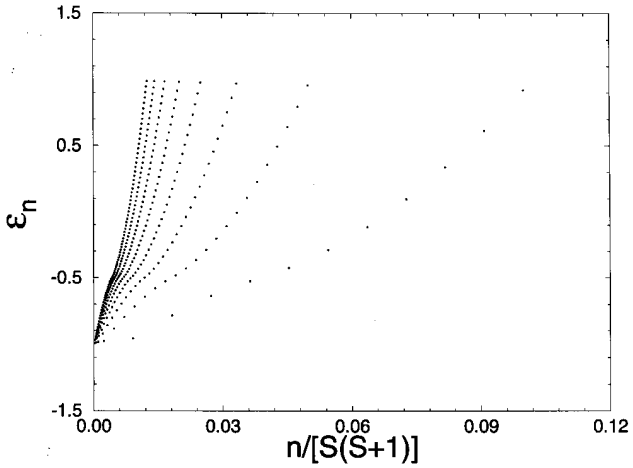

Figure 2. Energy spectra of the two-spin system for a single value of the anisotropy parameter, $\sigma = -0.5$, and several values of the spin size, $S = 10, 20, 30, ...80$.

The characteristics evidenced in ii) and iii) are obviously not completely general. They are, however, manifestations of a property common to a wide class of models, called Curie-Weiss type models, to which the Heisenberg systems belong [8]. Curie-Weiss models share the same kind of scaling behaviour for a certain parameter N (the spin size S in the case of the Heisenberg systems): in the large N limit, quantum fluctuations become negligible. Corrections to the semiclassical approximation can then be expressed as powers of $1/N$, showing that there is in fact a parameter which controls the limiting process [9].

The points presented in iii) will be the object of the next section where we explore the approximation to the classical limit for different values of σ.

III. The approximation to the classical limit

The control of the approximation to the classical limit by choosing the value of S means, in other words,

the choice of the Planck constant h. Qualitative at this level, this control can be made more quantitative and have a practical implementation after a simple manipulation of the energy levels present in a given spectrum. It suffices to take the (scaled) energy difference Δ_n between the $(n+1)$-th and n-th levels,

$$\Delta_n = \epsilon_{n+1} - \epsilon_n \tag{5}$$

and consider the plots $\Delta_n \times n$. Examples for the isotropic case ($\sigma = 0$) and an anisotropic case ($\sigma = -0.5$) for $S = 200$ are presented in Figs. 3 and 4.

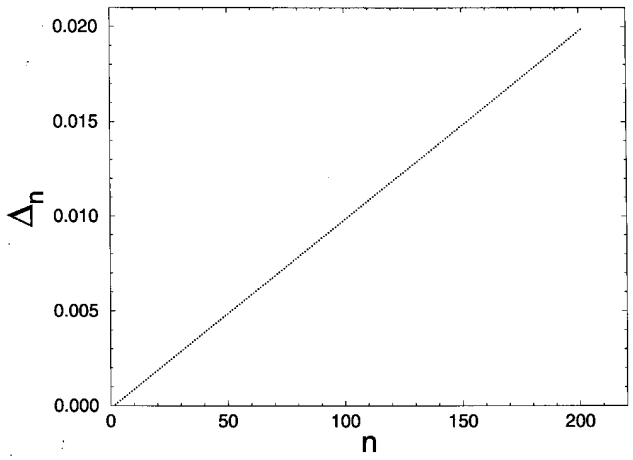

Figure 3. Energy level difference Δ_n versus the energy level index n for the isotropic case ($\sigma = 0.0$ and $S = 200$).

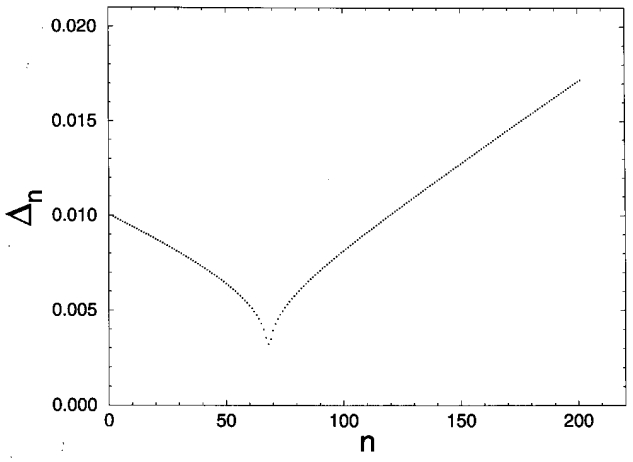

Figure 4. As in Fig. 3 here the anisotropic case ($\sigma = -0.5$ and $S = 200$) is illustrated.

The straight line appearing in Fig. 3 reveals a correlation between subsequent energy differences. In fact, numerical inspection of the spectrum in this case shows that

$$\Delta_{n+1} = \Delta_n + \Delta = \Delta_0 + n\delta, \tag{6}$$

where Δ_0 is a reference difference valid after a few first levels and δ depends only on $\hbar$ as

$$\delta = 4\hbar^2 \tag{7}$$

In this case the obtained constant δ gives $\hbar \approx 5 \times 10^{-3}$, confirming the scaling relation $\hbar\sqrt{S(S+1)} = 1$ for $S = 200$. Eqs. (6) and (7) can be easily inferred just by observing spectra obtained for different values of S with $\hbar$ held fixed, and comparing their energy differences with those for spectra obtained for different values of $\hbar$ with S held fixed.

These relations reveal a property that can be useful in numerical calculations where one has to simulate (semi)classical curves by what is in practice a discrete set of points. Using eqs. (6) and (7), by choosing a value for Δ sufficiently small as to prevent computational discrepancies with the classical results, one chooses in fact the Planck constant h, which can be accomplished by an appropriate value of S. This procedure should be followed for that part of the spectrum where Δ_n has its largest values: controlling this region we control the spectrum as a whole, since for smaller values of Δ_n the approximation would be still better. For the anisotropic case it can be seen from Fig. 4 that the correlation between subsequent energy differences is not so simple a relation as eq. (6) is for the isotropic case. Despite of the fact that in the anisotropic case this more complicated relation can still be found in principle, it is actually not necessary. As a matter of fact, for energy differences greater than 0.01 – this means, for that part of the spectrum where the differences are greater – Fig. 4 reproduces a straight line similar to that given by eq. (6). Hence, we can follow for this case the same procedure presented right above. In this sense, the isotropic case can be used to calibrate this checking of the value of Planck constant necessary to a good classical approximation.

Another point to be noticed is that, as expected, the energy difference between neighbouring levels is not constant for all levels. In these spin systems, however, it increases as the upper parts of the spectra are attained (see eq. (6)) and we could hardly point the highest energy levels as representative of the classical limit. In fact, in the isotropic case that would be the worst choice! The point is clarified by the plots $\Delta_n \times n$ themselves. Since Δ_n is an energy difference, for large values of S the plots reflect the density of levels per energy unity

$$\rho = \frac{1}{d\epsilon/dn} \tag{8}$$

which is higher at lower values of Δ_n. In this way, we can see that for the case $\sigma = 0$ ρ attains its maximum value at the lowest energy $\epsilon = -1.0$. For the anisotropic case $\sigma \neq 0$, it can be shown [10] that in the interval $-2.0 \leq \sigma \leq 0.0$, for example, the maximum value of ρ occurs at the energy

$$\epsilon = -1.0 + |\sigma|. \tag{9}$$

The last word about the classical approximation is always given by the limiting process $\hbar \to 0$. In practice, however, once we choose a value for $\hbar$, we see that in general the density of levels is not constant throughout the spectrum and, as we saw, its maximum value is not necessarily located at the upper extremum. Therefore, for a small *but fixed* value of $\hbar$, the best region of the spectrum to be taken as classical approximation should be the region where the density of levels is maximal. Thus we see that the energy levels corresponding to the region around the minimum of Δ_n (see Figs. 3 and 4) are in fact the good choice for representing the classical limit. In this sense, we can say that this region of higher density of levels meets classical characteristics before the other parts of the spectrum.

We finally recover what should be the correct rule: in looking for the classical limit we should look for the regions of the spectrum where the density of levels is maximal. Eq. (9) shows moreover that this maximum can be positioned anywhere between the extrema $\epsilon_{\min} = -1.0$ and $\epsilon_{\max} = 1.0$ for $-2.0 \leq \sigma \leq 0.0$ [11]. In few words, we can not freely associate the highest density of levels with the highest energies, or in general with the highest quantum numbers. This should be kept in mind whenever we are applying the highest-quantum-number prescription as it is presented in many textbooks on quantum mechanics. If it is true that this works in many practical cases, it is also true that it can be a conceptually misleading association.

IV. Concluding remarks

We have seen in this contribution that the simple two-spin system, viewed by Gade and Gade III as a possible paradigm in the study of generic classical integrable systems with two degrees of freedom, shows itself well suited to elucidate their properties also in the quantum domain. Shortly speaking, its quantum mechanical features are based on the discrete and finite SU(2) spin algebra and on the scaling behaviour of its classical limit. These two characteristics make the two-spin system an exactly solvable quantum system even in the classical limit - up to computational constraints - with energy spectra that at one time are easy to manipulate and provide instructive results.

We have concentrated our exposition on the classical limit showing how it can have a controllable approximation, a feature that gives us a feeling of what is going on with the levels as we get closer and closer to the limit. We also call attention to possible misconceptions concerning the traditional association between classical limit and high energy levels.

As a final remark, it is worth remembering that spin systems and the procedures here presented can be much further explored as a way to generate illustrations of other known results for Hamiltonian systems. It was shown [10] that the plots $\Delta_n \times n$ have an intuitive translation in terms of classical periodic orbits, the minimum of Δ_n marking the presence of a separatrix. Moreover, the simple manipulation leading to the differences Δ_n can be applied also to chaotic cases such as the anisotropic three-spin system [10], though in this case we must face the complications posed by the higher number of degrees of freedom and the degeneracies of the spectrum.

Acknowledgements

It is a pleasure to thank D. Alves for fruitful discussions and questions. This work was partially supported by the Brazilian agencies CAPES, FAPESP, CNPq and FINEP. G. Q. P. wishes to thank the hospitality of Instituto de Física "Gleb Wataghin" - Universidade Estadual de Campinas (Brasil), where the work was done, and Institut de Physique Nucléraire - Université Paris-Sud (France), where the paper was written.

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