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Understanding public debt dynamics

(Compreendendo a dinâmica da dívida pública)

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A brief account on the dynamics of public debt is given with the intent of explaining to high school students this crucial aspect in today's economic crisis. A simple model is proposed to argue that only for high enough values of an assumed constant rate ρ of increase of taxable resources there might exist a finite number of periods (years) for repaying an initial debt b_0 contracted by a government with its reference national bank.
Keywords: public debt, dynamic model, econophysics.

Apresenta-se um breve relato sobre a dinâmica da dívida pública com o objetivo de explicar para estudantes do ensino médio este aspecto crucial na crise econômica de hoje. Propõe-se um modelo simples para mostrar que apenas quando há valores suficientes altos de uma taxa constante ρ de aumento de recursos tributáveis é que pode ser estabelecido um número finito de períodos (anos) para pagar uma dívida inicial b_0 contratada por um governo com o seu banco nacional de referência
Palavras-chave: dívida pública; modelo dinâmico; econofísica.

1. Introduction

All Physics students are citizens and, as such, are subject to the effects of today's economic crisis. Moreover, most of them possess adequate analytical instruments to understand formation of public debt in a given country. On the other hand, high-school students might start systematic studies in Economics in advanced courses or may defer these studies to successive college years, while their curiosity about social aspects may arise even before this stage. In addition, Physics teachers are being exposed to interdisciplinary topics in the latest years. By coupling the words physics to economics, in fact, the subject of Econophysics [1] has been created in the nineties to apply statistical physics methods to the study of financial markets.

Students able to follow the actual debate on public debt formation are lead to develop a deeper awareness on the social complexity of today's world. Therefore, the present work intends to present an interdisciplinary lecture (which may be well given by a Physics instructor) by which the topic of public debt formation is introduced to high-school students. Starting from classical hypotheses, a simple expression for the volume of expenditure might be utilized to argue whether the rate of growth in expanding economies might be sufficient to assure public debt extinction. In other words, one would like to know whether the number of years necessary to pay back an initial debt b_0, assuming that the tax revenue τ_t collected by the government year after year increases steadily at the same rate of increase ρ of the tax pool Y_t, is finite or not. Despite one might set forth the optimistic assumption of economic growth, we shall see that the rate ρ needs to be sufficiently high for debt extinction to be possible. All these rather simple concepts are explained by a very simple analysis, whose difficulty does not go beyond the analytic properties of a curve on a Cartesian plane.

2. Origin of public debt

Consider a government which, at period $t = 0$, contracts a debt b_0 with its reference national bank. This amount of money is utilized for partially funding education, health, welfare, and all public activities, whose total cost is G_0. In this initial period, on the other hand, the government is able to collect taxes with a net revenue τ_0, in such a way the following balance is reached

$$G_0 = b_0 + \tau_0. \tag{1}$$

Considering that an economic growth is expected, with a corresponding expected increase in tax collection, the same government, to prevent unethical beha-

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viour by some citizens, is willing to spend an additional sum of money Z_t, for each successive period t ($t = 1$, 2, ...). In this way, the total expenditure can be described, for $t > 0$, as follows

$$G_t = G_0 + Z_t. \tag{2}$$

According to the homogeneous condition, by which Z_t is seen to double if the tax pool Y_t and the tax revenue τ_t are doubled [2], one may set

$$Z_t = \tau_t f\left(\frac{\tau_t}{Y_t}\right), \tag{3}$$

where f is an increasing function of the ratio $\frac{\tau_t}{Y_t}$. We may notice, however, that $Z_t << G_0$. Despite this relation, we shall retain the presence of Z_t in Eq. (2) for a more complete analysis. The simplest possible function f is given by the following linear dependence $f\left(\frac{\tau_t}{Y_t}\right) = \alpha\frac{\tau_t}{Y_t}$, where α is a positive real number, assumed to be very small with respect to unity. Therefore, we may set, for simplicity

$$Z_t = \alpha\frac{\tau_t^2}{Y_t}. \tag{4}$$

Notice that the assumption $\alpha << 1$ can be understood by arguing that $Z_t << G_0$; i.e., most of the government expenditure is contained in the value G_0.

3. Public debt dynamics

Consider now successive periods $t \geq 1$. The dynamical equation for the debt b_t in period t is given by the following difference Eq. (2)

$$G_t + rb_{t-1} = \tau_t + b_t - b_{t-1}. \tag{5}$$

where $b_t - b_{t-1}$ is the deficit at period t and r is a fixed rate of return on public and private debts. Equation (5) can be understood in the following elementary way. The total government expenditure is the sum of G_t and of the interests rb_{t-1} which need to be paid back at period t. These interests are calculated considering the debt b_{t-1} cumulated at the start of period t. In order to cover this total expenditure, governments may collect taxes (τ_t) and, eventually, may choose to increase their debts by an amount $b_t - b_{t-1}$ (deficit at period $t \geq 1$). Therefore, Eq. (5) can be seen as a balance equation and can be rewritten explicitly in terms of b_t, for $t \geq 1$, as follows

$$b_t = (1+r)\, b_{t-1} + \Delta_t. \tag{6}$$

where $\Delta_t = G_t - \tau_t$ and, according to Eq. (1), $\Delta_0 = b_0$. It is now easy to show, by successive substitutions, that the solution to Eq. (6) is given by

$$b_n = (1+r)^n \left[b_0 + \sum_{k=1}^{n} \frac{\Delta_k}{(1+r)^k}\right], \tag{7}$$

where the period t has been substituted by the more usual index n for an integer. By now considering the definition $\Delta_k = G_k - \tau_k$ (k =1, ... n) and by Eqs. (2) and (4), we can see that the quantity Δ_k takes on the following form

$$\Delta_k = \alpha\frac{\tau_k^2}{Y_k} - \tau_k + G_0. \tag{8}$$

4. Economic growth

Assume that a given country, whose debt at period n can be described by Eqs. (7) and (8), goes through a period of economic growth. Let then ρ be the constant rate of increase of the tax pool Y_t, so that

$$Y_k = (1+\rho)^k\, Y_0, \tag{9}$$

Y_0 being the tax pool at $k = 0$. Correspondingly, the tax revenue collected by the government will be taken to grow at the same rate, so that

$$\tau_k = (1+\rho)^k\, \tau_0. \tag{10}$$

Assume now that the period of observation of the dynamics of public debt is not too long. In this way, asymptotic economic constraints are not valid and the summation in Eq. (7) must be made considering finite values of n. Let us then define

$$S_n = \sum_{k=1}^{n} \frac{\Delta_k}{(1+r)^k} = G_0 \sum_{k=1}^{n} \frac{1}{(1+r)^k} - \tau_0\left(1 - \frac{\alpha\tau_0}{Y_0}\right) \sum_{k=1}^{n} \left(\frac{1+\rho}{1+r}\right)^k. \tag{11}$$

Recalling now that the partial summations can be expressed by the following general expression

$$\sum_{k=1}^{n} a^k = a\left(\frac{1-a^n}{1-a}\right), \tag{12}$$

a being a positive real number not equal to 1. By Eq. (12), considering $\rho \neq r$ ($a \neq 1$), we have

$$S_n = \frac{G_0}{r}\left[1 - \frac{1}{(1+r)^n}\right] - \tau_0\left(1 - \frac{\alpha\tau_0}{Y_0}\right)\frac{1+\rho}{r-\rho}\left[1 - \left(\frac{1+\rho}{1+r}\right)^n\right]. \tag{13}$$

Therefore, according to Eq. (13), we can write the solution Eq. (7) to the balance Eq. (5) as follows

$$b_n = (1+r)^n\, b_0 + \frac{G_0}{r}\left[(1+r)^n - 1\right] - \tau_0\gamma_\alpha\left(\frac{1+\rho}{r-\rho}\right)\left[(1+r)^n - (1+\rho)^n\right], \tag{14}$$

where $\gamma_\alpha = \left(1 - \frac{\alpha\tau_0}{Y_0}\right)$. Exact dynamics of public debt, under the hypotheses set forth in the previous sections, can thus be described by Eq. (14).

In Figs. 1a and 1b we report the continuous curves giving the interpolation of the discrete dynamics of public debt for $\gamma_\alpha = 1$ and for two different interest rates and various values of the growth rate parameter, fixing the remaining parameters to $\frac{b_0}{\tau_0} = 0.3$ and, according to Eq. (1), $\frac{G_0}{\tau_0} = 1.3$. Effects of the increase in the public expenditure for increasing values of α (decreasing values of γ_α) are given in Fig. 2, where full line curves are chosen from Fig. 1a ($\gamma_\alpha = 1$) and the adjacent dotted curves are plotted for the same parameters as in the full line curves except for the value of γ_α, taken to be 0.99. The overall upper shift of the curves for the chosen values of $\rho = 0.050,\ 0.025,\ 0.010$ and $\gamma_\alpha = 0.99$ signifies, for the model adopted in the present work, that the effect of the additional expenditure for tax collecting purposes can procrastinate the moment in which debt extinction occurs ($\rho = 0.050,\ 0.025$), or can increase the total debt, if extinction does not occur ($\rho = 0.010$).

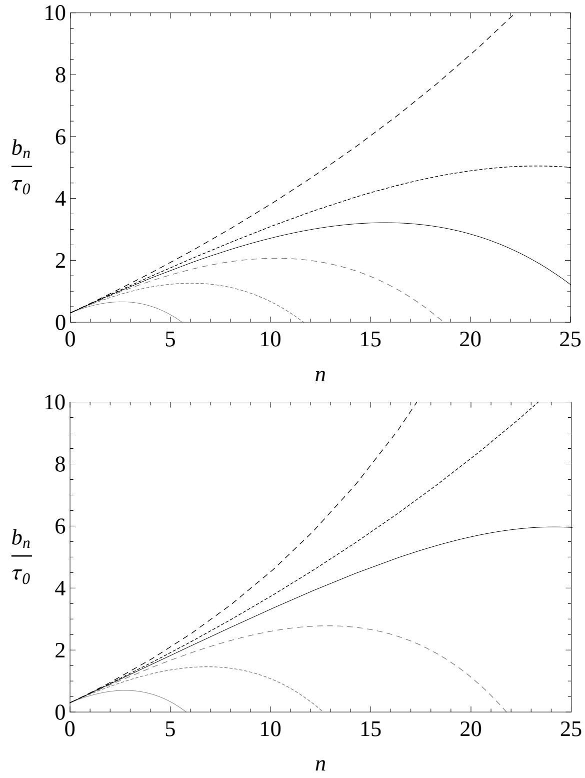

Figura 1 - Resulting dynamics for the following choice of parameters: $\gamma_\alpha = 1$, $\frac{b_0}{\tau_0} = 0.3$, $\frac{G_0}{\tau_0} = 1.3$, $r = 0.06$(a) and $r = 0.09$(b). In both figures (a) and (b) the choice of the growth rate ρ is as follows: 0.100 (gray full line), 0.050 (gray dotted line), 0.033 (gray dashed line), 0.025 (black full line), 0.02 (black dotted line), 0.01 (black dashed line). Notice how the higher interest rate in (b) delays the period in which public debt can be extinguished. This delay is higher for lower values of the growth rate ρ. Notice also the different behavior of the curves in (a) and (b) obtained for ρ =0.02; in this case, one sees how different interest rates can be crucial in determining whether debt extinction is possible or not.

5. Paying debts back

In the present section we shall explore the possibility of extinguishing public debt in a certain number of years n_0. For simplicity, we look at the qualitative behaviour of the curve in Figs. 1a and 1b, having specified, in Fig. 2, the effect of a decrease in the parameter γ_α. We start by noticing that two different behaviours of the curves are detectable. For high enough values of the growth rate ρ, a solution for $b_{n_0} = 0$ is possible. On the other hand, for low values of ρ, the curves show a positive curvature and no solution to the expression $b_{n_0} = 0$ can be found. Therefore, one can recognize that there might be situations in which, despite a growing economy, the rate of growth is not high enough to assure debt extinction.

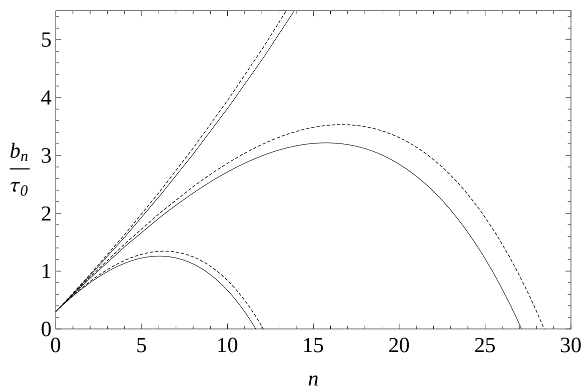

Figura 2 - Public debt dynamics for the following choice of fixed parameters:, $\frac{b_0}{\tau_0} = 0.3$, $\frac{G_0}{\tau_0} = 1.3$, $r = 0.06$. All full-line figures are obtained for $\gamma_\alpha = 1$; all dotted-line figures are obtained for $\gamma_\alpha = 0.99$. The values of the growth rate ρ are as follows: 0.050 (lower adjacent lines), 0.025 (middle adjacent lines), 0.010 (upper adjacent lines).

Looking at the curves in Fig. 1a and 1b we may argue that by setting the second derivative of b_n with respect to n, treating n as a real variable, less than zero, we might have assurance that a maximum occurs in the curves at $\hat{n}$ and that, for $n > \hat{n}$, periods of decreasing public debt will follow. Therefore, start by calculating the derivative with respect to n of b_n, so that

$$\frac{1}{\tau_0}\frac{\mathrm{d}b_n}{\mathrm{d}n} = (1+r)^n \ln(1+r) \left\{ \frac{b_0}{\tau_0} + \frac{G_0}{r\tau_0} - \gamma_\alpha \left(\frac{1+\rho}{r-\rho}\right) \left[1 - \frac{\ln(1+\rho)}{\ln(1+r)} \left(\frac{1+\rho}{1+r}\right)^n\right] \right\}. \tag{15}$$

Before calculating the inversion point $n = \hat{n}$, let us consider the second derivative, in order to make some distinction on the behaviour of the curves

$$\frac{1}{\tau_0}\frac{\mathrm{d}^2 b_n}{\mathrm{d}n^2} = (1+r)^n \ln^2(1+r) \left\{ \frac{b_0}{\tau_0} + \frac{G_0}{r\tau_0} - \gamma_\alpha \left(\frac{1+\rho}{r-\rho}\right) \left[1 - \frac{\ln^2(1+\rho)}{\ln^2(1+r)} \left(\frac{1+\rho}{1+r}\right)^n\right] \right\}. \tag{16}$$

By now imposing $\frac{\mathrm{d}^2 b_n}{\mathrm{d}n^2} < 0$, we obtain the following conditions. Define first the following important parameters of the model

$$\lambda_0 = \frac{\frac{1+\rho}{r-\rho}\gamma_\alpha - \frac{b_0}{\tau_0} - \frac{G_0}{r\tau_0}}{\frac{1+\rho}{r-\rho}\frac{\ln^2(1+\rho)}{\ln^2(1+r)}\gamma_\alpha}, \quad \text{and}$$

$$\mu_0 = \frac{\frac{1+\rho}{\rho-r}\gamma_\alpha + \frac{b_0}{\tau_0} + \frac{G_0}{r\tau_0}}{\frac{1+\rho}{\rho-r}\frac{\ln^2(1+\rho)}{\ln^2(1+r)}\gamma_\alpha}, \tag{17}$$

the first appropriate for the case $r > \rho$, the second for $r < \rho$. Notice that the ratio $\frac{b_0}{\tau_0}$ can be expressed in terms of $\frac{G_0}{\tau_0}$ by means of (1).

Let us start by analyzing the model behaviour for $r > \rho$. In this case, the condition $\frac{\mathrm{d}^2 b_n}{\mathrm{d}n^2} < 0$ is given by the following inequality

$$\left(\frac{1+\rho}{1+r}\right)^n < \lambda_0. \tag{18}$$

Therefore, if $\lambda_0 \leq 0$, it is not possible to satisfy Eq. (18) and, thus, the debt cannot be extinguished. On the other hand, if $\lambda_0 > 0$, we further make the following distinction. If $0 < \lambda_0 < 1$, the continuous version of the curves present a first portion of positive curvature before the descending inflection point x_I given by the following expression

$$x_I = \frac{\ln(\lambda_0)}{\ln\left(\frac{1+\rho}{1+r}\right)}. \tag{19}$$

In this case, debt extinction obviously occurs at $n_0 > x_I$. Instead, for $\lambda_0 \geq 1$, the curve has always a negative curvature. Looking back at the inversion point $\hat{n}$ (not necessarily an integer), calculated by setting to zero the first derivative of b_n in Eq. (15) for $\lambda_0 > 0$, we have

$$\hat{n} = \frac{\ln\left(\lambda_0 \frac{\ln(1+\rho)}{\ln(1+r)}\right)}{\ln\left(\frac{1+\rho}{1+r}\right)}. \tag{20}$$

Next, consider the case $r < \rho$. The condition $\frac{\mathrm{d}^2 b_n}{\mathrm{d}n^2} < 0$ is now given by the following inequality

$$\left(\frac{1+\rho}{1+r}\right)^n > \mu_0. \tag{21}$$

Since $\mu_0 > 0$ and $\frac{1+\rho}{1+r} > 1$, Eq. (21) is always satisfied for large enough values of n. On the other hand, it can be proven that indeed $0 < \mu_0 \leq 1$ by the following chain of inequalities

$$\hat{n} = \frac{\ln\left(\mu_0 \frac{\ln(1+\rho)}{\ln(1+r)}\right)}{\ln\left(\frac{1+\rho}{1+r}\right)}. \tag{22}$$

With these analytic results, in the next section we shall analyze the curves in Fig. 1b, in order to compare the predicted values of some characteristic quantities with the corresponding numerical evaluation extrapolated from the curves obtained by means of Eq. (14).

6. Comparing analytic and numerical results

In Fig. 1b various curves giving the exact dynamics of public debt have been calculated numerically. For these curves we have fixed some model parameters, namely, $\gamma_\alpha = 1$, $\frac{b_0}{\tau_0} = 0.3$, $\frac{G_0}{\tau_0} = 1.3$, and $r = 0.09$, while the growth rate parameter ρ takes on the following values: 0.100, 0.050, 0.033, 0.025, 0.020, 0.010.

Notice that only the lowest curve ($\rho = 0.100$) pertains to the case $r < \rho$. Let us thus analyze this curve first. In this case we calculate $\mu_0 = 0.9271$ and obtain negative curvature throughout. The inversion point, as calculated by Eq. (22), occurs at $\hat{n} = 2.74$, as can be verified from the lowest graph in Fig. 1b. All other values of pertain to the case $r > \rho$. For the curves with $\rho = 0.050,\ 0.033$, we have $\lambda_0 \geq 1$, so that the graphs show negative curvature for all value of n and inversion points, calculated by Eq. (20), at the respective values $\hat{n} = 6.85,\ 13.10$. For $\rho = 0.025$, on the other hand, we have $\lambda_0 = 0.791$. For this plot, therefore, an inflection point $x_I = 3.80$ is calculated by means of Eq. (19) and the inversion point $\hat{n} = 24.13$ is calculate by Eq. (20). For this particular curve we show a plot in an expanded range in Fig. 3. Finally, for $\rho = 0.020$ and $\rho = 0.010$ extinction of the debt is not possible, since the curves attain positive curvature for all n, given that $\lambda_0 = -0.22,\ -12.59$, respectively, and thus it is not possible to satisfy Eq. (18) at all in this case.

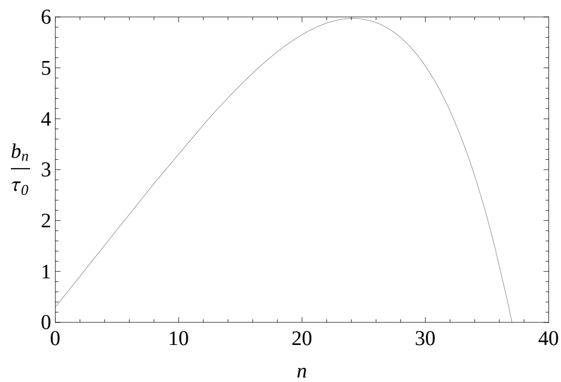

Figura 3 - Resulting public debt dynamics for the following choice of parameters: $\gamma_\alpha = 1.0$, $\frac{b_0}{\tau_0} = 0.3$, $\frac{G_0}{\tau_0} = 1.3$, $r = 0.09$, and $\rho = 0.025$. Notice that the inversion point occurs at $\hat{n} = 24.97$. Owing to the low value of the growth rate ρ, the extinction of the debt would be possible only after about 37 years.

7. Conclusions

By considering a simple model for economic growth, we have addressed the question of the extinction of public debt in a given country. The model, giving time evolution of public debt, is based upon classical hypotheses [2]. Assuming that the tax revenue τ_t collected by the government increases steadily at the same rate ρ of the correspondingly increasing tax pool Y_t, we find that not all countries with a growing economy may pay back the initial debt b_0. In fact, depending on the ratio G_0/τ_0 and on the interest rate r, the rate ρ needs to be sufficiently high for making debt extinction possible. In particular, for $\rho > r$, debt extinction is always possible within a given number of periods. For $\rho < r$, on the other hand, it is possible to observe either a finite debt extinction time, or the impossibility of paying back the initial debt b_0, despite a very long time series of periods of growing tax pools Y_t.

The present analysis is suitable for an interdisciplinary lecture to address to high-school students. Since time-evolution problems are very often encountered in elementary Physics, the present model, because of its simplicity, may be explained to Physics student eager to understand dynamical laws governing today's public finance. In order to account for the intrinsic complexity of today's economy, however, more detailed models are needed to account for time variation of all important parameters which, at this elementary stage, have been assumed to be constant.

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