Elementary Symmetry Considerations on Classical Electrodynamics

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A pedagogical symmetry analysis of electrodynamics is presented. We first discurss the tensorial character of the quantities which appear in electromagnetism. Then, we postulate that the laws of electrodynamics are described by linear equations containing those electromagnetic quantities and their first-order derivatives with respect to space and time coordinates. We also impose the condition that the laws are invariant under proper and improper space rotations and under time reversal. The Maxwell equations and the London equation are obtained. The Lorentz force, which contains the mon-linear $\nu \times H$ term cannot be obtained correctly. The same analysis is extended to the case in which magnetic monopoles are also present.

Classical electrodynamics can be deductively obtained starting from the form of the relativistic action of a set of particles i in an electromagnetic field [1] . In the Gaussian units this action can be written as

$$\int_{t_1}^{t_2} dt \left[\sum_i (-m_i c^2 \sqrt{1-\frac{\nu_i^2}{c^2}} + \frac{e_i}{c}\vec{A}\cdot\vec{\nu}_i - \epsilon_i \Phi + \frac{1}{8\pi}\int (E^2 - H^2 dV \right] . \tag{1}$$

In Eq.(1), m_i, e_i, $\vec{\nu}_i$ are mass, charge and velocity of the particles, respectively, and the electric and magnetic fields are defined as derivatives of the vector and scalar potentials:

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\Phi, \quad \vec{H} = \vec{\nabla}\times\vec{A} \tag{2}$$

It is difficult to be exaggerated when emphasizing the amount of information and predictive power contained in Eqs. (1), (2). The Maxwell equations

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t} \tag{3}$$

$$\vec{\nabla}\times\vec{H} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{j} \tag{4}$$

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \tag{5}$$

$$\vec{\nabla}\cdot\vec{H} = 0 \tag{6}$$

are deduced, and from Eqs. (4) and (5) the continuity equation

$$\vec{\nabla}\cdot\vec{j} + \frac{\partial P}{\partial t} = 0 \tag{7}$$

is obtained immediately, which implies that electric charges are conserved. Further, the equation of motion of a charged particle in an electromagnetic field is obtained

$$\frac{d}{dt}\frac{m\vec{\nu}}{\sqrt{1-\frac{\nu^2}{c^2}}} = e\vec{E} + e\vec{\nabla}\times\vec{H} . \tag{8}$$

Thus, all classical electromagnetic phenomena are a consequence of the following principle:

The dynamics of charged particles and electromagnetic fields are such as to make the integral in Eq. (1) an extremum, if the position of the particles and the value of the potentials are fixed at the time limits t_1, and t_2.

Very simple and economic, considering the enormous variety of electromagnetic phenomena which are involved. The fields created by charges at rest or in motion, the motion of charged particles under the influence of other charged particles, the way in which radiation in generated, propagate in free space and interact with matter, all those things have to obey only one rule, which is the least action principle stated above. That is the classical limit. Quantum mechanics allows some probability that the particles and fields move such that the action does not deviate too much from the classical extremum value. The Planck's constant is the reference standard for what means small or large action deviation.

Of course, the electrodynamical theory was not historically built that way. A tremendous experimental basis led to the formulation of severed physical laws. With those laws and some guess based on symmetry considerations Maxwell formulated the famous set of equations which carry his name. On the other hand, Newton's mechanics was mathematically manipulated by Leonard Euler and Lagrange, culminating in the least action principle formulation. At the turn of this century the work of Lorentz and Einstein resulted in the so-called covariant (relativistically invariant) formulation of mechanics and electromagnetism in one unified theoretical scheme: classical electrodynamics. Hence, this theory is the great synthesis encompassing about 250 years of scientific work. The action formula in Eq. (1) is the summit of this edification. From every point to there we are going up in our scientific adventure. From there to any other point we are going down just predicting details which are already contained in the seminal principle.

An important question concerning electrodynamics is the following: Okay, the action formula (1) contains as unique solution all the laws of electromagnetism. Suppose, however, that we are still on the valleys and do not know this summit. How much we must learn before trying to guess the rest of the scheme? This article is a little digression on this question. The first point to consider is wheter or not the question makes sense. It does! Contrarily to what happens with a jigsaw puzzle, when guessing the secrets of nature usually the knowledge of small pieces of the picture allows the solution of the whole game scheme. Recognizing the tail of a cat is the same as seeing the whole cat. If there is a lake here, there must be there a curled road climbing a hill up to a white castle.

Quantitative information is usually harder to obtain and less relevant for the game solution than qualitative information. And perhaps no other qualitative consideration carries more predictive power than symmetry. All those detailed measurements which resulted in the empirical laws by Coulomb, Ampere and Faraday are hard to perform and even harder to condense in the mathematical form describing the laws. For the modern physicist, guided by the contemporary paradigms, it could have been much easier.

The most obvious entity in electromagnetism is the electric charge. Very simple qualitative tests prove that the charge is a scalar quantity: the interaction between two given electric charges at rest depend only on their mutual separation and the force is along the line connecting them. Everything in electrostatics can be described in terms of two quantities, charge density ρ and electric field $\vec{E}$. The first is a scalar and the second is a vector.

Very fortunately, nature supplied us with rather simple means of manipulating electric charges. The availability of good electric conductors and insulators in nature is quite a piece of good luck. That allows the strict control of the flow of electric charges through metal wires and their stable setting in bodies in vacuum or inside insulators. An electric current is thus seen to have a distinctive action on other or in moving charges. The force of electric currents on a moving charge is always perpendicular the velocity of the latter. At a fixed point in space there is one unique direction along which the electric charge can move without being acted on by any force. Hence the field created by an electric current associates one directional quantity to each point in space. The direction of the magnetic field is the force-free direction for moving charges. We are thus tempted to say that the magnetic field $\vec{H}$ is a vector field. But that is not true. The magnetic force is allays perpendicular to the magnetic field $\vec{H}$, besides being perpendicular to the charge velocity $\vec{v}$. However, there is a subtle effect which manifests when we examine the field created by a current circulating in a coil or a ring. Fig. 1 shows a ring in which a current circulates counterclockwise. Tests indicate that the force-free direction for charges motion at the ring axis is along the axis, i.e., the z-direction. This is consequently the direction of the magnetic field. There is no objective way to decide between $+z$ and $-z$ as the positive direction of the field. We arbitrarily define it as the $+z$ direction, i.e., the direction to side from which the current is seen counterclockwise. However, one thing is completely objective: a positive charge moving along $+y$ is subject to a force along $+x$.

Now, let us reflect the ring with the current in a vertical mirror perpendicular to y, as shown in the figure. Now the current in the reflected ring will become clockwise as seen from the top and the reflected magnetic field will point along $-z$. The force along a positive charge moving along $+y$ will now be along $-x$! That is a very strange transformation law. A vector is not affected by a reflection in a plane parallel to its direction. We define the magnetic field as pseudo-vector. Fig. 2 shows how a vector ($\vec{E}$) and a pseudo-vector ($\vec{H}$) transform under reflection in a plane. The pseudo-vector does everything right as if it were a vector but at the end reverses the direction.

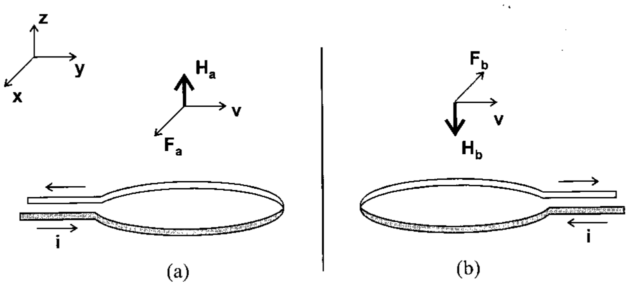

Figure 1. Part (a) shows a ring conducting a current counterclockwise as seen from the top. At a point along the ring axis at the top side, a positive electric particle moving on the $+y$ direction is subject to a force $\vec{F}_a$ on the $+x$ direction. Part (b) shows the same apparatus as seen reflected on a mirror. Right and left are exchanged and the electric charge moving on the $+y$ direction is forced along the $-x$ direction. If we consider that the electnc charge is a scalar, it is invariant under reflection and consequently we have to conclude that $\vec{H}_b = -\vec{H}_a$.

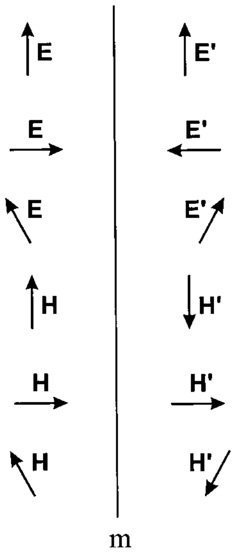

Figure 2. Reflection of a vector $\vec{E}$ and a pseudo-vector $\vec{H}$ on a mirror plane, at three orientations. Observe that if $\vec{H} = \vec{E}$, $\vec{H}' = -\vec{E}'$.

Hence, we have identified two field sources, ρ and $\vec{j}$ (the density of electric current) and two fields $\vec{E}$ and $\vec{H}$. We should look for equations involving these variables and their derivatives with respect to space and time coordinates. That is the Newtonian paradigm: the fundamental laws of nature are differential equations in space and time. The contemporary physicist knows that it is possible to write the fundamental equations as differential equations of first order, if we have choosen the appropriate dynamical variables. Let us hope that we did, and keep derivatives only of first order in space and time. We end up with a large number of quantities such as $\vec{E}$, $\frac{\partial \vec{E}}{\partial t}$, $\vec{\nabla}_i E_j$, $i, j = 1, 2, 3$, etc ... However, from then we have to build mathematical objects with well defined tensorial character, in order to obtain equations which are invariant under space rotations. Here again, we limit ourselves to tensors of zeroth and first order and the selected objects are $\vec{E}$, $\frac{\partial \vec{E}}{\partial t}$, $\vec{\nabla} \cdot \vec{E}$, $\vec{\nabla} \times \vec{E}$, etc...

Simple tests show that the sources ρ, $\vec{j}$ act independently and creat fields which add linearly (principle of superposition), which means that the equations should be linear. Finally, we make the assumption that electrodynamics obeys parity, i. e., its laws cannot distinguish between right and left. We also suppose that the laws are invariant under time reversal. That means that we cannot mix in the same equation vectors and pseudo-vectors, because if we reflect the world in a mirror the vectors and pseudo-vectors transform differently and the law will fail. We have also to consider the behavior of the mathematical objects under time reversal. After classifying our objects and collecting them is similar groups we have.

$\rho, \vec{\nabla} \cdot \vec{E}$ - scalar, symmetric (under time reversal) (SS)

$\vec{\nabla} \cdot \vec{H}$ - pseudo-scalar, antisymmetric (PSAS)

$\vec{\nabla} \times \vec{E}$, $\frac{\partial \vec{H}}{\partial t}$ - pseudo-vector, symmetric (PVS)

$\vec{\nabla} \times \vec{H}$, $\frac{\partial \vec{E}}{\partial t}, \vec{j}$, - vector, antisymmetric (VAS)

$\vec{\nabla} \cdot \vec{j}$, $\frac{\partial \rho}{\partial t}$ - scalar, antisymmetric (SAS)

$\vec{E}$, $\frac{\partial \vec{j}}{\partial t}, \vec{\nabla} \rho$ - vector, symmetric (VS)

$\vec{H}, \vec{\nabla} \times \vec{j}$ - pseudo-vector, antisymmetric (PVAS)

We can now write the fundamental laws of electromagnetism as linear relations of the variables above

$$\vec{\nabla} \cdot \vec{E} + a_1 \rho = 0 \tag{9}$$

$$\vec{\nabla} \cdot \vec{H} = 0 \tag{10}$$

$$\vec{\nabla} \times \vec{E} + a_2 \frac{\partial \vec{H}}{\partial t} = 0 \quad (11)$$

$$\vec{\nabla} \times \vec{H} + a_3 \frac{\partial \vec{E}}{\partial t} + a_4 \vec{j} = 0 \quad (12)$$

$$\vec{\nabla} \cdot \vec{j} + a_5 \frac{\partial \rho}{\partial t} = 0 \quad (13)$$

$$\frac{\partial \vec{j}}{\partial t} + a_6 \vec{E} + a_7 \vec{\nabla} \rho = 0 \quad (14)$$

$$\vec{H} + a_8 \vec{\nabla} \times \vec{j} = 0 \quad (15)$$

The relations 9-12 are the Maxwell equations. The constants a_1, a_2, a_3 and a_4 are arbitrarily defined, with the only restriction that $a_2 a_3 = -c^{-2}$, where c is the speed of propagation of the electromagnetic interactions. Combing the equations 1 and 4 we have

$$\vec{\nabla} \cdot \vec{j} - \frac{a_1 a_3}{a_4} \frac{\partial \rho}{\partial t} = 0 \quad (16)$$

which is equivalent to equation (13), i.e., a_5 is not an independent coefficient. The conservation of the total electric charge implies that $a_5 = -1$. Traditionally the conservation of electric charge is not stated as an independent principle in electromagnetism. The coefficients a_1, a_3 and a_4 are instead arbitrated in such a way that $a_1 a_3 = a_4$ and this conservation principle becomes implicit in the Maxwell equations.

Equation (15) is known to hold in a superconductor, i.e., a non-dissipative medium, and was postulated in 1935 by London and London [2] to explain the perfect diamagnetism in superconductors, known as Meissner effect.

In the present approach the equation (14) describes the motion of charges in the presence of an electromagnetic field. The term $a_6 \vec{E}$ is the contribution of the electric field to the change in current and the term $a_7 \vec{\nabla} \rho$ is the contribution of the charge density inhomogeneity. One should recall that for a heavily damped system such as an ohmic conductor the equivalent to Eq. (14) would be

$$\vec{j} = \sigma \vec{E} - A \vec{\nabla} \rho \ , \quad (17)$$

where σ is the electrical conductivity and the term $A \vec{\nabla} \rho$ (or $B \vec{\nabla} \mu$ where μ is the chemical potential) is the affinity which brings the system to thermodynamical equilibrium. One should observe that Eq. (17) is not invariant under time reversal, which is expected for a damped system.

In presence of a magnetic field, it is known that a term of the form $C \vec{j} \times \vec{H}$ appears in Eq. (14) as required by Eq. (8). This term was excluded from Eq. (14) due to the linearity postulate. One thus sees that the present approach results in imperfect prediction of eleckomagnetic phenomena. The existence of the term $\vec{j} \times \vec{H}$ breaks both the linearity and the homogeneity of the set os equations. A set of homogeneous equations on X_1, X_2, ... X_n is invariant to a change of scale, in the sense that if X_{10}, X_{20}, ... X_{n0} is a solution, λX_{10}, λX_{20}, ... λX_{n0} will also be, λ being an arbitrary constant. However, there is in nature an upper limit c to the possible speeds, and due to this the equations of motion cannot be strictly homogeneous. In fact, if they were homogeneous, given that a speed $\nu \neq 0$ is possible for the charge particles, then any speed $\lambda \nu$ would be also possible. In fact, the relativistic correction to the mass of the charged particles is also missing in Eq. (4).

Our system of equations 9 - 15 can be easily modified if we accept the existence of magnetic monoples. The density of magnetic monopoles ρ_m would be a pseudo-scalar antisymmetric under time reversal and $\vec{j}_m$ would be a symmetric pseudo-vector. The new set of equations would be

$$\vec{\nabla} \cdot \vec{E} + A_1 \rho = 0 \quad (18)$$

$$\vec{\nabla} \cdot \vec{H} + A_2 \rho_m = 0 \quad (19)$$

$$\vec{\nabla} \times \vec{E} + A_3 \frac{\partial \vec{H}}{\partial t} + A_4 \vec{j}_m = 0 \quad (20)$$

$$\vec{\nabla} \times \vec{H} + A_5 \frac{\partial \vec{E}}{\partial t} + A_6 \vec{j} = 0 \quad (21)$$

$$\vec{\nabla} \cdot \vec{j} + A_7 \frac{\partial \rho}{\partial t} = 0 \quad (22)$$

$$\vec{\nabla} \cdot \vec{j}_m + A_8 \frac{\partial \rho_m}{\partial t} = 0 \quad (23)$$

$$\frac{\partial \vec{j}}{\partial t} + A_9 \vec{E} + A_{10} \vec{\nabla} \rho + A_{11} \vec{\nabla} \times \vec{j}_m = 0 \quad (24)$$

$$\frac{\partial \vec{j}_m}{\partial t} + A_{12} \vec{H} + A_{13} \vec{\nabla} \rho_m + A_{14} \vec{\nabla} \times \vec{j} = 0 \quad (25)$$

It is clear that the introduction of magnetic monopoles brings great symmetry to the fundamental equations of electromagnetism.

If we add "by hand" the Lourentz forces $\vec{j} \times \vec{H}$ and $\vec{j}_m \times \vec{E}$ in the Eqs. (24), (25), we obtain

$$\frac{\partial \vec{j}}{\partial t} + A_9 \vec{E} + A_{10} \vec{\nabla} \rho + A_{11} \vec{\nabla} \times \vec{j}_m + A_{15} \vec{j} \times \vec{H} = 0 \quad (26)$$

$$\frac{\partial \vec{j}}{\partial t} + A_{12} \vec{H} + A_{10} \vec{\nabla} \rho_m + A_{14} \vec{\nabla} \times \vec{j} + A_{16} \vec{j}_m \times \vec{E} = 0 \quad (27)$$

The set of equations 18 - 23 and 26, 27 are supposedly the set of laws which govern classical electrodynamics in the presence of magnetic monopoles.

The magnetic monopoles are a pseudo-scalar, which means that they reverse sign when reflected in a mirror. It is important to catch the meaning of that in terms of observable phenomena. Suppose that someone holds a magnetic dipole and approaches it to a magnetic monopole. If the monopole is attracted to the south pole he concludes that it is a north monopole. But one of his colleagues is watching the experiment through a mirror, as indicated in Fig. 3. As reflected rigth becomes left and vice-versa, the magnetic dipole reverses sign and the reflected monopoles is attracted to north pole. For the other observer it is consequently a south monopole. That is the meaning of it: reflected north is south and reflected south is north, and reversing the sign of the magnetic charge transform north "sign" into south "sign", and vice-versa. It can look strange, but is it consistent. Monopoles of equal sign repel each other and monopoles of apposite sign are muttually attracted. That is the result of Eqs. (19) and (25) for A_2 and A_{12} negative. This behavior is invariant under reflection in a mirror and that is what matters. In fact, every law in electrodynamics in invariant with respect to space rotations, proper and inproper, and with respect to time reversal. The equations 18 - 27 were writen under this requeriment.

Thus, we have just seen that classical electrodynamics can be almost perfectly derived as the complete set of equations, invariant under space and time inversion, connecting the electric and magnetic charges, their currents, the electric and magnetic fields and the space and time first-order derivatives of these quantities. This simple prescription leads uniquely to the Eqs. (18) - (25). The purely relativistic side forces acting on currents, in form $\vec{j} \times \vec{H}$ and $\vec{j}_m \times \vec{E}$, are non-linear terms which have to be added "by hand", such that Eqs. (26), (27) are substituted for Eqs. (24), (25).

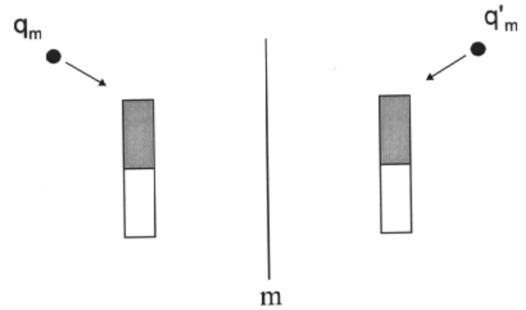

Figure 3. A magnetic monopole is attracted to the north pole of a magnet. After reflection on a mirror plane it is seen as attracted to the south pole. This means that the sign of the magnetic monopole is reversed under space reflection.

It is important to point out that the action principle, which is very effective on deducing the laws of electrodynamics in the absence of magnetic monopoles, cannot be easily extended to do the same in the presence of these poles. The problem was first discussed by Dirac in his pioneer work [3], and is related to difficulties on constructing the potentials Φ_m and $\vec{A}_m$ which are generated by the magnetic charges and currents. That is outside of the scope of this article. For a recent approach and references on the subject, see Cardoso de Mello et al [4].

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