# On the Energy Flow Vector of the E-M. Field

Onofre Rojo*
*PESTYC-IPN*
*México, D.F.*

Trabalho recebido em 19 de outubro de 1996

## I. Introduction

A renewed interest in the mathematical formalism and physical interpretation of the energy flow vector **S** of the electromagnetic field has emerged due to some apparently counter intuitive physical features of the orthodox interpretation of the Poynting vector and the Poynting flux [1].

Feynman [2] in his *Lectures on Physics II* analyzes three situations where these shortcomings are stressed. These are: (a) a constant current flowing through a wire, (b) a slowly-charging capacitor, and (c) a combination of non-parallel static electric and magnetic fields. Intuitively we expect in case (a) the energy flowing along the wire, in (b) the energy originating in the current charging the capacitor, and in (c) no energy flow. He calls these situations paradoxes; however as in all paradoxes a careful scrutiny of the issues will reveal their underlying meanings. Feynman hirmself, used to say that "a paradox is only a confusing in our understanding". The outcome is quite different when some other physical aspects of the current and energy flow vector are taken into consideration in particular the region space over which these magnitudes are considered.

Several authors trying to explain the obvious and observed Joule effect in a conductor with steady current have proposed alternative choices for the vector **S** , other than the standard definition $\mathbf{E}\times\mathbf{H}$ . However they fail to reproduce the (Larmor's) radiation formula for an accelerated charge [3]. More recently C.S. Lai [4] has proposed a truncated form of $\mathbf{E}\times\mathbf{H}$ with **E** defined through the vector potential that accounts for the Larmor radiation formula, but does not lead to a transparent interpretation of the equality of the rate of energy flow and the velocity of propagation of the energy carriers, neither follows directly from the relativistic theory of the electromagnetic field [1].

It has been emphasized [5] that in every physical situation the quantity which is physically significant is not **S** but rather the flux of **S** , however we will identify the Poynting vector with the energy flux at a given point in the field since this interpretation leads to many simple and obvious relationships, just mentioned, such as the equality of the rate of energy flow and the velocity of propagation of the energy carriers, and also to its deduction directly from the relativistic theory of the electromagnetic field [6].

We will examine cases (a), (b) and (c) proposed by Feynman, and it will be shown that one needs not depart from the classical and usual interpretation of the Poynting's vector in order to explain then in simple and physically clear terms: the Joule effect in a conductor with a steady current is due to the work of electromotive forces of non electrostatic origin, and the apparently meaningless continuos circulation of energy along closed paths in a static electromagnetic field acquires physical meaning if conservation of angular momentum is assumed and the hidden mechanical momentum is taken into account.

### Case (a)

In is discussion of case (a) Feymman ignores that the wire through which is flowing a constant current is part of a circuit that must be closed or pass away to infinity. In order to maintain a steady current in it is necessary that electromotwe forces of a non-electrostatic origin act in definite sections of a current

*Unidad Profesional Culhuacan, Edif. 2 - 3er. piso, Av. Santa Ana 1000, Col. San Francisco Culhuacan, Mexico, D.F. C.P. 04430, Fax: 607 64 80

circuit. It is their work that compensates the expenditure of electric energy liberated as Joule heat. In the same manner that when dealing with a system of electric charges, Earnsshaw's theorem stablishes that purely electrostatic systems cannot be stable and that in order to ensure the required stability (a real fact) we must introduce constraints of steady currents obliges us to assume that apart from electric forces of a stationary electric field a certain field of forces of non-electrostatic origin may also act on the electric charges in conductors. This forces will be called "extraneous" and its field denoted by $\mathbf{E_{ext}}$ . In the field of steady currents the intensity of an electromagnetic field and consequently its energy remains constant so that the work ofthe extraneous e.m.fs completely transforms into heat. This work is done, however, only in the portions of the circuit where $\mathbf{E_{ext}}$ differs from zero, whereas the joule heat is liberated in all the portions of the circuit. (Ref. 6, pag. 465), Hence the energy spent by the sources of extraneous e.m.f. flows to where it is consumed as electromagnetic energy.

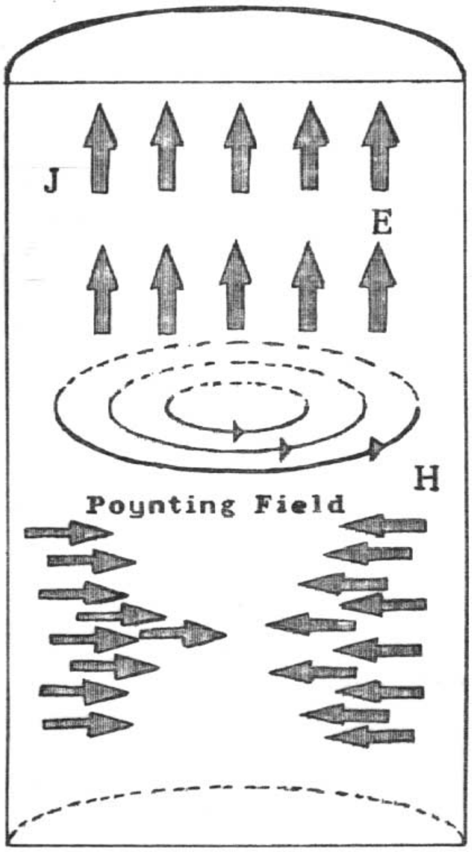


Figure 1. Poynting field for a cylindrical homogeneous conductor, confined between two cross sections to its axis.

To analyze this case (a) consider a segment of a cylindrical conductor of length 1 and cross-sectional area $\pi \mathbf{r}^2$ with current density $\mathbf{j}$ . (Fig. 1). Assume that the magnetic field coincides with the field of an infinite straight current having the same intensity the magnetic lines of force being circles concentric to the direction of the current, whose magnitude on the surface of the conductor is

$$H = \frac{2\pi\ rj}{c}\ .$$

First, assume $\mathbf{E_{ext}} = 0$ in the portion of the conductor being considered, then $\mathbf{E}$ points in the same direction as the current $\mathbf{j}$ and is equal to

$$\mathbf{E} = \frac{\mathbf{j}}{\sigma}$$

$\sigma$, being the conductivity. Since the vectors $\mathbf{j}$ and $\mathbf{H}$ are perpendicular, $\mathbf{S}$ is directed inward, normal to the surface of the conductor its magnitude given by

$$|S| = \frac{j^2 r}{2\sigma}$$

Hence, into the surface of the conductor, flows the energy

$$\int S_n da = \frac{j^2}{\sigma} V$$

from the sorrounding space, where $V$ is the volume of the portion being considered. This energy, coming from the electromagnetic field is liberated in this segment of the conductor as Joule heat. It flows into this region of space from those portions of the conductor in which work is done. To help the current flowing steadily, as supposed in our example, we must modify the original assumption and introduce an "extraneous" electromagnetic force such that $\mathbf{E_{ext}} \neq 0$. Then,

$$\mathbf{E} = \frac{\mathbf{j}}{\sigma} - \mathbf{E_{ext}}$$

and

$$\mathbf{S} = \frac{1}{\sigma}(\mathbf{j} \times \mathbf{H}) - \mathbf{E_{ext}} \times \mathbf{H}.$$

The first term on the right hand side of this equation is the energy flux directed into the conductor. The second term has a minus sign and therefore flows in the opposite direction, out of the conductor. This energy returns to other portions of the conductor to be liberated there in the form of heat. Hence, in general, it is the electromagnetic energy due to the current which is converted into heat. This energy is mainly localized in the external space surrounding the conductor and enters through its outer surface. This is more clearly manifested in fast varying currents produced by a rapidly varying field $\mathbf{E_{ext}}$ . When the field $\mathbf{E_{ext}}$ oscillates very rapidly the electromagnetic energy does not penetrate to the interior of the conductor and is converted into heat only in a thin surface layer where the current is concentrated. This kind of energy does not have time to reach the internal layers of the conductor and is converted into heat only within the surface layers of the conductor, in which the alternating currents are concentrated, (skin effect).

### Case (b)

This case is solved by Feynman [2] calling the attention of the reader to the fact that to check the calculation one has to assume that the energy is not coming down the wires but is coming in through the edges of the gap since the Poynting vector S points inward toward the axis parallel to the plates. The energy density in the region of space between the plates increases as the density of charge on the plates increase. According to Feynman, as the charge accumulates on the plates of the capacitor, electromagnetic energy flows in through the sides of the capacitor from far away regions of space. In his calculation, the Poynting **S** points inwards to an axis lying inside the capacitor parallel to the plates and perpendicular to the magnetic field. He argues that, as the charges come close together the field in the regions between the plates which was weak gets stronger, so the field energy which was way out moves toward the capacitor and eventually ends up between the plates.

### Case (c)

In this case, Feynman considers the seemingly paradoxical situation that arises when an electric static charge is near a bar magnet. In this case, **S** is different form zero and should represent a continuos circulation of energy along closed paths, which in a static electromagnetic field, as is the case here, is devoid of physical meaning.

The situation ceases to be peculiar if the electromagnetic momentum density **g** is taken into account. As we need to deal with the concept of angular momentum, consider the case of a long cylindrical capacitor placed in a uniform magnetic field, **H** , parallel to its axis (Fig. 2). We then have a configuration of electric and magnetic fields which is in a certain respect very similar to the case of a charge outside a bar magnet.

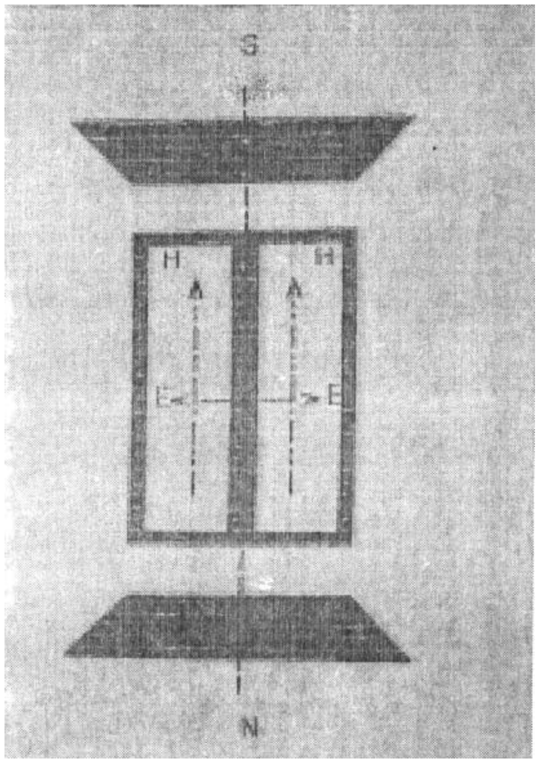


Figure 2. A cylindrical capacitor placed in a uniform magnetic field parallel to its axis. In the space between the capacitor plates there is a radial electric field of intensity $\mathbf{E} = \frac{2q\mathbf{r}}{1r^2}$. (1 = length of the capacitor; $r$, vector distance from the field to the capacitor).

The electric field in the space between the capacitor is radial and its magnitude is

$$\mathbf{E} = \frac{2qr}{1r^2}$$

Where $q$ is the charge on the inner plate of the capacitor, 1 the length and $r$ the distance from a field point to the capacitor axis. Thus, in the space between the plates the Poynting vector is different from zero and equals

$$\mathbf{S} = -\frac{c}{4\pi}(\mathbf{E}\times\mathbf{H}) = \frac{cq}{2\pi 1 r^2}(\mathbf{r} \ \times \mathbf{H})$$

The lines of Poynting's vector are concentric circles whose planes are perpendicular to the axis of the capacitor.

To examine this problem we take into account the density of electromagnetic momentum **g** which is proportional to the Poynting's vector **S** through, $\mathbf{g} = 1/c\mathbf{S}$, giving

$$\mathbf{g} = \frac{q}{2\pi c r^2 1}(\mathbf{r} \ \times \mathbf{H})$$

This momentum density is localized in the static field and its existence leads to results that can be verified experimentally. Knowing the space distribution of the electromagnetic momentum, the angular momentum **L** relative to the center of inertia can be determined from,

$$\mathbf{L} = \int (\mathbf{R} \times \mathbf{g}) dV \neq 0$$

Here, **R** is the distance from the volume element $dV$ (where the volume density is **g** ) to the center of inertia of the capacitor.

If we discharge the capacitor, then both the electric field **E** and the electromagnetic angular momentum **L** will vanish and consequently due to the law of conservation of the angular momentum the system [capacitor + magnet] will acquire in the course of the discharge a mechanical angular momentum equal to **L**. This could be verified experimentally by discharging the capacitor in the absence of external forces (or that the torque of these forces equals zero), condition that could actually be attained by bringing close to the capacitor (filled with, say gas) some radioactive substance causing ionization of the gas between the plates. The system [capacitor + magnet] acquires a mechanical angular momentum equal to **L**. If the magnet is fixed and the capacitor can rotate freely, it should acquire an angular velocity $\omega$ equal to $L/I$, where $I$ is the moment of inertia of the capacitor relative to its axis.

## Conclusion

Apparently paradoxical results obtained by considering the Poynting vector **S** as the energy flow per unit area at any given point are dispelled if one is consistent in accepting the existence of an external electromagnetic field for any steady current and the conservation of angular momentum (mechanical + electrical) in the electromagnetic field.

## References


1. C.S. Lai, Am. J. Phys.**49**, 9 (1981).
2. R.P. Feynman (1964) *Lectures of Physics II*, chap. 27. Addison-Wesley-Reading, MA.
3. C.O. Hines, Can. J. Phys **30**, 123 (1952).
4. W. Pauli. (1981) *Theory of Relativity,* Dover Publications, New York, pag. 86.
5. M. Born and E. Wolf, (1965) *Principles of Optics,* 3rd. Ed. J. Wiley & Sons. New York, NY, p. 9.
6. I.E. Tam, (1979) *Fundamentals of the Theory of Electricity,* MIR Publishers. Moscow, Secs. 7.2 - 7.14 Also H.H. Skilling, (1984) *Fundamentals of Electric Waves,* J. Wiley & Sons, New York, NY. 133.