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Green's functions for the wave, Helmholtz and Poisson equations in a two-dimensional boundless domain

(Funções de Green para as equações da onda, Helmholtz e Poisson num domínio bidimensional sem fronteiras)

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In this work, Green's functions for the two-dimensional wave, Helmholtz and Poisson equations are calculated in the entire plane domain by means of the two-dimensional Fourier transform. New procedures are provided for the evaluation of the improper double integrals related to the inverse Fourier transforms that furnish these Green's functions. The integrals are calculated by using contour integration in the complex plane. The method consists basically in applying the correct prescription for circumventing the real poles of the integrand as well as in using well-known integral representations of some Bessel functions.
Keywords: Green's function, Helmholtz equation, two dimensions.

Neste trabalho, as funções de Green para as equações bidimensionais da onda, Helmholtz e Poisson são calculadas na totalidade do domínio plano por meio da transformada de Fourier bidimensional. São apresentados novos modos de se efetuarem as integrais duplas impróprias relacionadas às transformadas de Fourier inversas que fornecem essas funções de Green. As integrais são calculadas a partir de integrais no plano complexo. O método consiste basicamente em determinar o caminho de integração que se desvia corretamente dos polos reais do integrando bem como em usar representações integrais bem conhecidas de algumas funções de Bessel.
Palavras-chave: função de Green, equação de Helmholtz, duas dimensões.

1. Introduction

Green's functions for the wave, Helmholtz and Poisson equations in the absence of boundaries have well known expressions in one, two and three dimensions. A standard method to derive them is based on the Fourier transform. Nevertheless, its derivation in two dimensions (the most difficult one), unlike in one and three, is hardly found in the literature, this being particularly true for the Helmholtz equation.[2] This work aims at providing new ways of performing the improper double integrals related to the inverse Fourier transforms that furnish those Green's functions in the bidimensional case.

It is assumed that the reader is acquainted with the usual prescriptions for circumventing the real poles of a function being integrated over the real axis: the $i\epsilon$-prescription [2–5] and that which leads to the Cauchy principal value. In this respect, Ref. [6] is closely followed. As described in this reference, in physical applications, the improper integrals that arise are often not well defined mathematically, being necessary to consider the physical conditions to determine the correct prescription. For this reason, the wave and Helmholtz equations solved in this work refer to concrete situations.

Sections 2, 3 and 4 are devoted to the wave, Helmholtz and Poisson equations, respectively. Section 5 concludes the body of the paper with final comments.

2. Wave equation

For the reasons given in the Introduction, in order to calculate Green's function for the wave equation, let us consider a concrete problem, that of a vibrating, stretched, boundless membrane

$$\nabla^2 z(\mathbf{r},t) - c^{-2} z_{tt} = -T^{-1} f(\mathbf{r},t), \qquad \left[\,\mathbf{r} \text{ in } \mathbb{R}^2,\ \ t \text{ in } \mathbb{R}\,\right]. \tag{1}$$

In this, $z(\mathbf{r},t)$ is the membrane vertical displacement at the point $\mathbf{r}$ of the xy-plane at the time t, $f(\mathbf{r},t)$ is the external vertical force per unit area, and $c \equiv \sqrt{T/\sigma}$,

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[2]For this equation, the author is aware of the procedure sketched in Ref. [1], p. 173-176

where T is the stretching force per unit length (uniform and isotropic) and σ is the mass per unit area. We admit that the membrane has *always* been (since $t \to -\infty$) lying at rest on the xy-plane until the source of vibrations f starts generating waves.

The associated Green's function $G(\mathbf{r}, t\,|\,\mathbf{r}', t')$ is the solution of (1) with a unit, point, instantaneous source at the point $\mathbf{r}'$ at the time t'

$$\nabla^2 G(\mathbf{r}, t\,|\,\mathbf{r}', t') - c^{-2} G_{tt} = -T^{-1}\,\delta(\mathbf{r}-\mathbf{r}')\,\delta(t-t') \qquad \big[\,\mathbf{r} \text{ and } \mathbf{r}' \text{ in } \mathbb{R}^2,\;\; t \text{ and } t' \text{ in } \mathbb{R}\,\big]\,. \tag{2}$$

We consider here the *causal* Green's function, for which

$$G(\mathbf{r}, t\,|\,\mathbf{r}', t') = 0 \quad \text{if} \quad t < t' \;\;, \tag{3}$$

meaning that, before the instantaneous action of the unit point source, the membrane is found in its original state - horizontally and at rest - in accordance with the causality principle.

To solve the problem defined by Eqs. (2) and (3), we Fourier transform (2), both in the spatial and time variables - according to the definitions

$$\mathcal{F}_{\mathbf{r}}\left\{G(\mathbf{r}, t\,|\,\mathbf{r}', t')\right\} \equiv \frac{1}{2\pi}\int_{\mathbb{R}^2} d^2r\, e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} G(\mathbf{r}, t\,|\,\mathbf{r}', t') \equiv \bar{G}(\mathbf{k}, t\,|\,\mathbf{r}', t') \tag{4}$$

$$\mathcal{F}_t\{\bar{G}(\mathbf{k}, t\,|\,\mathbf{r}', t')\} \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, e^{\mathrm{i}\omega t}\bar{G}(\mathbf{k}, t\,|\,\mathbf{r}', t') \equiv \tilde{\bar{G}}(\mathbf{k}, \omega\,|\,\mathbf{r}', t'),$$

and solve the resulting equation to obtain $\tilde{\bar{G}}(\mathbf{k}, \omega\,|\,\mathbf{r}', t')$

$$\nabla^2 G(\mathbf{r}, t\,|\,\mathbf{r}', t') - c^{-2} G_{tt} = -T^{-1}\,\delta(\mathbf{r}-\mathbf{r}')\,\delta(t-t') \quad \overset{\mathcal{F}_{\mathbf{r}}}{\Longrightarrow}$$

$$-k^2\bar{G}(\mathbf{k}, t\,|\,\mathbf{r}', t') - c^{-2} d^2\bar{G}/dt^2 = [-T^{-1} e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}'}/(2\pi)]\,\delta(t-t') \quad \overset{\mathcal{F}_t}{\Longrightarrow}$$

$$-k^2\tilde{\bar{G}}(\mathbf{k}, \omega\,|\,\mathbf{r}', t') + (\omega/c)^2\tilde{\bar{G}} = [-T^{-1}\, e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}'}/(2\pi)]\, e^{\mathrm{i}\omega t'}/\sqrt{2\pi} \quad \Longrightarrow$$

$$\tilde{\bar{G}}(\mathbf{k}, \omega\,|\,\mathbf{r}', t') = \frac{-c^2\, T^{-1}\, e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}'} e^{\mathrm{i}\omega t'}/\,(2\pi)^{3/2}}{\omega^2 - k^2c^2}.$$

We then calculate the inverse Fourier transforms, $\mathcal{F}_t^{-1}\{\tilde{\bar{G}}\} = \bar{G}$ first [carried out below, in Eq. (5)]. This is an integral of the type discussed in Ref. [6], whose evaluation as part of a contour integral in the ω-plane imposes the need of prescribing the way to circumvent the two real poles at $\omega = \pm kc$.

For $t < t'$, in closing the contour with a semicircle C_R of radius $R \to \infty$, we obtain a vanishing integral along C_R if we let it lying in the upper half-plane (as Fig. 1(a) shows), according to Jordan's lemma. Therefore, fulfillment of Eq. (3) necessitates the path of integration along the real axis to be that in Fig. 1(a), approaching both poles from above (which is equivalent to calculate the limit as $\epsilon \to 0^+$ of the Fourier integral with both poles shifted of $-\mathrm{i}\epsilon$), thus leaving both poles outside the contour and leading to the expected null result. For $t > t'$, we adopt this same prescription indicating how the path along the real axis circumvents the poles, but, because of Jordan's lemma, we close the contour as in Fig. 1(b), with C_R in the lower half-plane, thus enclosing both poles, at which the residues now contribute to the result.

In the notation of Ref. [6], such a way of calculating the improper integral leads to its $\mathcal{D}_{--}$-value[3]

$$\bar{G}(\mathbf{k}, t\,|\,\mathbf{r}', t') = \mathcal{F}_t^{-1}\{\tilde{\bar{G}}(\mathbf{k}, \omega\,|\,\mathbf{r}', t')\} = \frac{-c^2T^{-1}}{(2\pi)^2}\, e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}'}\mathcal{D}_{--}\!\!\int_{-\infty}^{\infty}\frac{e^{-\mathrm{i}\omega(t-t')}}{\omega^2-k^2c^2}\,d\omega = \tag{5}$$

$$\left[\frac{c^2T^{-1}}{(2\pi)^2}\, e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}'}\cdot 2\pi\mathrm{i}\cdot\begin{cases} 0 & (t<t') \\ \underbrace{\mathrm{Res}(-kc)}_{e^{\mathrm{i}kc\tau}/(-2kc)} + \underbrace{\mathrm{Res}(kc)}_{e^{-\mathrm{i}kc\tau}/(2kc)} & (t>t')\end{cases}\right] = \frac{c}{2\pi T}\, e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{r}'}\,\frac{\sin kc\tau}{k}\,\mathcal{U}(\tau)\;\;,$$

where $\tau \equiv t - t'$ and $\mathcal{U}(\tau)$ is the unit step function (equal to 0 for $\tau < 0$ and to 1 for $\tau > 0$).

[3] "$\mathcal{D}_{--}$" means that, in applying the iϵ-prescription method, both the left and right poles are shifted of $-\mathrm{i}\epsilon$. Cauchy's $\mathcal{P}$-value as well as the $\mathcal{D}_{++}$-, $\mathcal{D}_{+-}$- and $\mathcal{D}_{-+}$-value of the improper integral are not physically acceptable, because they do not vanish for $t < t'$ and are thus inconsistent with the causality principle.

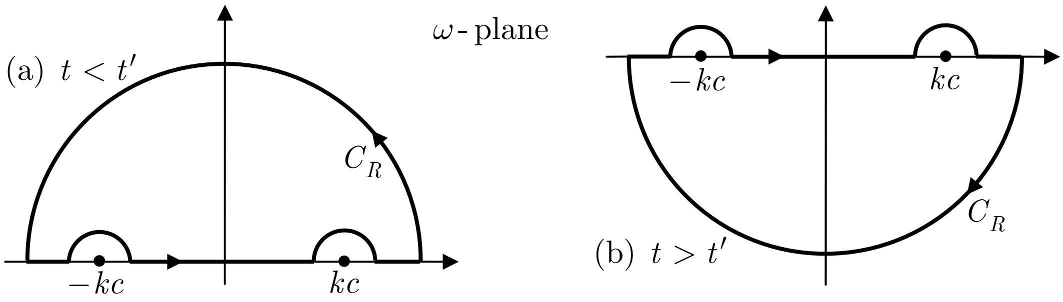

Figure 1 - The contours used to evaluate the integral in Eq. (5) for (a) $t < t'$ and (b) $t > t'$.

Now evaluating $\mathcal{F}_{\mathbf{r}}^{-1}$ of the result above, we obtain

$$G(\mathbf{r},t\,|\,\mathbf{r}',t') = \mathcal{F}_{\mathbf{r}}^{-1}\{\bar{G}(\mathbf{k},t\,|\,\mathbf{r}',t')\} = \frac{c\,\mathcal{U}(\tau)}{(2\pi)^2 T}\int_{\mathbb{R}^2} d^2k\; e^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\,\frac{\sin kc\tau}{k}. \qquad (6)$$

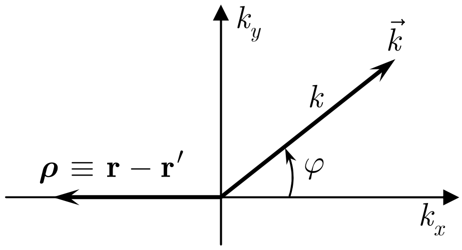

Figure 2 - The axes of the Cartesian coordinates k_x and k_y of the vector $\mathbf{k}$ used to evaluate the double integrals in Eqs. (6), (14), (19) and (25).

This double integral becomes easier to perform if we express the area element of the $\mathbf{k}$-plane in the polar coordinates, $d^2k = k\,dk\,d\varphi$ (instead of the Cartesian ones, $d^2k = dk_x\,dk_y$), and choose the k_x-axis in the opposite direction of the vector $\boldsymbol{\rho} \equiv \mathbf{r} - \mathbf{r}'$, as shown in Fig. 2. In addition, noticing that $\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}') = k\rho\cos(\pi-\varphi)$, we can write

$$G(\mathbf{r},t\,|\,\mathbf{r}',t') = \frac{c\,\mathcal{U}(\tau)}{(2\pi)^2 T}\int_0^{2\pi}\int_0^{\infty} e^{\mathrm{i}k\rho\cos\varphi}\sin kc\tau\,dk\,d\varphi$$

$$= \frac{c\,\mathcal{U}(\tau)}{2\pi T}\int_0^{\infty} dk\;\sin kc\tau\left[\frac{1}{2\pi}\int_0^{2\pi} d\varphi\; e^{\mathrm{i}k\rho\cos\varphi}\right]. \qquad (7)$$

The brackets in this equation enclose an integral representation of the Bessel function $\mathrm{J}_0(k\rho)$, which also admits another well-known integral representation; that is,[4]

$$\frac{1}{2\pi}\int_0^{2\pi} d\varphi\; e^{\mathrm{i}k\rho\cos\varphi} = \mathrm{J}_0(k\rho) = \frac{2}{\pi}\int_1^{\infty} du\,\frac{\sin k\rho u}{\sqrt{u^2-1}}\;. \qquad (8)$$

Therefore, with the substitution of this latter integral representation of $\mathrm{J}_0(k\rho)$ for the former one in Eq. (7), we can continue the calculation as follows

$$G(\mathbf{r},t\,|\,\mathbf{r}',t') = \frac{c\,\mathcal{U}(\tau)}{2\pi T}\int_0^{\infty} dk\;\sin kc\tau\left[\frac{2}{\pi}\int_1^{\infty} du\,\frac{\sin k\rho u}{\sqrt{u^2-1}}\right]$$

$$= \frac{c\,\mathcal{U}(\tau)}{2\pi T}\int_1^{\infty}\frac{du}{\sqrt{u^2-1}}\left[\frac{2}{\pi}\int_0^{\infty} dk\;\sin k\rho u\sin kc\tau\right]\;. \qquad (9)$$

Now recognizing that the last pair of brackets encloses an integral representation of the delta function $\delta(\rho u - c\tau)$ [8,9] and then changing the variable of integration from u to $\xi = \rho u - c\tau$, we can write

$$G(\mathbf{r},t\,|\,\mathbf{r}',t') = \frac{c\,\mathcal{U}(\tau)}{2\pi T}\int_1^{\infty}\frac{du\,\delta(\rho u - c\tau)}{\sqrt{u^2-1}} = \frac{c\mathcal{U}(\tau)}{2\pi T\rho}\int_{\rho-c\tau}^{\infty}\frac{d\xi\;\delta(\xi)}{\sqrt{\left(\frac{\xi+c\tau}{\rho}\right)^2-1}}.$$

This integral is easily evaluated by using the sifting property of the delta function;[5] we obtain

$$G(\mathbf{r},t\,|\,\mathbf{r}',t') = \frac{c\mathcal{U}(\tau)}{2\pi T\rho}\times\begin{cases} 0 & \text{if}\quad \rho - c\tau > 0 \quad (i.e.\;\; \tau - \rho/c < 0) \\ 1\Big/\sqrt{\left(\frac{c\tau}{\rho}\right)^2-1} & \text{if}\quad \rho - c\tau < 0 \quad (i.e.\;\; \tau - \rho/c > 0) \end{cases}$$

$$= \frac{\mathcal{U}(\tau)}{2\pi T\sqrt{\tau^2-(\rho/c)^2}}\times\underbrace{\begin{cases} 0 & \text{if}\quad \tau-\rho/c<0 \\ 1 & \text{if}\quad \tau-\rho/c>0\end{cases}}_{\mathcal{U}(\tau-\rho/c)} = \frac{\mathcal{U}(\tau)\,\mathcal{U}(\tau-\rho/c)}{2\pi T\sqrt{\tau^2-(\rho/c)^2}}\;.$$

[4]The first integral representation, by bisecting the range of integration and making the changing of variable $\varphi \to 2\pi - \varphi$ in the latter part, becomes the Eq. (2) in Ref. [7], §2.3, with $n = 0$ and $z = k\rho$. The second one is also found in this reference, §6.13, Eq. (3), with $\nu = 0$ and $x = k\rho$.

[5]That is, $\int_a^b dx\,\delta(x)\,f(x)$ is equal to $f(0)$ if $a < 0 < b$ and is zero otherwise.

But, in the product $\mathcal{U}(\tau)\mathcal{U}(\tau-\rho/c)$, we can drop the first unit step function without altering the result (this is readily seen by plotting them both). We then obtain the final result

$$G(\mathbf{r},t\,|\,\mathbf{r}',t') = G(\rho,\tau) = \frac{1}{2\pi T}\,\frac{\mathcal{U}(\tau-\rho/c)}{\sqrt{\tau^2-(\rho/c)^2}} \quad [\,\rho\equiv|\,\mathbf{r}-\mathbf{r}'|\,,\ \ \tau\equiv t-t'\,]\ .$$

3. Helmholt equation

In this section, we calculate Green's function for the Helmholtz equation in an unbounded two-dimensional domain. Being more specific, we solve Eq. (13), which arises in the following concrete problem.

Suppose that, in the membrane problem of the previous section, the source term is given by

$$f(\mathbf{r},t) = \phi(\mathbf{r})\,e^{-\mathrm{i}\omega_0 t} \quad (t\geq t_0), \tag{10}$$

that is, the external vertical force per unit area varies harmonically with time with frequency ω_0 at all points of the membrane, being $\phi(\mathbf{r})$ (a non-negative function everywhere) its maximum magnitude at the point $\mathbf{r}$. In addition, we admit that we are only interested in the *stationary* solution $z_{st}(\mathbf{r},t)$ that will prevail after a very long time has elapsed as a consequence of the forced harmonic oscillation.

It is well-known that, when this steady-state motion is reached, all points of the membrane will be vibrating harmonically with the same frequency ω_0, but with amplitudes described by some function $\zeta(\mathbf{r})$, that is

$$z_{st}(\mathbf{r},t) = \zeta(\mathbf{r})\,e^{-\mathrm{i}\omega_0 t} \quad (t\gg t_0). \tag{11}$$

Therefore, the desired stationary solution becomes determined as soon as $\zeta(\mathbf{r})$ is calculated. By substituting Eqs. (10) and (11) in Eq. (1), we verify that $\zeta(\mathbf{r})$ is the solution of the two-dimensional Helmholtz equation

$$\nabla^2\zeta + k_0^2\zeta(\mathbf{r}) = -T^{-1}\phi(\mathbf{r}) \quad [k_0\equiv\omega_0/c], \tag{12}$$

whose solution is given by $\zeta(\mathbf{r}) = \int_{\mathbb{R}^2} d^2r'\,\Gamma(\mathbf{r}\,|\,\mathbf{r}')\,\phi(\mathbf{r}')$, where $\Gamma(\mathbf{r}\,|\,\mathbf{r}')$ is the Green's function for the above equation, the solution of

$$\nabla^2\Gamma + k_0^2\Gamma(\mathbf{r}\,|\,\mathbf{r}') = -T^{-1}\delta(\mathbf{r}-\mathbf{r}') \quad [\,\mathbf{r}\text{ and }\mathbf{r}'\text{ in }\mathbb{R}^2\,]. \tag{13}$$

Since Eq. (13) is Eq. (12) with $\phi(\mathbf{r}) = \delta(\mathbf{r}-\mathbf{r}')$, we see that $\Gamma(\mathbf{r}\,|\,\mathbf{r}')$ describes the amplitude of the stationary motion when the external vertical force is concentrated at $\mathbf{r}'$ and oscillates with unit amplitude.

Let us solve Eq. (13). We first take the same Fourier transform $\mathcal{F}_{\mathbf{r}}$ defined in Eq. (4), obtaining

$$-k^2\bar{\Gamma}(\mathbf{k}\,|\,\mathbf{r}') + k_0^2\bar{\Gamma}(\mathbf{k}\,|\,\mathbf{r}') = -T^{-1}e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}'}/(2\pi) \quad \Longrightarrow$$
$$\bar{\Gamma}(\mathbf{k}\,|\,\mathbf{r}') = \frac{T^{-1}}{2\pi}\,\frac{e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}'}}{k^2-k_0^2},$$

and then calculate the inverse Fourier transform

$$\Gamma(\mathbf{r}\,|\,\mathbf{r}') = \frac{T^{-1}}{(2\pi)^2}\int_{\mathbb{R}^2} d^2k\,\frac{e^{-\mathrm{i}\mathbf{k}\cdot\boldsymbol{\rho}}}{k^2-k_0^2} \quad [\,\boldsymbol{\rho}\equiv\mathbf{r}-\mathbf{r}'\,]. \tag{14}$$

We present two methods of calculating this integral in the next two subsections.

3.1. First method

To evaluate Eq. (14), we again choose the k_x-axis in the opposite direction of the vector $\boldsymbol{\rho}$, as in Fig. 2, but we now proceed with the Cartesian coordinates of $\mathbf{k} = k_x\,\mathbf{e}_x + k_y\,\mathbf{e}_y$

$$\Gamma(\mathbf{r}\,|\,\mathbf{r}') = \frac{T^{-1}}{(2\pi)^2}\int_{-\infty}^{\infty}dk_x\,e^{\mathrm{i}k_x\rho}\underbrace{\int_{-\infty}^{\infty}\frac{dk_y}{k_x^2+k_y^2-k_0^2}}_{I(k_x)}\,. \tag{15}$$

The integral denoted by $I(k_x)$ above can be calculated as part of a contour integral in the k_y-plane. It does not matter if we close the contour with a semicircle of radius $R\to\infty$ through the upper or lower half-plane; let us close it through the upper half-plane (Figs. 3 and 4). However, to investigate the poles of the integrand, we need to consider two separate cases: $|k_x|>k_0$ and $|k_x|<k_0$.

For $|k_x|>k_0$, by writing the denominator of the integrand in the form

$$k_y^2 + \left(k_x^2-k_0^2\right) = \left(k_y+\mathrm{i}\sqrt{k_x^2-k_0^2}\,\right)\left(k_y-\mathrm{i}\sqrt{k_x^2-k_0^2}\,\right),$$

we verify that only the pole $\mathrm{i}\sqrt{k_x^2-k_0^2}$ is inside the contour (Fig. 3); therefore

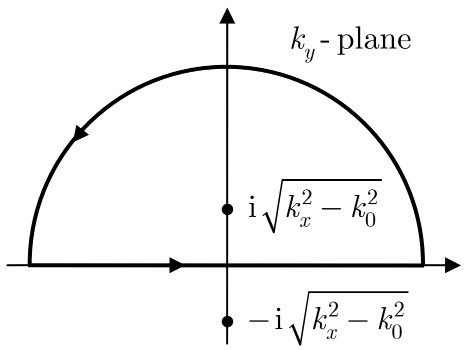

Figure 3 - The closed contour used to evaluate the integral $I(k_x)$ defined in Eq. (15) for $|k_x|>k_0$.

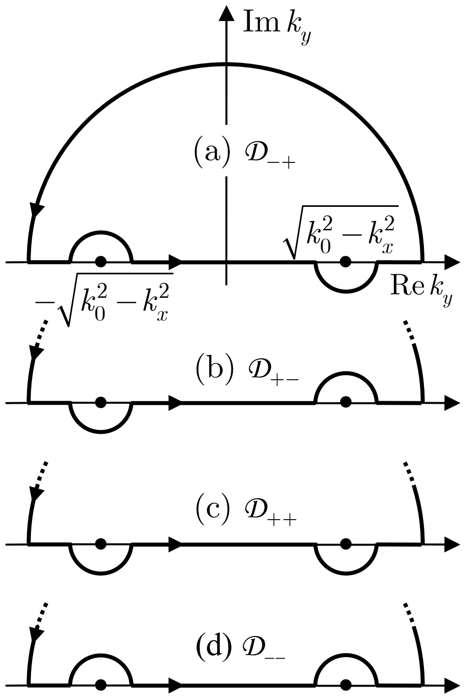

Figure 4 - Four ways of circumventing the real poles in the evaluation of the integral $I(k_x)$ defined in Eq. (15) for $|k_x| < k_0$. These are the possible iϵ-prescriptions; for each one, the corresponding $\mathcal{D}$-value ($\mathcal{D}_{-+}$, etc) is indicated.

$$\begin{aligned} I(k_x) &= 2\pi \mathrm{i} \,\mathrm{Res}\left(\mathrm{i}\sqrt{k_x^2-k_0^2}\,\right) = 2\pi\mathrm{i}\,\frac{1}{2\mathrm{i}\sqrt{k_x^2-k_0^2}} \\ &= \frac{\pi}{\sqrt{k_x^2-k_0^2}} \qquad [\,|k_x| > k_0\,]\,. \end{aligned} \tag{16}$$

For $|k_x| < k_0$, that denominator in the form

$$k_y^2 - \left(k_0^2-k_x^2\right) = \left(k_y + \sqrt{k_0^2-k_x^2}\,\right)\left(k_y - \sqrt{k_0^2-k_x^2}\,\right)$$

clearly shows the existence of the two real poles $\pm\sqrt{k_0^2-k_x^2}$. Since the residues at them are given by $\mathrm{Res}\left(\pm\sqrt{k_0^2-k_x^2}\,\right) = \pm 1/\left(2\sqrt{k_0^2-k_x^2}\,\right)$, we can deduce that

$$I(k_x) = \pm\,\pi\mathrm{i}\Big/\sqrt{k_0^2-k_x^2} \qquad [\,|k_x| < k_0\,] \tag{17}$$

are possible values for $I(k_x)$. In fact, the + and − signs correspond, in the notation of Ref. [6], to the $\mathcal{D}_{-+}$- and $\mathcal{D}_{+-}$-value of $I(k_x)$, obtained with the first and second prescriptions shown in Fig. 4. The $\mathcal{D}_{++}$- and $\mathcal{D}_{--}$-value (obtained with the third and fourth prescriptions) as well as the Cauchy $\mathcal{P}$-value of $I(k_x)$ all vanish. We proceed with both signs in Eq. (17), because only at the end, by physically analyzing the two corresponding solutions, we will be able to take the correct sign.

Let us now perform the integral with respect to k_x in Eq. (15). In view of the distinct results in Eq. (16) and Eq. (17) for $I(k_x)$, we split the interval of integration in two (for $|k_x| = |k_0|$, convergence occurs when $I(k_x) = \mathcal{P}\int_{-\infty}^{\infty} dk_y/k_y^2 = 0$, which makes no contribution)

$$\begin{aligned} \mathit{\Gamma}(\mathbf{r}\,|\,\mathbf{r}') &= \frac{T^{-1}}{(2\pi)^2}\left[\int\limits_{|k_x|>k_0} dk_x + \int\limits_{|k_x|<k_0} dk_x\right] e^{\mathrm{i}k_x\rho}\, I(k_x) \\ &= \frac{T^{-1}}{(2\pi)^2}\left[\int\limits_{|k_x|>k_0} \frac{dk_x\,\pi\, e^{\mathrm{i}k_x\rho}}{\sqrt{k_x^2-k_0^2}} \pm \int\limits_{|k_x|<k_0} \frac{dk_x\,\pi\mathrm{i}\, e^{\mathrm{i}k_x\rho}}{\sqrt{k_0^2-k_x^2}}\right]. \end{aligned}$$

The intervals of integration of both integrals above are symmetric with respect to the origin. Therefore, if we substitute $\cos k_x\rho + \mathrm{i}\sin k_x\rho$ for $e^{\mathrm{i}k_x\rho}$, we get integrals of odd terms (exhibiting $\sin k_x\rho$), which vanish, and of even terms (exhibiting $\cos k_x\rho$), which can be replaced by twice the integral over the positive values of k_x

$$\mathit{\Gamma}(\mathbf{r}\,|\,\mathbf{r}') = \frac{T^{-1}}{(2\pi)^2}\left[2\pi\int_{k_0}^{\infty} dk_x\,\frac{\cos k_x\rho}{\sqrt{k_x^2-k_0^2}} \pm 2\pi\mathrm{i}\int_0^{k_0} dk_x\,\frac{\cos k_x\rho}{\sqrt{k_0^2-k_x^2}}\right].$$

The first integral, with the change of variable given by $k_x = k_0 u$, becomes a known integral representation of the Neumann function of order zero[6]

$$\int_{k_0}^{\infty} dk_x\,\frac{\cos k_x\rho}{\sqrt{k_x^2-k_0^2}} = \int_1^{\infty} du\,\frac{\cos k_0\rho u}{\sqrt{u^2-1}} = -\frac{\pi}{2}\,\mathrm{N}_0(k_0\rho).$$

The second integral, with the change of variable $k_x = k_0\cos\vartheta$, also becomes a known integral representation[7]

$$\int_0^{k_0} dk_x\,\frac{\cos k_x\rho}{\sqrt{k_0^2-k_x^2}} = \int_0^{\pi/2} d\vartheta\,\cos(k_0\rho\cos\vartheta) = \frac{\pi}{2}\,\mathrm{J}_0(k_0\rho).$$

We thus get

$$\begin{aligned} \mathit{\Gamma}(\mathbf{r}\,|\,\mathbf{r}') &= \frac{T^{-1}}{2\pi}\left[-\frac{\pi}{2}\,\mathrm{N}_0(k_0\rho) \pm \mathrm{i}\,\frac{\pi}{2}\,\mathrm{J}_0(k_0\rho)\right] = \\ &\qquad \frac{\pm\,\mathrm{i}\,T^{-1}}{4}\left[\pm\,\mathrm{i}\,\mathrm{N}_0(k_0\rho) + \mathrm{J}_0(k_0\rho)\right], \end{aligned}$$

which is equal to $(\mathrm{i}\,T^{-1}/4)\,\mathrm{H}_0^{(1)}(k_0\rho)$ for the plus sign and to $(-\mathrm{i}\,T^{-1}/4)\,\mathrm{H}_0^{(2)}(k_0\rho)$ for the minus sign.

However, the stationary membrane motion $\mathit{\Gamma}(\mathbf{r}\,|\,\mathbf{r}')\,e^{-\mathrm{i}\omega_0 t}$, according to the *radiation condition* (see Ref. [10], p. 471), must be a wave moving away from the unit point source (*i.e.*, from the harmonic force of unit amplitude at $\mathbf{r}'$). This imposes the choice of the first Hankel function.[8] The final result is then

$$\mathit{\Gamma}(\mathbf{r}\,|\,\mathbf{r}') = \mathit{\Gamma}(\rho) = \frac{\mathrm{i}}{4T}\,\mathrm{H}_0^{(1)}(k_0\rho) \qquad [\,\rho \equiv |\,\mathbf{r}-\mathbf{r}'|\,]\,. \tag{18}$$

[6]Ref. [7], §6.21, Eq. (15), where this function is denoted by Y_0 instead of N_0.

[7]Ref. [7], §2.2, Eq. (9), with $n = 0$ and $z = k_0\rho$.

[8]A good explanation of the representation of outgoing and incoming cylindrical waves by means of the Hankel functions is given in Ref. [10], Sec. 9.12, p. 470. Notice, however, that, in this reference, the outward-traveling wave is formed with the second Hankel function $\mathrm{H}_0^{(2)}$, because the time factor used there is $e^{\mathrm{i}\omega_0 t}$, instead of the $e^{-\mathrm{i}\omega_0 t}$ used here.

3.2. Second method

The integral in Eq. (14) can also be calculated in polar coordinates. The first steps are similar to those performed in going from Eq. (6) to Eq. (9)

$$\begin{aligned}\Gamma(\mathbf{r}\,|\,\mathbf{r}') &= \frac{T^{-1}}{(2\pi)^2}\int_{\mathbb{R}^2} d^2k\,\frac{e^{-\mathrm{i}\mathbf{k}\cdot\boldsymbol{\rho}}}{k^2-k_0^2} = \frac{T^{-1}}{(2\pi)^2}\int_0^{2\pi}\int_0^{\infty} e^{\mathrm{i}k\rho\cos\varphi}\,\frac{k\,dk\,d\varphi}{k^2-k_0^2}\\ &= \frac{T^{-1}}{2\pi}\int_0^{\infty}\frac{dk\,k}{k^2-k_0^2}\underbrace{\left[\frac{1}{2\pi}\int_0^{2\pi}d\varphi\,e^{\mathrm{i}k\rho\cos\varphi}\right]}_{\mathrm{J}_0(k\rho)} = \frac{T^{-1}}{2\pi}\int_0^{\infty}\frac{dk\,k}{k^2-k_0^2}\underbrace{\left[\frac{2}{\pi}\int_1^{\infty}du\,\frac{\sin k\rho u}{\sqrt{u^2-1}}\right]}_{\mathrm{J}_0(k\rho)}\\ &= \frac{T^{-1}}{\pi^2}\int_1^{\infty}\frac{du}{\sqrt{u^2-1}}\underbrace{\left[\int_0^{\infty}dk\,\frac{k\sin k\rho u}{k^2-k_0^2}\right]}_{\equiv S(u)} = \frac{T^{-1}}{\pi^2}\int_1^{\infty}\frac{du\,S(u)}{\sqrt{u^2-1}}\ .\end{aligned} \tag{19}$$

To evaluate the integral $S(u)$ defined above, we write it in a more appropriate form

$$S(u) = \frac{1}{2}\int_{-\infty}^{\infty}dk\,\frac{k\left(e^{\mathrm{i}k\rho u}-e^{-\mathrm{i}k\rho u}\right)}{2\mathrm{i}\,(k^2-k_0^2)} = \frac{E^+(u)-E^-(u)}{4\mathrm{i}} \qquad \left[E^{\pm}(u) \equiv \int_{-\infty}^{\infty}dk\,\frac{k\,e^{\pm\mathrm{i}k\rho u}}{k^2-k_0^2}\right]\ .$$

The residues of the integrands in $E^+(u)$ and $E^-(u)$ (denoted by Res_+ and Res_-, respecively) at the poles $\pm k_0$ are

$$\mathrm{Res}_+(\pm k_0) = e^{\pm\mathrm{i}k_0\rho u}/2 \qquad \text{and} \qquad \mathrm{Res}_-(\pm k_0) = e^{\mp\mathrm{i}k_0\rho u}/2\ .$$

Therefore, referring to the contours shown in Fig. 5 (which are closed with infinite semicircles that, according to Jordan's lemma, do not contribute to the integrals), we calculate the four possible $\mathcal{D}$-values of $S(u)$ as follows

$$\begin{aligned}\mathcal{D}_{-+}S(u) &= \left[\mathcal{D}_{-+}E^+(u)-\mathcal{D}_{-+}E^-(u)\right]/(4\mathrm{i}) = \left[2\pi\mathrm{i}\mathrm{Res}_+(k_0)-(-2\pi\mathrm{i})\mathrm{Res}_-(-k_0)\right]/(4\mathrm{i})\\ &= (\pi/2)\left[e^{\mathrm{i}k_0\rho u}/2+e^{\mathrm{i}k_0\rho u}/2\right] = (\pi/2)\,e^{\mathrm{i}k_0\rho u}\end{aligned} \tag{20}$$

$$\begin{aligned}\mathcal{D}_{+-}S(u) &= \left[\mathcal{D}_{+-}E^+(u)-\mathcal{D}_{+-}E^-(u)\right]/(4\mathrm{i}) = \left[2\pi\mathrm{i}\mathrm{Res}_+(-k_0)-(-2\pi\mathrm{i})\mathrm{Res}_-(k_0)\right]/(4\mathrm{i})\\ &= (\pi/2)\left[e^{-\mathrm{i}k_0\rho u}/2+e^{-\mathrm{i}k_0\rho u}/2\right] = (\pi/2)\,e^{-\mathrm{i}k_0\rho u}\end{aligned} \tag{21}$$

$$\begin{aligned}\mathcal{D}_{++}S(u) &= \Big[\mathcal{D}_{++}E^+(u)-\overbrace{\mathcal{D}_{++}E^-(u)}^{0}\Big]/(4\mathrm{i}) = 2\pi\mathrm{i}\left[\mathrm{Res}_+(-k_0)+\mathrm{Res}_+(k_0)\right]/(4\mathrm{i})\\ &= (\pi/2)\left[e^{-\mathrm{i}k_0\rho u}/2+e^{\mathrm{i}k_0\rho u}/2\right] = (\pi/2)\cos(k_0\rho u)\end{aligned} \tag{22}$$

$$\begin{aligned}\mathcal{D}_{--}S(u) &= \Big[\overbrace{\mathcal{D}_{--}E^+(u)}^{0}-\mathcal{D}_{--}E^-(u)\Big]/(4\mathrm{i}) = -(-2\pi\mathrm{i})\left[2\pi\mathrm{i}\mathrm{Res}_-(-k_0)+\mathrm{Res}_-(k_0)\right]/(4\mathrm{i})\\ &= (\pi/2)\left[e^{\mathrm{i}k_0\rho u}/2+e^{-\mathrm{i}k_0\rho u}/2\right] = (\pi/2)\cos(k_0\rho u)\end{aligned} \tag{23}$$

In order to determine the $\mathcal{P}$-value of $S(u)$, let us first calculate the $\mathcal{P}$-values of $E^+(u)$ and $E^-(u)$ employing the contours in Figs. 5(c) and 5(h), respectively. Denoting the semicircles of radius $r \to 0$ used to circumvent the poles at $\pm k_0$ by $(\pm k_0, r)$ and the semicircle of radius $R \to \infty$ centered at the origin by $(0, R)$, we can write

$$\begin{aligned}\mathcal{P}E^{\pm}(u) &= \lim_{\substack{r\to 0\\ R\to\infty}}\left[\oint_{C^{\pm}} - \int_{(-k_0,r)} - \int_{(+k_0,r)} - \int_{(0,R)}^{0}\right]\frac{k\,e^{\pm\mathrm{i}k_0\rho u}\,dk}{k^2-k_0^2}\\ &= \pm 2\pi\mathrm{i}\left[\mathrm{Res}_{\pm}(-k_0)+\mathrm{Res}_{\pm}(k_0)\right] - (\pm\pi\mathrm{i})\mathrm{Res}_{\pm}(-k_0) - (\pm\pi\mathrm{i})\mathrm{Res}_{\pm}(k_0) - 0\\ &= \pm\pi\mathrm{i}\left[\mathrm{Res}_{\pm}(-k_0)+\mathrm{Res}_{\pm}(k_0)\right] = \pm\pi\mathrm{i}\left[e^{\mp\mathrm{i}k_0\rho u}/2+e^{\pm\mathrm{i}k_0\rho u}/2\right] = \pm\pi\mathrm{i}\cos(k_0\rho u)\ ,\end{aligned}$$

from which

$$\mathcal{P}S(u) = \left[\mathcal{P}E^+(u)-\mathcal{P}E^-(u)\right] = (\pi/2)\cos(k_0\rho u). \tag{24}$$

Among the several results for $S(u)$ calculated above, given by Eqs. (20) to (24), it is its $\mathcal{D}_{-+}$-value, given by Eq. (20), which is to be taken and substituted in Eq. (19), because, by doing so, we obtain the correct result, that given by Eq. (18)

$$\varGamma(\mathbf{r}\,|\,\mathbf{r}') = \frac{T^{-1}}{\pi^2}\int_1^\infty \frac{du\;(\pi/2)\;e^{\mathrm{i}k_0\rho u}}{\sqrt{u^2-1}} = \frac{\mathrm{i}}{4T}\,\mathrm{H}_0^{(1)}(k_0\rho).$$

Here, in the last step, we used the integral representation of the first Hankel function of order zero $\mathrm{H}_0^{(1)}(x) = (-2\mathrm{i}/\pi)\int_1^\infty du\, e^{\mathrm{i}ux}/\sqrt{u^2-1}$, given in Ref. [7], §6.13, Eq. (1).

4. Poisson equation

To calculate Green's function for the Poisson equation in an unbounded two-dimensional domain, that is, the solution of

$$\nabla^2\varOmega(\mathbf{r}\,|\,\mathbf{r}') = 2\pi\,\delta(\mathbf{r}-\mathbf{r}') \quad \left[\,\mathbf{r}\text{ and }\mathbf{r}'\text{ in }\mathbb{R}^2\,\right],$$

we again take the Fourier transform defined in Eq. (4), obtaining

$$-k^2\bar{\varOmega}(\mathbf{k}\,|\,\mathbf{r}') = e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}'} \quad\Longrightarrow\quad \bar{\varOmega}(\mathbf{k}\,|\,\mathbf{r}') = -e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}'}/k^2,$$

and then calculate the inverse Fourier transform:

$$\varOmega(\mathbf{r}\,|\,\mathbf{r}') = -\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{e^{-\mathrm{i}\mathbf{k}\cdot\boldsymbol{\rho}}}{k^2}\,d^2k \quad [\,\boldsymbol{\rho}\equiv\mathbf{r}-\mathbf{r}'\,]. \qquad (25)$$

To evaluate this integral, we act as in the case of the wave equation, that is, we choose the k_x-axis in the opposite direction of the vector $\boldsymbol{\rho}$, as in Fig. 2, and adopt the polar coordinates k and φ of $\mathbf{k}$. Next, we recognize the integral with respect to φ as the first integral representation of $\mathrm{J}_0(k\rho)$ given in Eq. (8). The result is the following function of ρ only

$$\varOmega(\mathbf{r}\,|\,\mathbf{r}') = -\frac{1}{2\pi}\int_0^{2\pi}\!\!\int_0^\infty \frac{e^{\mathrm{i}k\rho\cos\varphi}}{k^2}\,k\,dk\,d\varphi =$$

$$-\int_0^\infty\frac{dk}{k}\,\underbrace{\frac{1}{2\pi}\int_0^{2\pi}e^{\mathrm{i}k\rho\cos\varphi}}_{\mathrm{J}_0(k\rho)} = -\int_0^\infty dk\,\frac{\mathrm{J}_0(k\rho)}{k} = \varOmega(\rho).$$

The integral with respect to k becomes straightforward if we differentiate it with respect to ρ and then use the chain rule to change to a derivative with respect to k

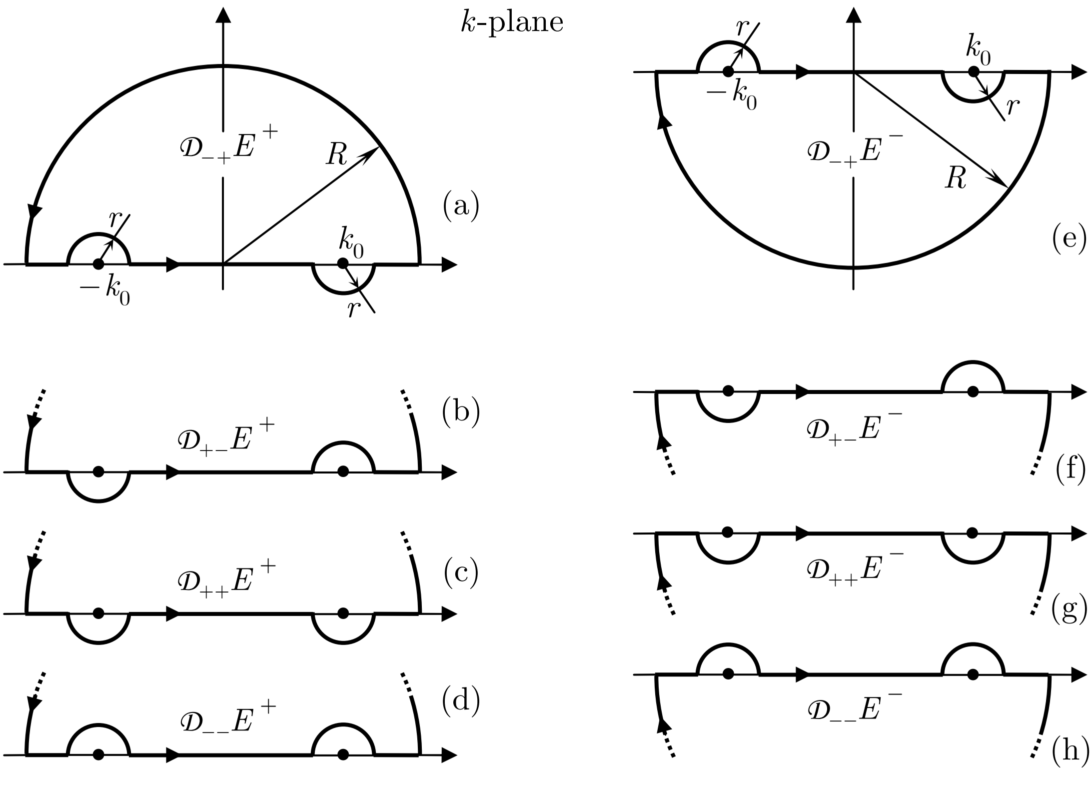

Figure 5 - The contours in the k-plane associated to the four possible $\mathcal{D}$-values of the integrals $E^+(u)$ (at the left) and $E^-(u)$ (at the right).

$$\Omega'(\rho) = -\int_0^\infty dk\,\frac{\partial}{\partial\rho}\frac{J_0(k\rho)}{k} =$$
$$-\int_0^\infty dk\,\frac{\partial}{\partial k}\frac{J_0(k\rho)}{\rho} = -\left[\frac{J_0(k\rho)}{\rho}\right]_{k=0}^\infty = \frac{1}{\rho}\ ,$$

since $J_0(x)$ equals 1 at $x = 0$ and goes to 0 as $x \to \infty$. Now, integrating with respect to ρ, we obtain the final result

$$\Omega(\mathbf{r}\,|\,\mathbf{r}') = \Omega(\rho) = \ln\rho + \text{constant} \quad [\,\rho \equiv |\,\mathbf{r} - \mathbf{r}'|\,]\,,$$

also obtained in Ref. [1], p. 169-170 (by a much more complicated method). The arbitrary additive constant is physically irrelevant. In fact, Green's function for the Poisson equation can be interpreted, for instance, as the electrostatic potential at $\mathbf{r}$ due to electrical charge concentrated at $\mathbf{r}'$, and only potential differences are relevant. However, unlike in the three-dimensional case, in which such constant is found to be zero by choosing the zero potential at infinite distances away from the *point charge* at $\mathbf{r}'$, this cannot be done in the two-dimensional case, because the potential diverges (logarithmically) as the distance from the *line charge* at $\mathbf{r}'$ increases.

5. Final comments

As said in the Introduction, the Green's functions considered here have well known expressions, which are obtained in a number of ways (*e.g.*, by descenting from the easier three-dimensional case). Therefore, we did not aimed at presenting new results but new methods. In this respect, concerning the footnote in the first page, we should say that, for the Helmholtz equation, the procedure adopted here differs considerably from that in Ref. [1], where that equation is converted into an ordinary differential equation (in the y variable) by means of a one-dimensional Fourier transform (in the x variable).

For the Helmholtz equation, two procedures for evaluating the inverse Fourier transform which furnishes the Green's functions were presented. They differ on the coordinates used to carry on the double integrals. It seems that the calculation employing the Cartesian coordinates is somewhat less cumbersome than that based on the polar coordinates.

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