

A simple derivation of the Lindblad equation

(Uma derivação simples da equação de Lindblad)

Carlos Alexandre Brasil¹, Felipe Fernandes Fanchini², Reginaldo de Jesus Napolitano³

¹*Instituto de Física “Gleb Wataghin”, Universidade Estadual de Campinas, Campinas, SP, Brasil*

²*Faculdade de Ciências de Bauru, Universidade Estadual Paulista “Julio de Mesquita Filho”, Bauru, SP, Brasil*

³*Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, SP, Brasil*

Recebido em 2/1/2012; Aceito em 29/8/2012; Publicado em 18/2/2013

We present a derivation of the Lindblad equation - an important tool for the treatment of nonunitary evolutions - that is accessible to undergraduate students in physics or mathematics with a basic background on quantum mechanics. We consider a specific case, corresponding to a very simple situation, where a primary system interacts with a bath of harmonic oscillators at zero temperature, with an interaction Hamiltonian that resembles the Jaynes-Cummings format. We start with the Born-Markov equation and, tracing out the bath degrees of freedom, we obtain an equation in the Lindblad form. The specific situation is very instructive, for it makes it easy to realize that the Lindblads represent the effect on the main system caused by the interaction with the bath, and that the Markov approximation is a fundamental condition for the emergence of the Lindbladian operator. The formal derivation of the Lindblad equation for a more general case requires the use of quantum dynamical semi-groups and broader considerations regarding the environment and temperature than we have considered in the particular case treated here.

Keywords: Lindblad equation, open quantum systems.

Apresentamos uma derivação da equação de Lindblad - uma ferramenta importante no tratamento de evoluções não-unitárias - acessível a estudantes de graduação em física ou matemática com noções básicas de mecânica quântica. Consideramos aqui um caso específico, correspondente a uma situação bem simples, onde o sistema principal interage com um banho de osciladores harmônicos à temperatura nula, com hamiltoniano de interação que se assemelha ao modelo de Jaynes-Cummings. Iniciamos com a equação de Born-Markov e, através do traço parcial dos graus de liberdade do banho, obtemos uma equação na forma de Lindblad. Essa situação específica é bem instrutiva, pois permite verificar que os lindblads representam a contribuição do sistema principal ao hamiltoniano de interação com o banho, e que a aproximação markoviana é vital para o surgimento do lindbladiano. A dedução formal da equação de Lindblad para situações gerais requer o uso do formalismo de semigrupos dinâmicos quânticos e considerações mais abrangentes sobre o ambiente e a temperatura do que as utilizadas aqui.

Palavras-chave: equação de Lindblad, sistemas quânticos abertos.

1. Introduction

The Lindblad equation [1] is the most general form for a Markovian master equation, and it is very important for the treatment of irreversible and non-unitary processes, from dissipation and decoherence [2] to the quantum measurement process [3,4]. For the latter, in recent applications [4,5], the Lindblad equation was used in the introduction of time in the interaction between the measured system and the measurement apparatus. Then, the measurement process is no longer treated as instantaneous, but finite, with the duration of that interaction changing the probabilities - diagonal elements of the density operator - associated to the possible final

results. On the other hand, in quantum optics, the analysis of spontaneous emission on a two-level system conducts to the Lindblad equation [6]. At last, in the case of quantum Brownian movement, it is possible to transform the Caldeira-Leggett equation [7] into Lindblad with the addition of a term that becomes small in the high-temperature limit [2]. These are a couple of many applications of the Lindblad equation, justifying its understanding by students in the early levels.

Contrasting against its importance and wide range of applications, its original deduction [1] involves the formalism of quantum dynamical semigroups [8,9], which is quite unfamiliar to most of the students and researchers. Other more recent ways to derive it in-

¹E-mail: carlosbrasil.physics@gmail.com.

volve the use of Itô stochastic calculus [10, 11] or, in the specific case of quantum measurements, considerations about the interaction between the system and the meter [12]. Another deduction, where the quantum dynamical semigroups are not explicitly used can be found on Ref. [2]. These methods, their assumptions, their applications, and, more importantly, their physical meanings appear very intimidating to beginning students.

To make the Lindblad equation more understandable, this article presents its deduction in the specific case of two systems: S , the principal system, and B , which can be the environment or the measurement apparatus, at zero temperature, with an interaction between them that resembles the one of the Jaynes-Cummings model [2]. Initially we derive the Born-Markov master equation [2] and then we trace out the degrees of freedom of system B . The Lindbladian emerges naturally as a consequence of the Markov approximation. Each Lindblad represents the effect on system S caused by the $S - B$ interaction.

Clearly, the present approach does not prove the general validity of the Lindblad equation. Our intention is simply to provide an accessible illustration of the validity of the Lindblad equation to non-specialists. The only prerequisite to follow the arguments exposed here is a basic knowledge of quantum mechanics, including a familiarity with the concepts of the density operator and the Liouville-von Neumann equation, at the level of Ref. [13], for example.

The paper is structured as follows: in the sec. 2 we derive the Born-Markov master equation by tracing out the degrees of freedom of system B , starting from the Liouville-von Neumann equation; in sec. 3 we derive the Lindblad equation; and in sec. 4 we present the conclusion.

2. The Born-Markov master equation

Let us consider a physical situation where a *principal* system S , whose dynamics is the object of interest, is coupled with another quantum system B , called *bath*. Here, \mathcal{H}_S and \mathcal{H}_B are, respectively, the Hilbert spaces of principal system S and bath B ; the global Hilbert space $S + B$ will be represented by the tensor-product space $\mathcal{H}_S \otimes \mathcal{H}_B$. The total Hamiltonian is

$$\hat{H}(t) = \hat{H}_S \otimes \hat{1}_B + \hat{1}_S \otimes \hat{H}_B + \alpha \hat{H}_{SB}, \quad (1)$$

where \hat{H}_S describes the principal system S , \hat{H}_B describes the bath B , \hat{H}_{SB} is the Hamiltonian for the system-bath interaction and $\hat{1}_B$ and $\hat{1}_S$ are the corresponding identities in the Hilbert spaces. Here, we will considerate \hat{H}_S and \hat{H}_B both time-independent. For the sake of simplicity, let us ignore the symbol \otimes and write

$$\hat{H}(t) = \hat{H}_S + \hat{H}_B + \alpha \hat{H}_{SB}. \quad (2)$$

Here, α is a real constant that provides the intensity of interaction between the principal system and the bath. Writing $\hat{\rho}_{SB}$ for the global density operator ($S + B$), the Liouville-von Neumann equation will be:

$$\frac{d}{dt} \hat{\rho}_{SB} = -\frac{i}{\hbar} \left[\hat{H}_S + \hat{H}_B + \alpha \hat{H}_{SB}, \hat{\rho}_{SB} \right]. \quad (3)$$

It is convenient to write Eq. (3) in the interaction picture of $\hat{H}_S + \hat{H}_B$. With the definitions of the new density operator and Hamiltonian:

$$\hat{H}(t) = e^{\frac{i}{\hbar}(\hat{H}_S + \hat{H}_B)t} \hat{H}_{SB} e^{-\frac{i}{\hbar}(\hat{H}_S + \hat{H}_B)t} \quad (4)$$

and

$$\hat{\rho}(t) = e^{\frac{i}{\hbar}(\hat{H}_S + \hat{H}_B)t} \hat{\rho}_{SB}(t) e^{-\frac{i}{\hbar}(\hat{H}_S + \hat{H}_B)t}, \quad (5)$$

the new equation for $\hat{\rho}(t)$ will be

$$\frac{d}{dt} \hat{\rho}(t) = -\frac{i}{\hbar} \alpha \left[\hat{H}(t), \hat{\rho}(t) \right]. \quad (6)$$

Here and in the following, we will use the time argument explicitated ((t)) to indicate the interaction-picture transformation.

We want to find the evolution for $\hat{\rho}_S(t) = \text{tr}_B \{ \hat{\rho}_{SB}(t) \}$ where, according Eq. (5),

$$\hat{\rho}_{SB}(t) = e^{-\frac{i}{\hbar}(\hat{H}_S + \hat{H}_B)t} \hat{\rho} e^{\frac{i}{\hbar}(\hat{H}_S + \hat{H}_B)t}. \quad (7)$$

Equation (6) is the starting point of our iterative approach. Its time derivative yields

$$\hat{\rho}(t) = \hat{\rho}(0) - \frac{i}{\hbar} \alpha \int_0^t \left[\hat{H}(t'), \hat{\rho}(t') \right] dt'. \quad (8)$$

Replacing Eq. (8) into Eq. (6), we have

$$\begin{aligned} \frac{d}{dt} \hat{\rho}(t) &= -\frac{i}{\hbar} \alpha \left[\hat{H}(t), \hat{\rho}(0) \right] - \\ &\frac{1}{\hbar^2} \alpha^2 \left[\hat{H}(t), \int_0^t \left[\hat{H}(t'), \hat{\rho}(t') \right] dt' \right]. \end{aligned} \quad (9)$$

For the *Born approximation*, Eq. (9) is enough. Then, we take the partial trace of the bath degrees of freedom,

$$\begin{aligned} \frac{d}{dt} \hat{\rho}_S(t) &= -\frac{i}{\hbar} \alpha \text{tr}_B \left\{ \left[\hat{H}(t), \hat{\rho}(0) \right] \right\} \\ &- \frac{1}{\hbar^2} \alpha^2 \text{tr}_B \left\{ \left[\hat{H}(t), \int_0^t \left[\hat{H}(t'), \hat{\rho}(t') \right] dt' \right] \right\}. \end{aligned} \quad (10)$$

By the definition in Eq. (4), $\hat{H}(t)$ depends on \hat{H}_{SB} , and \hat{H}_{SB} can always be defined in a manner in which the first term of the right hand side of Eq. (10) is zero. Hence, we obtain

$$\frac{d}{dt} \hat{\rho}_S(t) = -\frac{1}{\hbar^2} \alpha^2 \text{tr}_B \left\{ \left[\hat{H}(t), \int_0^t \left[\hat{H}(t'), \hat{\rho}(t') \right] dt' \right] \right\}. \quad (11)$$

Integrating Eq. (11) from t to t' yields

$$\hat{\rho}_S(t) - \hat{\rho}_S(t') = -\frac{1}{\hbar^2} \alpha^2 \int_t^{t'} dt' \text{tr}_B \left\{ \left[\hat{H}(t'), \int_0^{t'} [\hat{H}(t''), \hat{\rho}(t'')] dt'' \right] \right\},$$

which shows that the difference between $\hat{\rho}_S(t)$ and $\hat{\rho}_S(t')$ is of the second order of magnitude in α and, therefore, we can write $\hat{\rho}_S(t)$ in the integrand of Eq. (11), obtaining a time-local equation for the density operator, without violating the Born approximation

$$\frac{d}{dt} \hat{\rho}_S(t) = -\frac{1}{\hbar^2} \alpha^2 \text{tr}_B \left\{ \left[\hat{H}(t), \int_0^t [\hat{H}(t'), \hat{\rho}(t')] dt' \right] \right\}. \quad (12)$$

The α constant was introduced in Eq. (2) only for clarifying the order of magnitude of each term in the iteration and, now, it can be suppressed, that is, let us take $\alpha = 1$ (full interaction). Thus, let us write

$$\frac{d}{dt} \hat{\rho}_S(t) = -\frac{1}{\hbar^2} \text{tr}_B \left\{ \left[\hat{H}(t), \int_0^t [\hat{H}(t'), \hat{\rho}(t')] dt' \right] \right\}. \quad (13)$$

For this approximation, we can write $\hat{\rho}(t) = \hat{\rho}_S(t) \otimes \hat{\rho}_B$ inside the integral and obtain the equation that will be used in the next calculations (again, for the sake of simplicity, let us ignore the symbol \otimes)

$$\frac{d}{dt} \hat{\rho}_S(t) = -\frac{1}{\hbar^2} \int_0^\infty dt' \text{tr}_B \left\{ \left[\hat{H}(t), \left[\hat{H}(t'), \hat{\rho}_S(t) \hat{\rho}_B \right] \right] \right\}, \quad (14)$$

where we assume that the integration can be extended to infinity without changing its result. Equation (14) is the *Born-Markov master equation* [2].

3. Lindblad equation

3.1. The master equation commutator

Let us consider that system-bath interaction is of the following form,

$$\hat{H}_{SB} = \hbar \left(\hat{S} \hat{B}^\dagger + \hat{S}^\dagger \hat{B} \right), \quad (15)$$

where \hat{S} is a general operator that acts only on the principal system S , and \hat{B} is an operator that acts only on the bath B . Now, we consider that \hat{S} commutes with \hat{H}_S , i.e.,

$$[\hat{S}, \hat{H}_S] = 0,$$

resulting in

$$\hat{S}(t) = \hat{S} \quad (16)$$

(\hat{S} is not affected by the interaction-picture transformation). Let us consider the bath hamiltonian defined by a bath of bosons,

$$\hat{H}_B = \hbar \sum_k \omega_k \hat{a}_k^\dagger \hat{a}_k \quad (17)$$

where \hat{a}_k e \hat{a}_k^\dagger are the annihilation and creation bath operators, the ω_k are the characteristic frequencies of each mode, and the \hat{B} operator on Eq. (15) defined by

$$\hat{B} = \sum_k g_k^* \hat{a}_k, \quad (18)$$

where g_k are complex coefficients representing coupling constantes. Then, in the interaction picture,

$$\hat{B}(t) = e^{\frac{i}{\hbar} \hat{H}_B t} \hat{B} e^{-\frac{i}{\hbar} \hat{H}_B t}. \quad (19)$$

Expanding each exponential and using the commutator relations, Eq. (19) will result in

$$\hat{B}(t) = \sum_k g_k^* \hat{a}_k e^{-i\omega_k t}. \quad (20)$$

The interaction (15) with the definition (18) resembles the *Jaynes-Cummings* one, who represents a single two-level atom interacting with a single mode of the radiation field [2, 14].

With this, the commutator in Eq. (14), $[\hat{H}(t), [\hat{H}(t'), \hat{\rho}_S(t) \hat{\rho}_B]]$, will be evaluated. Firstly,

$$\begin{aligned} [\hat{H}(t), [\hat{H}(t'), \hat{\rho}_B \hat{\rho}_S(t)]] &= \hbar \left[\hat{S} \hat{B}^\dagger(t) + \hat{S}^\dagger \hat{B}(t), [\hat{H}(t'), \hat{\rho}_B \hat{\rho}_S(t)] \right] \\ &= \hbar \left[\hat{S} \hat{B}^\dagger(t), [\hat{H}(t'), \hat{\rho}_B \hat{\rho}_S(t)] \right] \\ &+ \hbar \left[\hat{S}^\dagger \hat{B}(t), [\hat{H}(t'), \hat{\rho}_B \hat{\rho}_S(t)] \right]. \end{aligned} \quad (21)$$

The gradual expansion of each term in Eq. (21) will result in

$$\begin{aligned}
\left[\hat{S}\hat{B}^\dagger(t), \left[\hat{H}(t'), \hat{\rho}_B\hat{\rho}_S(t) \right] \right] &= \hbar \left[\hat{S}\hat{B}^\dagger(t), \left[\hat{S}\hat{B}^\dagger(t') + \hat{S}^\dagger\hat{B}(t'), \hat{\rho}_B\hat{\rho}_S(t) \right] \right] \\
&= \hbar\hat{S}\hat{B}^\dagger(t) \left[\hat{S}\hat{B}^\dagger(t') + \hat{S}^\dagger\hat{B}(t') \right] \hat{\rho}_B\hat{\rho}_S(t) \\
&\quad - \hbar\hat{S}\hat{B}^\dagger(t) \hat{\rho}_B\hat{\rho}_S(t) \left[\hat{S}\hat{B}^\dagger(t') + \hat{S}^\dagger\hat{B}(t') \right] \\
&\quad - \hbar \left[\hat{S}\hat{B}^\dagger(t') + \hat{S}^\dagger\hat{B}(t') \right] \hat{\rho}_B\hat{\rho}_S(t) \hat{S}\hat{B}^\dagger(t) \\
&\quad + \hbar\hat{\rho}_B\hat{\rho}_S(t) \left[\hat{S}\hat{B}^\dagger(t') + \hat{S}^\dagger\hat{B}(t') \right] \hat{S}\hat{B}^\dagger(t)
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
\left[\hat{S}^\dagger\hat{B}(t), \left[\hat{H}(t'), \hat{\rho}_B\hat{\rho}_S(t) \right] \right] &= \hbar \left[\hat{S}^\dagger\hat{B}(t), \left[\hat{S}\hat{B}^\dagger(t') + \hat{S}^\dagger\hat{B}(t'), \hat{\rho}_B\hat{\rho}_S(t) \right] \right] \\
&= \hbar\hat{S}^\dagger\hat{B}(t) \left[\hat{S}\hat{B}^\dagger(t') + \hat{S}^\dagger\hat{B}(t') \right] \hat{\rho}_B\hat{\rho}_S(t) \\
&\quad - \hbar\hat{S}^\dagger\hat{B}(t) \hat{\rho}_B\hat{\rho}_S(t) \left[\hat{S}\hat{B}^\dagger(t') + \hat{S}^\dagger\hat{B}(t') \right] \\
&\quad - \hbar \left[\hat{S}\hat{B}^\dagger(t') + \hat{S}^\dagger\hat{B}(t') \right] \hat{\rho}_B\hat{\rho}_S(t) \hat{S}^\dagger\hat{B}(t) \\
&\quad + \hbar\hat{\rho}_B\hat{\rho}_S(t) \left[\hat{S}\hat{B}^\dagger(t') + \hat{S}^\dagger\hat{B}(t') \right] \hat{S}^\dagger\hat{B}(t),
\end{aligned} \tag{23}$$

or, expanding Eqs. (22) and (23) and grouping the similar terms in S and B , we have

$$\begin{aligned}
\left[\hat{S}\hat{B}^\dagger(t), \left[\hat{H}(t'), \hat{\rho}_B\hat{\rho}_S(t) \right] \right] &= \hbar\hat{S}\hat{S}\hat{\rho}_S(t) \hat{B}^\dagger(t) \hat{B}^\dagger(t') \hat{\rho}_B + \hbar\hat{S}\hat{S}^\dagger\hat{\rho}_S(t) \hat{B}^\dagger(t) \hat{B}(t') \hat{\rho}_B \\
&\quad - \hbar\hat{S}\hat{\rho}_S(t) \hat{S}\hat{B}^\dagger(t) \hat{\rho}_B\hat{B}^\dagger(t') - \hbar\hat{S}\hat{\rho}_S(t) \hat{S}^\dagger\hat{B}^\dagger(t) \hat{\rho}_B\hat{B}(t') \\
&\quad - \hbar\hat{S}\hat{\rho}_S(t) \hat{S}\hat{B}^\dagger(t') \hat{\rho}_B\hat{B}^\dagger(t) - \hbar\hat{S}^\dagger\hat{\rho}_S(t) \hat{S}\hat{B}(t') \hat{\rho}_B\hat{B}^\dagger(t) \\
&\quad + \hbar\hat{\rho}_S(t) \hat{S}\hat{S}\hat{\rho}_B\hat{B}^\dagger(t') \hat{B}^\dagger(t) + \hbar\hat{\rho}_S(t) \hat{S}^\dagger\hat{S}\hat{\rho}_B\hat{B}(t') \hat{B}^\dagger(t)
\end{aligned} \tag{24}$$

and

$$\begin{aligned}
\left[\hat{S}^\dagger\hat{B}(t), \left[\hat{H}(t'), \hat{\rho}_B\hat{\rho}_S(t) \right] \right] &= \hbar\hat{S}^\dagger\hat{S}\hat{\rho}_S(t) \hat{B}(t) \hat{B}^\dagger(t') \hat{\rho}_B + \hbar\hat{S}^\dagger\hat{S}^\dagger\hat{\rho}_S(t) \hat{B}(t) \hat{B}(t') \hat{\rho}_B \\
&\quad - \hbar\hat{S}^\dagger\hat{\rho}_S(t) \hat{S}\hat{B}(t) \hat{\rho}_B\hat{B}^\dagger(t') - \hbar\hat{S}^\dagger\hat{\rho}_S(t) \hat{S}^\dagger\hat{B}(t) \hat{\rho}_B\hat{B}(t') \\
&\quad - \hbar^2\hat{S}\hat{\rho}_S(t) \hat{S}^\dagger\hat{B}^\dagger(t') \hat{\rho}_B\hat{B}(t) - \hbar\hat{S}^\dagger\hat{\rho}_S(t) \hat{S}^\dagger\hat{B}(t') \hat{\rho}_B\hat{B}(t) \\
&\quad + \hbar\hat{\rho}_S(t) \hat{S}\hat{S}^\dagger\hat{\rho}_B\hat{B}^\dagger(t') \hat{B}(t) + \hbar\hat{\rho}_S(t) \hat{S}^\dagger\hat{S}^\dagger\hat{\rho}_B\hat{B}(t') \hat{B}(t).
\end{aligned} \tag{25}$$

3.2. The partial trace

Now we are in a position to trace out the bath degrees of freedom in Eqs. (24) and (25). As we can verify with Eq. (20),

$$tr_B \left\{ \hat{B}(t) \hat{B}(t') \hat{\rho}_B \right\} = tr_B \left\{ \hat{B}^\dagger(t) \hat{B}^\dagger(t') \hat{\rho}_B \right\} = 0, \forall t, t'.$$

With this, then,

$$\begin{aligned}
tr_B \left\{ \left[\hat{S}\hat{B}^\dagger(t), \left[\hat{H}(t'), \hat{\rho}_B\hat{\rho}_S(t) \right] \right] \right\} &= \hbar\hat{S}\hat{S}^\dagger\hat{\rho}_S(t) tr_B \left\{ \hat{B}^\dagger(t) \hat{B}(t') \hat{\rho}_B \right\} \\
&\quad - \hbar\hat{S}\hat{\rho}_S(t) \hat{S}^\dagger tr_B \left\{ \hat{B}^\dagger(t) \hat{\rho}_B\hat{B}(t') \right\} \\
&\quad - \hbar\hat{S}^\dagger\hat{\rho}_S(t) \hat{S} tr_B \left\{ \hat{B}(t') \hat{\rho}_B\hat{B}^\dagger(t) \right\} \\
&\quad + \hbar\hat{\rho}_S(t) \hat{S}^\dagger\hat{S} tr_B \left\{ \hat{\rho}_B\hat{B}(t') \hat{B}^\dagger(t) \right\}
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
tr_B \left\{ \left[\hat{S}^\dagger \hat{B}(t), \left[\hat{H}(t'), \hat{\rho}_B \hat{\rho}_S(t) \right] \right] \right\} &= \hbar \hat{S}^\dagger \hat{S} \hat{\rho}_S(t) tr_B \left\{ \hat{B}(t) \hat{B}^\dagger(t') \hat{\rho}_B \right\} \\
&- \hbar \hat{S}^\dagger \hat{\rho}_S(t) \hat{S} tr_B \left\{ \hat{B}(t) \hat{\rho}_B \hat{B}^\dagger(t') \right\} \\
&- \hbar \hat{S} \hat{\rho}_S(t) \hat{S}^\dagger tr_B \left\{ \hat{B}^\dagger(t') \hat{\rho}_B \hat{B}(t) \right\} \\
&+ \hbar \hat{\rho}_S(t) \hat{S} \hat{S}^\dagger tr_B \left\{ \hat{\rho}_B \hat{B}^\dagger(t') \hat{B}(t) \right\}, \tag{27}
\end{aligned}$$

where, if we use the cyclic properties of the trace,

$$\begin{aligned}
tr_B \left\{ \left[\hat{S} \hat{B}^\dagger(t), \left[\hat{H}(t'), \hat{\rho}_B \hat{\rho}_S(t) \right] \right] \right\} &= \hbar \left[\hat{S} \hat{S}^\dagger \hat{\rho}_S(t) - \hat{S}^\dagger \hat{\rho}_S(t) \hat{S} \right] tr_B \left\{ \hat{B}^\dagger(t) \hat{B}(t') \hat{\rho}_B \right\} \\
&+ \hbar \left[\hat{\rho}_S(t) \hat{S}^\dagger \hat{S} - \hat{S} \hat{\rho}_S(t) \hat{S}^\dagger \right] tr_B \left\{ \hat{B}(t') \hat{B}^\dagger(t) \hat{\rho}_B \right\} \tag{28}
\end{aligned}$$

and

$$\begin{aligned}
tr_B \left\{ \left[\hat{S}^\dagger \hat{B}(t), \left[\hat{H}(t'), \hat{\rho}_B \hat{\rho}_S(t) \right] \right] \right\} &= \hbar \left[\hat{S}^\dagger \hat{S} \hat{\rho}_S(t) - \hat{S} \hat{\rho}_S(t) \hat{S}^\dagger \right] tr_B \left\{ \hat{B}(t) \hat{B}^\dagger(t') \hat{\rho}_B \right\} \\
&+ \hbar \left[\hat{\rho}_S(t) \hat{S} \hat{S}^\dagger - \hat{S}^\dagger \hat{\rho}_S(t) \hat{S} \right] tr_B \left\{ \hat{B}^\dagger(t') \hat{B}(t) \hat{\rho}_B \right\}. \tag{29}
\end{aligned}$$

The terms represented by Eqs. (28) and (29) allow us to return to the Eq. (21)

$$\begin{aligned}
tr_B \left\{ \left[\hat{H}(t), \left[\hat{H}(t'), \hat{\rho}_B \hat{\rho}_S(t) \right] \right] \right\} &= \hbar^2 \left[\hat{S} \hat{S}^\dagger \hat{\rho}_S(t) - \hat{S}^\dagger \hat{\rho}_S(t) \hat{S} \right] tr_B \left\{ \hat{B}^\dagger(t) \hat{B}(t') \hat{\rho}_B \right\} \\
&+ \hbar^2 \left[\hat{\rho}_S(t) \hat{S}^\dagger \hat{S} - \hat{S} \hat{\rho}_S(t) \hat{S}^\dagger \right] tr_B \left\{ \hat{B}(t') \hat{B}^\dagger(t) \hat{\rho}_B \right\} \\
&+ \hbar^2 \left[\hat{S}^\dagger \hat{S} \hat{\rho}_S(t) - \hat{S} \hat{\rho}_S(t) \hat{S}^\dagger \right] tr_B \left\{ \hat{B}(t) \hat{B}^\dagger(t') \hat{\rho}_B \right\} \\
&+ \hbar^2 \left[\hat{\rho}_S(t) \hat{S} \hat{S}^\dagger - \hat{S}^\dagger \hat{\rho}_S(t) \hat{S} \right] tr_B \left\{ \hat{B}^\dagger(t') \hat{B}(t) \hat{\rho}_B \right\}. \tag{30}
\end{aligned}$$

3.3. The expansion of the integrand of the master equation

With the results of the preceding paragraphs, the integrand in Eq. (14) becomes

$$\begin{aligned}
tr_B \left\{ \left[\hat{H}(t), \left[\hat{H}(t'), \hat{\rho}_B \hat{\rho}_S(t) \right] \right] \right\} &= \hbar^2 \left[\hat{S} \hat{S}^\dagger \hat{\rho}_S(t) - \hat{S}^\dagger \hat{\rho}_S(t) \hat{S} \right] tr_B \left\{ \hat{B}^\dagger(t) \hat{B}(t') \hat{\rho}_B \right\} \\
&+ \hbar^2 \left[\hat{\rho}_S(t) \hat{S}^\dagger \hat{S} - \hat{S} \hat{\rho}_S(t) \hat{S}^\dagger \right] tr_B \left\{ \hat{B}(t') \hat{B}^\dagger(t) \hat{\rho}_B \right\} \\
&+ \hbar^2 \left[\hat{S}^\dagger \hat{S} \hat{\rho}_S(t) - \hat{S} \hat{\rho}_S(t) \hat{S}^\dagger \right] tr_B \left\{ \hat{B}(t) \hat{B}^\dagger(t') \hat{\rho}_B \right\} \\
&+ \hbar^2 \left[\hat{\rho}_S(t) \hat{S} \hat{S}^\dagger - \hat{S}^\dagger \hat{\rho}_S(t) \hat{S} \right] tr_B \left\{ \hat{B}^\dagger(t') \hat{B}(t) \hat{\rho}_B \right\}. \tag{31}
\end{aligned}$$

For convenience, let us define the functions

$$\begin{aligned}
F(t) &= \int_0^t dt' tr_B \left\{ \hat{B}(t) \hat{B}^\dagger(t') \hat{\rho}_B \right\}, \\
G(t) &= \int_0^t dt' tr_B \left\{ \hat{B}^\dagger(t') \hat{B}(t) \hat{\rho}_B \right\}. \tag{32}
\end{aligned}$$

Then,

$$\begin{aligned}
F^*(t) &= \int_0^t dt' tr_B \left\{ \hat{B}(t') \hat{B}^\dagger(t) \hat{\rho}_B \right\}, \\
G^*(t) &= \int_0^t dt' tr_B \left\{ \hat{B}^\dagger(t) \hat{B}(t') \hat{\rho}_B \right\}.
\end{aligned}$$

Replacing Eq. (31) in Eq. (14) yields

$$\begin{aligned} \frac{d}{dt} \hat{\rho}_S(t) &= - \left[\hat{S} \hat{S}^\dagger \hat{\rho}_S(t) - \hat{S}^\dagger \hat{\rho}_S(t) \hat{S} \right] G^*(t) - \left[\hat{\rho}_S(t) \hat{S}^\dagger \hat{S} - \hat{S} \hat{\rho}_S(t) \hat{S}^\dagger \right] F^*(t) \\ &\quad - \left[\hat{S}^\dagger \hat{S} \hat{\rho}_S(t) - \hat{S} \hat{\rho}_S(t) \hat{S}^\dagger \right] F(t) - \left[\hat{\rho}_S(t) \hat{S} \hat{S}^\dagger - \hat{S}^\dagger \hat{\rho}_S(t) \hat{S} \right] G(t). \end{aligned} \quad (33)$$

Actually, the usual Lindblad equation emerges when $G(t) = 0$ and $F(t) = F^*(t)$. In the following, we make some specifications about the environment to discuss these approximations in detail.

3.4. The bath specification

Furthermore, for the initial state of the thermal bath, we consider the vacuum state

$$\hat{\rho}_B = (|0\rangle |0\rangle \dots) \otimes (\langle 0| \langle 0| \dots). \quad (34)$$

The evaluation of the $F(t)$ and $G(t)$ functions defined in Eq. (32) are done considering the $\hat{B}(t)$ and $\hat{\rho}_B$ definitions in Eqs. (20) and (34). By Eq. (20), $\hat{B}^\dagger(t)$ is

$$\hat{B}^\dagger(t) = \sum_k g_k \hat{a}_k^\dagger e^{i\omega_k t}. \quad (35)$$

Then, the partial trace in $F(t)$ and $G(t)$ can be evaluated

$$tr_B \left\{ \hat{B}(t) \hat{B}^\dagger(t') \hat{\rho}_B \right\} = tr_B \left\{ \hat{B}(t) \hat{B}^\dagger(t') (|0\rangle |0\rangle \dots) \otimes (\langle 0| \langle 0| \dots) \right\} \quad (36)$$

and

$$tr_B \left\{ \hat{B}^\dagger(t') \hat{B}(t) \hat{\rho}_B \right\} = tr_B \left\{ \hat{B}^\dagger(t') \hat{B}(t) (|0\rangle |0\rangle \dots) \otimes (\langle 0| \langle 0| \dots) \right\}. \quad (37)$$

If we use some bath state basis $\{|b\rangle\}$, Eqs. (36) and (37) become

$$\begin{aligned} tr_B \left\{ \hat{B}(t) \hat{B}^\dagger(t') \hat{\rho}_B \right\} &= \sum_b \langle b| \hat{B}(t) \hat{B}^\dagger(t') (|0\rangle |0\rangle \dots) \otimes (\langle 0| \langle 0| \dots) |b\rangle \\ &= \sum_b (\langle 0| \langle 0| \dots) |b\rangle \langle b| \hat{B}(t) \hat{B}^\dagger(t') (|0\rangle |0\rangle \dots) \\ &= (\langle 0| \langle 0| \dots) \sum_b |b\rangle \langle b| \hat{B}(t) \hat{B}^\dagger(t') (|0\rangle |0\rangle \dots) \\ &= (\langle 0| \langle 0| \dots) \hat{B}(t) \hat{B}^\dagger(t') (|0\rangle |0\rangle \dots) \end{aligned} \quad (38)$$

and

$$\begin{aligned} tr_B \left\{ \hat{B}^\dagger(t') \hat{B}(t) \hat{\rho}_B \right\} &= \sum_b \langle b| \hat{B}^\dagger(t') \hat{B}(t) (|0\rangle |0\rangle \dots) \otimes (\langle 0| \langle 0| \dots) |b\rangle \\ &= \sum_b (\langle 0| \langle 0| \dots) |b\rangle \langle b| \hat{B}^\dagger(t') \hat{B}(t) (|0\rangle |0\rangle \dots) \\ &= (\langle 0| \langle 0| \dots) \sum_b |b\rangle \langle b| \hat{B}^\dagger(t') \hat{B}(t) (|0\rangle |0\rangle \dots) \\ &= (\langle 0| \langle 0| \dots) \hat{B}^\dagger(t') \hat{B}(t) (|0\rangle |0\rangle \dots). \end{aligned} \quad (39)$$

Let us expand $\hat{B}^\dagger(t)$ and $\hat{B}(t)$ using Eqs. (20) and (35)

$$\begin{aligned} tr_B \left\{ \hat{B}(t) \hat{B}^\dagger(t') \hat{\rho}_B \right\} &= (\langle 0| \langle 0| \dots) \sum_k g_k^* \hat{a}_k e^{-i\omega_k t} \sum_{k'} g_{k'} \hat{a}_{k'}^\dagger e^{i\omega_{k'} t'} (|0\rangle |0\rangle \dots) \\ &= \sum_{k, k'} g_k^* g_{k'} e^{-i(\omega_k t - \omega_{k'} t')} (\langle 0| \langle 0| \dots) \hat{a}_k \hat{a}_{k'}^\dagger (|0\rangle |0\rangle \dots) \end{aligned} \quad (40)$$

and

$$\begin{aligned} \text{tr}_B \left\{ \hat{B}^\dagger(t') \hat{B}(t) \rho_B \right\} &= (\langle 0| \langle 0| \dots) \sum_{k'} g_{k'} \hat{a}_{k'}^\dagger e^{i\omega_{k'} t'} \sum_k g_k^* \hat{a}_k e^{-i\omega_k t} (|0\rangle |0\rangle \dots) \\ &= \sum_{k, k'} g_k^* g_{k'} e^{-i(\omega_k t - \omega_{k'} t')} (\langle 0| \langle 0| \dots) \hat{a}_{k'}^\dagger \hat{a}_k (|0\rangle |0\rangle \dots) = 0 \end{aligned} \quad (41)$$

Hence, we can rewrite Eq. (40) with the \hat{a}_k^\dagger operators on the left of the \hat{a}_k operators. We know that

$$\hat{a}_k \hat{a}_{k'}^\dagger = \delta_{k, k'} + \hat{a}_{k'}^\dagger \hat{a}_k. \quad (42)$$

Then

$$\begin{aligned} \text{tr}_B \left\{ \hat{B}(t) \hat{B}^\dagger(t') \hat{\rho}_B \right\} &= \sum_{k, k'} g_k^* g_{k'} e^{-i(\omega_k t - \omega_{k'} t')} \delta_{k, k'} + \sum_{k, k'} g_k^* g_{k'} e^{-i(\omega_k t - \omega_{k'} t')} (\langle 0| \langle 0| \dots) \hat{a}_{k'}^\dagger \hat{a}_k (|0\rangle |0\rangle \dots) \\ &= \sum_k |g_k|^2 e^{-i\omega_k (t-t')}. \end{aligned} \quad (43)$$

Therefore, from Eqs. (41) and (43), it follows that

$$\begin{aligned} F(t) &= \sum_k |g_k|^2 \int_0^t dt' e^{-i\omega_k (t-t')}, \\ G(t) &= 0. \end{aligned} \quad (44)$$

3.5. Transition to the continuum

In the expression of $F(t)$ in Eq. (44), if we adopt the general definition of the density of states as

$$J(\omega) = \sum_l |g_l|^2 \delta(\omega - \omega_l), \quad (45)$$

then the sum over k can be replaced by an integral over a continuum of frequencies

$$F(t) = \int_0^\infty d\omega J(\omega) \int_0^t dt' e^{-i\omega (t-t')}.$$

Let us introduce the new variable

$$\begin{aligned} \tau &= t - t', \\ d\tau &= -dt', \end{aligned}$$

with

$$\int_0^t dt' = -\int_t^0 d\tau = \int_0^t d\tau,$$

yielding

$$F(t) = \int_0^\infty d\omega J(\omega) \int_0^t d\tau e^{-i\omega \tau}.$$

3.6. The Markov approximation

In the Markov approximation, the limit $t \rightarrow \infty$ is taken in the time integral, as we have mentioned regarding

Eq. (14), that is, we take

$$\int_0^\infty d\tau e^{-i\omega \tau}.$$

As the integrand oscillates, we will use the device

$$\begin{aligned} \int_0^\infty d\tau e^{-i\omega \tau} &= \lim_{\eta \rightarrow 0^+} \int_0^\infty d\tau e^{-i\omega \tau - \eta \tau} \\ &= \lim_{\eta \rightarrow 0^+} \frac{1}{\eta + i\omega} \\ &= \lim_{\eta \rightarrow 0^+} \frac{\eta - i\omega}{\eta^2 + \omega^2} \\ &= \lim_{\eta \rightarrow 0^+} \frac{\eta}{\eta^2 + \omega^2} - \lim_{\eta \rightarrow 0^+} \frac{i\omega}{\eta^2 + \omega^2} \\ &= \pi \delta(\omega) - iP \frac{1}{\omega}, \end{aligned}$$

where P stands for the *Cauchy principal part*. Then,

$$F = \pi \int_0^\infty d\omega J(\omega) \delta(\omega) - iP \int_0^\infty d\omega \frac{J(\omega)}{\omega}.$$

3.7. The final form

For a general density of states, F yields

$$F = \frac{\gamma + i\varepsilon}{2}, \quad (46)$$

where

$$\gamma \equiv 2\pi \int_0^\infty d\omega J(\omega) \delta(\omega), \quad (47)$$

$$\varepsilon \equiv -2P \int_0^\infty d\omega \frac{J(\omega)}{\omega}. \quad (48)$$

As we have verified that $G = 0$, let us replace Eq. (46) in Eq. (33)

$$\begin{aligned} \frac{d}{dt} \hat{\rho}_S(t) &= - \left[\hat{\rho}_S(t) \hat{S}^\dagger \hat{S} - \hat{S} \hat{\rho}_S(t) \hat{S}^\dagger \right] \frac{\gamma - i\varepsilon}{2} - \left[\hat{S}^\dagger \hat{S} \hat{\rho}_S(t) - \hat{S} \hat{\rho}_S(t) \hat{S}^\dagger \right] \frac{\gamma + i\varepsilon}{2} \\ &= - \frac{\gamma}{2} \left[\hat{\rho}_S(t) \hat{S}^\dagger \hat{S} - \hat{S} \hat{\rho}_S(t) \hat{S}^\dagger + \hat{S}^\dagger \hat{S} \hat{\rho}_S(t) - \hat{S} \hat{\rho}_S(t) \hat{S}^\dagger \right] \\ &\quad + i \frac{\varepsilon}{2} \left[\hat{\rho}_S(t) \hat{S}^\dagger \hat{S} - \hat{S} \hat{\rho}_S(t) \hat{S}^\dagger - \hat{S}^\dagger \hat{S} \hat{\rho}_S(t) - \hat{S} \hat{\rho}_S(t) \hat{S}^\dagger \right]. \end{aligned}$$

If the density of states is chosen to yield $\varepsilon = 0$ (a *Lorentzian*, for example, where we can extend the lower limit of integration to $-\infty$), the final result is

$$\frac{d}{dt} \hat{\rho}_S(t) = \gamma \left[\hat{S} \hat{\rho}_S(t) \hat{S}^\dagger - \frac{1}{2} \left\{ \hat{S}^\dagger \hat{S}, \hat{\rho}_S(t) \right\} \right]. \quad (49)$$

Let us, then, return to the original picture. Since

$$\hat{\rho}_S(t) = e^{\frac{i}{\hbar} \hat{H}_S t} \hat{\rho}_S e^{-\frac{i}{\hbar} \hat{H}_S t}, \quad (50)$$

then

$$\begin{aligned} \frac{d}{dt} \hat{\rho}_S(t) &= \frac{i}{\hbar} e^{\frac{i}{\hbar} \hat{H}_S t} \hat{H}_S \hat{\rho}_S e^{-\frac{i}{\hbar} \hat{H}_S t} + e^{\frac{i}{\hbar} \hat{H}_S t} \frac{d\hat{\rho}_S}{dt} e^{-\frac{i}{\hbar} \hat{H}_S t} - \frac{i}{\hbar} e^{\frac{i}{\hbar} \hat{H}_S t} \hat{\rho}_S \hat{H}_S e^{-\frac{i}{\hbar} \hat{H}_S t} \\ &= e^{\frac{i}{\hbar} \hat{H}_S t} \frac{d\hat{\rho}_S}{dt} e^{-\frac{i}{\hbar} \hat{H}_S t} + \frac{i}{\hbar} e^{\frac{i}{\hbar} \hat{H}_S t} \left[\hat{H}_S, \hat{\rho}_S \right] e^{-\frac{i}{\hbar} \hat{H}_S t}. \end{aligned} \quad (51)$$

Performing the same operation on the right-hand side of Eq. (49) gives

$$\left[\hat{S} \hat{\rho}_S(t) \hat{S}^\dagger - \frac{1}{2} \left\{ \hat{S}^\dagger \hat{S}, \hat{\rho}_S(t) \right\} \right] = e^{\frac{i}{\hbar} \hat{H}_S t} \left[\hat{S} \hat{\rho}_S \hat{S}^\dagger - \frac{1}{2} \left\{ \hat{S}^\dagger \hat{S}, \hat{\rho}_S \right\} \right] e^{-\frac{i}{\hbar} \hat{H}_S t}. \quad (52)$$

Replacing Eqs. (51) and (52) in Eq. (49), we obtain

$$\frac{d\hat{\rho}_S}{dt} = -\frac{i}{\hbar} \left[\hat{H}_S, \hat{\rho}_S \right] + \gamma \left[\hat{S} \hat{\rho}_S \hat{S}^\dagger - \frac{1}{2} \left\{ \hat{S}^\dagger \hat{S}, \hat{\rho}_S \right\} \right]. \quad (53)$$

4. Conclusion

In summary, in this paper we consider an interaction that resembles the *Jaynes-Cummings interaction* [2], Eq. (15), between a bath and a system S , assuming that the operator \hat{S} commutes with the system Hamiltonian, \hat{H}_S , at zero temperature. We substituted Eq. (15) in the Born-Markov master Eq. (14) and took the partial trace of the degrees of freedom of B . The $T = 0$ hypothesis is necessary to simplify the calculations, making them more accessible to the students, simplifying the treatment of Eqs. (36) and (37),

and avoiding complications such as the Lamb shift in Eq. (46). The Markov approximation in sec. 3-F was also vital to obtain the final result, Eq. (53). All these simplifications limit the validity of our derivation to more general cases, but it provides a detailed illustration of the physical meaning of each term appearing in the Lindblad equation.

Equation (53) is commonly presented with several \hat{S} operators, usually denoted by \hat{L} , in a linear combination of Lindbladian operators. The \hat{L} operators are named *Lindblad operators* and, in the general case, the Lindblad equation takes the form

$$\frac{d\hat{\rho}_S}{dt} = -\frac{i}{\hbar} [\hat{H}_S, \hat{\rho}_S] + \gamma \sum_j \left[\hat{L}_j \hat{\rho}_S \hat{L}_j^\dagger - \frac{1}{2} \{ \hat{L}_j^\dagger \hat{L}_j, \hat{\rho}_S \} \right]. \quad (54)$$

If we consider only the first term on the right hand side of Eq. (54), we obtain the *Liouville-von Neumann equation*. This term is the *Liouillian* and describes the *unitary evolution* of the density operator. The second term on the right hand side of the Eq. (53) is the *Lindbladian* and it emerges when we take the partial trace - a *non-unitary operation* - of the degrees of freedom of system B . The Lindbladian describes the *non-unitary evolution* of the density operator. By the interaction form adopted here, Eq. (15), the physical meaning of the Lindblad operators can be understood: they represent the system S contribution to the $S-B$ interaction - remembering once more that the Lindblad equation was derived from the Liouville-von Neumann one by tracing the bath degrees of freedom. This conclusion is also achieved with the more general derivation [1,2]. It is important to emphasize that, due to our simplifying assumptions, the summation appearing in Eq. (54) was not obtained in our derivation of Eq. (53).

If the Lindblad operators \hat{L}_j are Hermitian (observables), the Lindblad equation can be used to treat the measurement process. A simple application in this sense is the Hamiltonian $\hat{H}_S \propto \hat{\sigma}_z$ ($\hat{\sigma}_z$ is the 2-level z -Pauli matrix) when we want to measure one specific component of the spin ($\hat{L} \propto \hat{\sigma}_\alpha$, $\alpha = x, y, z$, without the summation) [3,5]. If the Lindblads are non-Hermitian, the equation can be used to treat dissipation, decoherence or decays. For this, a simple example is the same Hamiltonian $\hat{H}_S \propto \hat{\sigma}_z$ with the Lindblad $\hat{L} \propto \hat{\sigma}_-$ ($\hat{\sigma}_- = \frac{\hat{\sigma}_x - i\hat{\sigma}_y}{2}$), where γ will be the spontaneous emission rate [6].

Acknowledgments

C.A. Brasil acknowledges support from Coordenação de Aperfeiçoamento de Pessoal de Nível Superior

(CAPES) and Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) project number 2011/19848-4, Brazil. F.F. Fanchini acknowledges support from FAPESP and Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) through the Instituto Nacional de Ciência e Tecnologia - Informação Quântica (INCT-IQ), Brazil. R. d. J. Napolitano acknowledges support from CNPq, Brazil.

References

- [1] Göran Lindblad, *Commun. Math. Phys.* **48**, 119 (1976).
- [2] Heinz-Peter Breuer and Francesco Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002).
- [3] Ian Percival, *Quantum State Diffusion* (Cambridge University Press, Cambridge, 1998).
- [4] Carlos A. Brasil and Reginaldo de J. Napolitano, *Eur. Phys. J. Plus* **126**, 91 (2011).
- [5] Carlos A. Brasil, Leonardo A. de Castro and Reginaldo de J. Napolitano, *Phys. Rev. A* **84**, 022112 (2011).
- [6] Michael A. Nielsen and Isaac L. Chuang, *Computação Quântica e Informação Quântica* (Bookman, Porto Alegre, 2005).
- [7] Amir O. Caldeira and Anthony J. Leggett, *Physica A* **121**, 587 (1983).
- [8] Edward Brian Davies, *Quantum Theory of Open Systems* (Academic Press, London, 1976).
- [9] Robert Alicki and Karl Lendi, *Quantum Dynamical Semigroups and Applications* (Springer-Verlag, Berlin, 2007).
- [10] Stephen L. Adler, *Phys. Lett. A* **265**, 58 (2000).
- [11] Stephen L. Adler, *Phys. Lett. A* **267**, 212 (2000).
- [12] Asher Peres, *Phys. Rev. A* **61**, 022116 (2000).
- [13] Claude Cohen-Tannoudji, Bernard Diu and Franck Laló, *Quantum Mechanics* (Wiley, New York, 1977).
- [11] Wolfgang P. Schleich, *Quantum Optics in Phase Space* (Wiley, Berlin, 2001).