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On equivalent expressions for the Faraday's law of induction

(Formulações equivalentes da lei de Faraday)

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In this paper we give a rigorous proof of the equivalence of some different forms of Faraday's law of induction clarifying some misconceptions on the subject and emphasizing that many derivations of this law appearing in textbooks and papers are only valid under very special circunstances, thus are not satisfactory under a mathematical point of view. We show also that Faraday´s law of induction is a relativistic invariant law in a very precise mathematical sense.
Keywords: Faraday's law, principle of equivalence, relativistic invariance.

Neste trabalho é dada uma prova de equivalência entre diferentes formas de se escrever a lei de indução de Faraday, elucidando alguns equívocos sobre o tópico e enfatizando que muitas derivações desta lei apresentadas na maioria dos livros e artigos são válidas somente sob circunstâncias muito particulares e, portanto, não satisfatórias sob o ponto de vista matemático. Também mostramos que a lei de indução de Faraday é relativisticamente invariante em um sentido matematicamente bem preciso.
Palavras-chave: lei de Faraday, princípio de equivalência, invariância relativistica.

1. Introduction

Let $\mathbf{\Gamma}_t$ a smooth closed curve in $\mathbb{R}^3$ with parametrization $\mathbf{x}(t,\ell)$ which is here supposed to represent a filamentary closed circuit which is moving in an a convex and simply-connected (open) region $U \subset \mathbb{R}^3$ where at time t as measured in an inertial frame,[2] there are an electric and a magnetic fields $\mathbf{E} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, $(t, \mathbf{x}) \mapsto \mathbf{E}(t, \mathbf{x}) \in \mathbb{R}^3$ and $\mathbf{B} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, $(t, \mathbf{x}) \mapsto \mathbf{B}(t, \mathbf{x}) \in \mathbb{R}^3$. We suppose that when in motion the closed circuit may be eventually *deforming.* Let $\mathbf{\Gamma}$ be a smooth closed curve in $\mathbb{R}^3$ with parametrization $\mathbf{x}(\ell)$ representing the filamentary circuit at $t = 0$. Then, the smooth curve $\mathbf{\Gamma}_t$ is given by $\mathbf{\Gamma}_t = \sigma_t(\mathbf{\Gamma})$ where σ_t (see details below) is the *flow* of a velocity vector field $\mathbf{v} : \mathbb{R}\times\mathbb{R}^3 \mapsto \mathbb{R}^3$, which describes the motion (and deformation) of the closed circuit. It is an empirical fact known as Faraday's law of induction that on the closed loop $\mathbf{\Gamma}_t$ acts an induced *electromotive force,* $\mathcal{E}$, such that

$$\mathcal{E} = -\frac{d}{dt}\int_{\mathcal{S}_t} \mathbf{B}\cdot\mathbf{n}\; da, \tag{1}$$

where $\mathcal{S}_t$ is a smooth surface on $\mathbb{R}^3$ such that $\mathbf{\Gamma}_t$ is its boundary and $\mathbf{n}$ is the normal vector field on $\mathcal{S}_t$. We write $\mathbf{\Gamma}_t = \partial\mathcal{S}_t$ with $\mathbf{\Gamma} = \partial\mathcal{S}$. Now, on each element of $\mathbf{\Gamma}_t$ the force acting on a unit charge which is moving with velocity $\mathbf{v}(t, \mathbf{x}(t, s))$ is given by the Lorentz force law. Thus[3] the $\mathcal{E}$ is by definition

$$\mathcal{E} = \int_{\mathbf{\Gamma}_t} (\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}, \tag{2}$$

where $d\mathbf{l} := \frac{\partial \mathbf{x}(t,s)}{\partial s} d\ell$ and Faraday's law reads [3–7]

$$\int_{\mathbf{\Gamma}_t} (\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot d\mathbf{l} = -\frac{d}{dt}\int_{\mathcal{S}_t} \mathbf{B}\cdot\mathbf{n}\; da. \tag{3}$$

Note that $(\mathbf{E} + \mathbf{v}\times\mathbf{B})$ is the Lorentz force acting on a unity charged carrier in the circuit according to the laboratory observers and it is sometimes called the *effective electric field* [3]. In the appendix we show how the first term of Eq. (3) is related with measurements done by observers at rest in an inertial reference frame commoving at a given instant t with velocity $\mathbf{v}$ relative to the laboratory frame.

We want to prove that Eq. (3) is equivalent to

$$\int_{\mathbf{\Gamma}_t} \mathbf{E}\cdot d\mathbf{l} = -\int_{\mathcal{S}_t} \frac{\partial \mathbf{B}}{\partial t}\cdot\mathbf{n}\; da, \tag{4}$$

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[2]For a mathematical defintion of an inertial reference frame in Minkowski spacetime see, *e.g.*, Refs. [1, 2].
[3]In this paper we use a system of units such that the numerical value of the speed of light is $c = 1$.

from where it trivially follows the differential form of Faraday's law, *i.e.*,

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0. \tag{5}$$

Those statements will be proved in Section 3, but first we shall need to recall a few mathematical results concerning differentiable vector fields, in Section 2.

2. Some identities involving the integration of differentiable vector fields

Let $U \subset \mathbb{R}^3$ be a convex and simply-connected (open) region, $\mathbf{X} : \mathbb{R}\times U \to \mathbb{R}^3,(t,\mathbf{x}) \mapsto \mathbf{X}(t,\mathbf{x})$ be a generic differentiable vector field and let $\mathbf{v} : \mathbb{R}\times U \to \mathbb{R}^3$ be a differentiable *velocity vector field* of a *fluid flow*. An *integral line*[4] of $\mathbf{v}$ passing through a given $\mathbf{x} \in \mathbb{R}^3$ is a smooth curve $\sigma_\mathbf{x} : \mathbb{R} \to \mathbb{R}^3$, $t \to \sigma_\mathbf{x}(t) = \sigma(t,\mathbf{x})$ which at $t = 0$ is at $\mathbf{x}$ (*i.e.*, $\sigma_\mathbf{x}(0) = \mathbf{x}$) and such that its tangent vector at $\sigma(t,\mathbf{x})$ is

$$\frac{\partial}{\partial t}\sigma(t,\mathbf{x}) = \mathbf{v}(t,\sigma(t,\mathbf{x})). \tag{6}$$

Let moreover $\sigma_t : U \to \mathbb{R}^3, \sigma_t(\mathbf{x}) = \sigma(t,\mathbf{x})$. We call σ_t the *fluid flow map*. Let $J = (0,1) \in \mathbb{R}$ and let $\mathbf{\Gamma}$ be a closed loop parametrized by $\mathbf{\Gamma} : J \to \mathbb{R}^3$, $\ell \mapsto \mathbf{\Gamma}(\ell) := \mathbf{x}(\ell)$ and denote by $\Gamma_t = \sigma_t(\Gamma)$ the loop *transported* by the flow (see Fig. 1). Then

$$\sigma(t,\mathbf{x}(\ell)) := \mathbf{x}(t,\ell) \tag{7}$$

is clearly a parametrization of $\mathbf{\Gamma}_t$. We have the proposition:

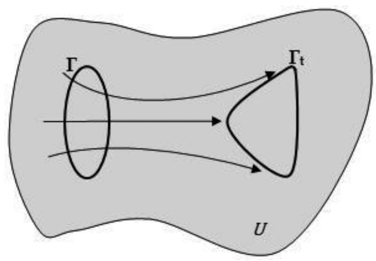

Figure 1 - The closed loop Γ being transported by the fluid flow σ_t.

Proposition

$$\frac{d}{dt}\int_{\Gamma_t}\mathbf{X}\cdot d\mathbf{l} = \int_{\Gamma_t}\frac{D}{Dt}\mathbf{X}\cdot d\mathbf{l} + \int_{\Gamma_t}\mathbf{X}\cdot[(d\mathbf{l}\cdot\nabla)\mathbf{v}], \tag{8a}$$

$$= \int_{\Gamma_t}\frac{D}{Dt}\mathbf{X}\cdot d\mathbf{l} + \int_{\Gamma_t}[\mathbf{X}\times(\nabla\times\mathbf{v})]\cdot d\mathbf{l} + \int_{\Gamma_t}[(\mathbf{X}\cdot\nabla)\mathbf{v})]\cdot d\mathbf{l}, \tag{8b}$$

$$= \int_{\Gamma_t}\frac{\partial}{\partial t}\mathbf{X}\cdot d\mathbf{l} - \int_{\Gamma_t}[\mathbf{v}\times(\nabla\times\mathbf{X})]\cdot d\mathbf{l}, \tag{8c}$$

where

$$\frac{d}{dt}\mathbf{X} = \frac{D}{Dt}\mathbf{X} := \frac{\partial}{\partial t}\mathbf{X} + (\mathbf{v}\cdot\nabla)\mathbf{X} \tag{9}$$

is the so-called *material derivative*[5] and

$$d\mathbf{l} = \frac{\partial}{\partial \ell}\sigma(t,\mathbf{x}(\ell))d\ell = \frac{\partial \mathbf{x}(t,\ell)}{\partial \ell}d\ell \tag{10}$$

is the tangent line element[6] of Γ_t at $\sigma(t,\mathbf{x}(\ell))$.

Proof. We can write

$$\frac{d}{dt}\int_{\Gamma_t}\mathbf{X}\cdot d\mathbf{l} = \frac{d}{dt}\int_0^1 \mathbf{X}(t,\sigma(t,\mathbf{x}(\ell)))\cdot\frac{\partial}{\partial \ell}\sigma(t,\mathbf{x}(\ell))d\ell$$

$$= \int_0^1 \frac{d}{dt}\left[\mathbf{X}(t,\sigma(t,\mathbf{x}(\ell)))\right]\cdot\frac{\partial}{\partial \ell}\sigma(t,\mathbf{x}(\ell))d\ell + \int_0^1 \mathbf{X}(t,\sigma(t,\mathbf{x}(\ell)))\cdot\frac{\partial}{\partial t}\frac{\partial}{\partial \ell}\sigma(t,\mathbf{x}(\ell))d\ell. \tag{11}$$

Now, taking into account that for each $\mathbf{x}(\ell)$, $\frac{\partial}{\partial t}\sigma(t,\mathbf{x}) = \mathbf{v}(t,\sigma(t,\mathbf{x}(\ell)))$ we have

$$\frac{D}{Dt}\left[\mathbf{X}(t,\sigma(t,\mathbf{x}(\ell)))\right] = \frac{\partial}{\partial t}\mathbf{X}(t,\sigma(t,\mathbf{x}(\ell))) + (\mathbf{v}\cdot\nabla)\mathbf{X}(t,\sigma(t,\mathbf{x}(\ell))), \tag{12}$$

[4] Also called a stream line.

[5] Mind that the *material derivative* is a derivative taken along a path σ_t with tangent vector $\mathbf{v}|_{\sigma_x}$. It is frequently used in fluid mechanics, where it describes the total time rate of change of a given quantity as viewed by a fluid particle moving on σ_x. In the present case it appears because in the integral $\int_{\Gamma_t}\mathbf{X}\cdot d\mathbf{l}$ we need the values of $\mathbf{X}$ for each t at all points of Γ_t, *i.e.*, $\mathbf{X}(t,\sigma(t,\mathbf{x}(\ell)))$.

[6] Take notice that $d\mathbf{l}$ is not an explicit function of the cartesian coordinates (x,y,z).

hence, the first term in the right side of Eq. (11) can be written as

$$\int_0^1 \frac{d}{dt}\left[\mathbf{X}(t,\sigma(t,\mathbf{x}(\ell)))\right]\cdot\frac{\partial}{\partial\ell}\sigma(t,\mathbf{x}(\ell))d\ell = \int_{\Gamma_t}\left[\frac{\partial}{\partial t}\mathbf{X}+(\mathbf{v}\cdot\nabla)\mathbf{X}\right]\cdot d\mathbf{l} = \int_{\Gamma_t}\frac{D}{Dt}\mathbf{X}\cdot d\mathbf{l}. \tag{13}$$

Also writing $\sigma(t,\mathbf{x}(\ell)) = (x^1(t,\ell), x^2(t,\ell), x^3(t,\ell))$ we see that the last term in Eq. (11) can be written as

$$\int_0^1 \mathbf{X}(t,\sigma(t,\mathbf{x}(\ell)))\cdot\frac{\partial}{\partial t}\frac{\partial}{\partial\ell}\sigma(t,\mathbf{x}(\ell))d\ell = \int_0^1 \mathbf{X}(t,\sigma(t,\mathbf{x}(\ell)))\cdot\left[\frac{\partial}{\partial\ell}\mathbf{v}(t,\sigma(t,\mathbf{x}(\ell)))d\ell\right] = \int_{\Gamma_t}\mathbf{X}\cdot[(d\mathbf{l}\cdot\nabla)\mathbf{v}]. \tag{14}$$

We now recall that for arbitrary differentiable vector fields $\mathbf{a},\mathbf{b}: U\to\mathbb{R}^3$ it holds

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b}+(\mathbf{b}\cdot\nabla)\mathbf{a}+\mathbf{a}\times(\nabla\times\mathbf{b})+\mathbf{b}\times(\nabla\times\mathbf{a}). \tag{15}$$

Setting $\mathbf{a} = d\mathbf{l}$ and $\mathbf{b} = \mathbf{v}$ and noting that $(\mathbf{v}\cdot\nabla)d\mathbf{l} = \mathbf{v}\times(\nabla\times d\mathbf{l}) = \mathbf{0}$, it implies that

$$(d\mathbf{l}\cdot\nabla)\mathbf{v} = \nabla(d\mathbf{l}\cdot\mathbf{v}) - d\mathbf{l}\times(\nabla\times\mathbf{v}). \tag{16}$$

We need also to recall the well known identity

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}), \tag{17}$$

which implies setting $\mathbf{a} = \mathbf{X}$, $\mathbf{b} = d\mathbf{l}$ and $\mathbf{c} = (\nabla\times\mathbf{v})$, that

$$-\mathbf{X}\cdot[d\mathbf{l}\times(\nabla\times\mathbf{v})] = -d\mathbf{l}\cdot[(\nabla\times\mathbf{v})\times\mathbf{X}], \tag{18}$$

and also the *not* so well known identity[7]

$$\mathbf{X}\cdot[\nabla(d\mathbf{l}\cdot\mathbf{v})] = [(\mathbf{X}\cdot\nabla)\mathbf{v}]\cdot d\mathbf{l}, \tag{19}$$

to write that

$$\int_{\Gamma_t}\mathbf{X}\cdot[(d\mathbf{l}\cdot\nabla)\mathbf{v}] = -\int_{\Gamma_t}\mathbf{X}\cdot[d\mathbf{l}\times(\nabla\times\mathbf{v})] + \int_{\Gamma_t}[(\mathbf{X}\cdot\nabla)\mathbf{v}]\cdot d\mathbf{l} = \int_{\Gamma_t}[\mathbf{X}\times(\nabla\times\mathbf{v})]\cdot d\mathbf{l} + \int_{\Gamma_t}[(\mathbf{X}\cdot\nabla)\mathbf{v})]\cdot d\mathbf{l}. \tag{20}$$

Finally, using Eq. (13) and Eq. (20) completes the proof of Eq. (8a) and Eq. (8b). Also, from Eq. (8b) it follows if we recall Eq. (15) that

$$\begin{aligned}\frac{d}{dt}\int_{\Gamma_t}\mathbf{X}\cdot d\mathbf{l} &= \int_{\Gamma_t}\frac{\partial}{\partial t}\mathbf{X}\cdot d\mathbf{l} + \int_{\Gamma}[(\mathbf{v}\cdot\nabla)\mathbf{X})]\cdot d\mathbf{l} + \int_{\Gamma_t}[\mathbf{X}\times(\nabla\times\mathbf{v})]\cdot d\mathbf{l} + \int_{\Gamma}[(\mathbf{X}\cdot\nabla)\mathbf{v})]\cdot d\mathbf{l} \\ &= \int_{\Gamma_t}\frac{\partial}{\partial t}\mathbf{X}\cdot d\mathbf{l} - \int_{\Gamma_t}[\mathbf{v}\times(\nabla\times\mathbf{X})]\cdot d\mathbf{l}.\end{aligned}$$

from where the proof of Eq. (8c) follows immediately. ■

Remark 1 *Before proceeding, we recall that if* $\mathbf{X} = \mathbf{v}$ *we have*

$$\frac{d}{dt}\int_{\Gamma_t}\mathbf{v}\cdot d\mathbf{l} = \int_{\Gamma_t}\frac{D}{Dt}\mathbf{v}\cdot d\mathbf{l}, \tag{21}$$

a result that is known in fluid mechanics as Kelvin's circulation theorem (see, *e.g.*, Refs. [8,9]).

Now,

$$\frac{d}{dt}\int_{\Gamma_t}\mathbf{X}\cdot d\mathbf{l} = \frac{d}{dt}\int_{\mathcal{S}_t}(\nabla\times\mathbf{X})\cdot\mathbf{n}\,da, \tag{22}$$

where, if $\mathcal{S}$ is a smooth surface such that $\partial\mathcal{S} = \Gamma$, then $\mathcal{S}_t = \sigma_t(\mathcal{S})$. Also $\mathbf{n}$ is the normal vector field to S_t. Then using Eq. (8c) we can write

[7] See the Appendix for a proof of this identity.

$$\frac{d}{dt}\int_{\mathcal{S}_t}(\nabla\times\mathbf{X})\cdot\mathbf{n}\,da=\int_{\Gamma_t}\frac{\partial}{\partial t}\mathbf{X}\cdot d\mathbf{l}-\int_{\Gamma_t}[\mathbf{v}\times(\nabla\times\mathbf{X})]\cdot d\mathbf{l}=\int_{\mathcal{S}_t}\frac{\partial}{\partial t}(\nabla\times\mathbf{X})\cdot\mathbf{n}\,da-\int_{\mathcal{S}_t}\nabla\times[\mathbf{v}\times(\nabla\times\mathbf{X})]\cdot\mathbf{n}\,da. \tag{23}$$

Also, denoting $\mathbf{Y} := \nabla\times\mathbf{X}$ we can write

$$\frac{d}{dt}\int_{\mathcal{S}_t}\mathbf{Y}\cdot\mathbf{n}\,da=\int_{\mathcal{S}_t}\left[\frac{\partial}{\partial t}\mathbf{Y}-\nabla\times(\mathbf{v}\times\mathbf{Y})\right]\cdot\mathbf{n}\,da. \tag{24}$$

Despite Eq. (24), for a general differentiable vector field $\mathbf{Z}:\mathbb{R}\times U\to\mathbb{R}^3$ such that $\nabla\cdot\mathbf{Z}\neq 0$ we have

$$\frac{d}{dt}\int_{\mathcal{S}_t}\mathbf{Z}\cdot\mathbf{n}\,da=\int_{\mathcal{S}_t}\left[\frac{\partial}{\partial t}\mathbf{Z}+\mathbf{v}(\nabla\cdot\mathbf{Z})-\nabla\times(\mathbf{v}\times\mathbf{Z})\right]\cdot\mathbf{n}\,da, \tag{25}$$

the so-called *Helmholtz identity* [10]. Note that the identity is also mentioned in [11]. A proof of Helmholtz identity can be obtained using arguments similar to the ones used in the proof of Eq. (8a). Some textbooks quoting Helmholtz identity are [12–16]. However, we emphasize that the proof of Faraday's law of induction presented in all the textbooks just quoted are always for very particular situations and definitively not satisfactory from a mathematical point of view.

We now want to use the above results to prove Eq. (3) and Eq. (4).

3. Proofs of Eq. (3) and Eq. (4)

We start remembering that in Maxwell theory we have that the $\mathbf{E}$ and $\mathbf{B}$ fields are derived from potentials, *i.e.*,

$$\begin{aligned}\mathbf{E}&=-\nabla\phi-\frac{\partial\mathbf{A}}{\partial t},\\ \mathbf{B}&=\nabla\times\mathbf{A},\end{aligned} \tag{26}$$

where $\phi:\mathbb{R}\times\mathbb{R}^3\to\mathbb{R}$ is the scalar potential and $\mathbf{A}:\mathbb{R}\times\mathbb{R}^3\mapsto\mathbb{R}$ is the (magnetic) vector potential. If Eq. (26) is taken into account we can immediately derive Eq. (3). All we need is to use the results just derived in Section 2 taking $\mathbf{X}=\mathbf{A}$. Indeed, the first line of Eq. (23) then becomes

$$\frac{d}{dt}\int_{\mathcal{S}_t}(\nabla\times\mathbf{A})\cdot\mathbf{n}\,da=\int_{\mathbf{\Gamma}_t}\frac{\partial}{\partial t}\mathbf{A}\cdot d\mathbf{l}-\int_{\mathbf{\Gamma}_t}[\mathbf{v}\times(\nabla\times\mathbf{A})]\cdot d\mathbf{l},$$

or

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{S}_t}\mathbf{B}\cdot\mathbf{n}\,da&=\int_{\mathbf{\Gamma}_t}\frac{\partial}{\partial t}\mathbf{A}\cdot d\mathbf{l}-\int_{\mathbf{\Gamma}_t}(\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}\\ &=\int_{\mathbf{\Gamma}_t}\left(\frac{\partial}{\partial t}\mathbf{A}+\nabla\phi-\mathbf{v}\times\mathbf{B}\right)\cdot d\mathbf{l}\\ &=-\int_{\mathbf{\Gamma}_t}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}.\end{aligned} \tag{27}$$

To obtain Eq. (4) we recall that from the second line of Eq. (23) we can write (using Stokes theorem)

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{S}_t}\mathbf{B}\cdot\mathbf{n}\,da&=\int_{\mathcal{S}_t}\frac{\partial}{\partial t}\mathbf{B}\cdot\mathbf{n}\,da-\int_{\mathcal{S}_t}\nabla\times[\mathbf{v}\times\mathbf{B}]\cdot\mathbf{n}\,da\\ &=\int_{\mathcal{S}_t}\frac{\partial}{\partial t}\mathbf{B}\cdot\mathbf{n}\,da-\int_{\mathbf{\Gamma}_t}(\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}.\end{aligned} \tag{28}$$

Comparing the second member of Eq. (27) and Eq. (28) we get Eq. (4), *i.e.*,

$$\int_{\mathbf{\Gamma}_t}\mathbf{E}\cdot d\mathbf{l}=-\int_{\mathcal{S}_t}\frac{\partial}{\partial t}\mathbf{B}\cdot\mathbf{n}\,da, \tag{29}$$

from where the differential form of Faraday's law follows.

Remark 2 *We end this section by recalling that in the physical world the real circuits are not filamentary and worse, are not described by smooth closed curves. However, if the closed curve representing a 'filamentary circuit' is made of finite number of sections that are smooth, we can yet apply the above formulas with the integrals meaning Lebesgue integrals.*

4. Conclusions

Recently a paper [17] titled 'Faraday's law via the magnetic vector potential', has been commented in Ref. [18] and replied in Ref. [19]. Thus, the author of Ref. [17], claims to have presented an "alternative" derivation for Faraday's law for a filamentary circuit which is moving with an arbitrary velocity and which is changing its shape, using directly the vector potential $\mathbf{A}$ instead of the magnetic field $\mathbf{B}$ and the electric field $\mathbf{E}$ (which is the one presented in almost all textbooks).

Now, Ref. [18] correctly identified that the derivation in Ref. [17] is wrong, and that author agreed with that in Ref. [19]. Here we want to recall that a presentation of Faraday's law in terms of the magnetic vector potential $\mathbf{A}$ already appeared in Maxwell treatise [20], using big formulas involving the components of the vector fields involved. We recall also that a formulation of Faraday's law in terms of $\mathbf{A}$ using modern vector calculus has been given by Gamo more than 30 years ago [21]. In Gamo's paper (not quoted in Refs. [17–19]) Eqs. (8c) appear for the special case in which $\mathbf{X}=\mathbf{A}$ (the vector potential) and $\mathbf{B}=\nabla\times\mathbf{A}$ (the magnetic field), *i.e.*,

$$\frac{d}{dt}\int_{\Gamma_t}\mathbf{A}\cdot d\mathbf{l}=\int_{\Gamma_t}\frac{\partial}{\partial t}\mathbf{A}\cdot d\mathbf{l}-\int_{\Gamma_t}[\mathbf{v}\times(\nabla\times\mathbf{A})]\cdot d\mathbf{l}. \quad (30)$$

Thus, Eq. (30) also appears in Ref. [17] (it is there Eq. (9)). However, in footnote 3 of [17] it is said that Eq. (30) is equivalent to "$\frac{d}{dt}\int_{\Gamma_t}\mathbf{A}\cdot d\mathbf{l}=\int_{\Gamma_t}\frac{D}{Dt}\mathbf{A}\cdot d\mathbf{l}$", where the term $\int_{\Gamma}[(\mathbf{A}\cdot\nabla)\mathbf{v})]\cdot d\mathbf{l}$ is missing. This is the error that has been observed by authors [18], which also presented a proof of Eq. (8b), which however is not very satisfactory from a mathematical point of view, that being one of the reasons why we decided to write this note presenting a correct derivation of Faraday's law in terms of $\mathbf{A}$ and its relation with Helmholtz formula. Another reason is that there are still people (*e.g.*, Ref. [22]) that do not understand that Eq. (3) and Eq. (4) are equivalent and think that Eq. (3) implies the form of Maxwell equations as given by Hertz, something that we know since a long time that is wrong [23].

We also want to observe that Jackson's proof of Faraday's law using 'Galilean invariance' is valid only for a filamentary circuit moving without deformation with a constant velocity. The proof we presented is general and valid in Special Relativity, since it is based on trustful mathematical identities and in the Lorentz force law applied in the laboratory frame with the motion and deformation of the filamentary circuit mathematically well described.

A Proof of the identity in Eq. (19)

We know from Eq. (16) that

$$\nabla(d\mathbf{l}\cdot\mathbf{v})=(d\mathbf{l}\cdot\nabla)\mathbf{v}+d\mathbf{l}\times(\nabla\times\mathbf{v}). \quad (31)$$

Let $\{e^1,e^2,e^3\}$ be an orthonormal base of $\mathbb{R}^3$. We can write, using *Einstein convention,*

$$(\nabla\times\mathbf{v})=e^i\partial_i\times\mathbf{v}=e^i\times\partial_i\mathbf{v}, \quad (32)$$

where $\nabla=(\partial_1,\partial_2,\partial_3)=e^1\frac{\partial}{\partial x^1}+e^2\frac{\partial}{\partial x^2}+e^3\frac{\partial}{\partial x^3}=e^i\partial_i$, with $\partial_i=\frac{\partial}{\partial x^i}$ and $\{x^i\}$, $i=1,2,3$ are Cartesian coordinates. It follows then

$$d\mathbf{l}\times(\nabla\times\mathbf{v})=d\mathbf{l}\times(e^i\times\partial_i\mathbf{v}). \quad (33)$$

Using the known identity $\mathbf{a}\times\mathbf{b}\times\mathbf{c}=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$ in Eq. (33), we obtain

$$d\mathbf{l}\times(e^i\times\partial_i\mathbf{v})=(d\mathbf{l}\cdot\partial_i\mathbf{v})e^i-(d\mathbf{l}\cdot e^i)\partial_i\mathbf{v}. \quad (34)$$

On the other hand, considering $d\mathbf{l}=(dl_1,dl_2,dl_3)=dl_ie^i$, we have

$$(d\mathbf{l}\cdot\nabla)\mathbf{v}=(dl_i\partial_i)\mathbf{v}=(d\mathbf{l}\cdot e^i)\partial_i\mathbf{v}. \quad (35)$$

Hence, substituting Eq. (34) and Eq. (35) in Eq. (31), we can rewrite it as

$$\begin{aligned}\nabla(d\mathbf{l}\cdot\mathbf{v})&=(d\mathbf{l}\cdot e^i)\partial_i\mathbf{v}+(d\mathbf{l}\cdot\partial_i\mathbf{v})e^i-(d\mathbf{l}\cdot e^i)\partial_i\mathbf{v}\\&=(d\mathbf{l}\cdot\partial_i\mathbf{v})e^i.\end{aligned} \quad (36)$$

From this last result, it is easy to see that

$$\begin{aligned}&\mathbf{X}\cdot[\nabla(d\mathbf{l}\cdot\mathbf{v})]=\mathbf{X}\cdot\left[(d\mathbf{l}\cdot\partial_i\mathbf{v})e^i\right]=X^i(d\mathbf{l}\cdot\partial_i)\mathbf{v}\\&=d\mathbf{l}\cdot(X^i\partial_i)\mathbf{v}=d\mathbf{l}\cdot[(\mathbf{X}\cdot\nabla)\mathbf{v}]=[(\mathbf{X}\cdot\nabla)\mathbf{v}]\cdot d\mathbf{l},\end{aligned}$$

where $\mathbf{X}=(X^1,X^2,X^3)=X^ie_i$, $e_i\cdot e^j=\delta_i^j$.

B Phenomenological interpretation of the first member of Eq. (3)

Let $\boldsymbol{S}$ be the inertial laboratory frame and $\boldsymbol{S}'$ the inertial frame that at time t has velocity $\mathbf{u}=\mathbf{v}\left(t,\sigma(t,\mathbf{x}(\ell))\right)$ (which is the velocity of an element of the circuit Γ_t).

The electric and magnetic fields observed in $\boldsymbol{S}'$ are [3,4]

$$\begin{aligned}&\mathbf{E}'_{\|}=\mathbf{E}_{\|},\quad \mathbf{B}'_{\|}=\mathbf{B}_{\|}\\&\mathbf{E}'_{\perp}=\gamma\left(\mathbf{E}_{\perp}+\mathbf{u}\times\mathbf{B}_{\perp}\right),\quad \mathbf{B}'_{\perp}=\gamma\left(\mathbf{B}_{\perp}-\mathbf{u}\times\mathbf{E}_{\perp}\right)\end{aligned} \quad (37)$$

where $\gamma=\sqrt{1-\mathbf{u^2}}$ is the *Lorentz factor* and the symbols $\|$ and $\perp$ denotes the components parallel and orthogonal to $\mathbf{u}$. Then, taking into account that $d\mathbf{l}'_{\|}=\gamma d\mathbf{l}$ and $d\mathbf{l}'_{\perp}=d\mathbf{l}_{\perp}$ and by letting $\mathbf{u}=\mathbf{v}\left(t,\sigma(t,\mathbf{x}(\ell))\right)$ it follows that

$$\int_{\Gamma_t}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}=\frac{1}{\gamma}\int_{\Gamma_t}\mathbf{E}'\cdot d\mathbf{l}'=-\frac{d\Phi}{dt}, \quad (38)$$

where $\Phi=\int_{\mathcal{S}_t}\mathbf{B}\cdot\mathbf{n}\;da$ is the flux of $\mathbf{B}$.

Using the right-side identity in Eq. (38), we can write

$$\int_{\Gamma_t}\mathbf{E}'\cdot d\mathbf{l}'=-\gamma\frac{d\Phi}{dt}=-\frac{d\Phi}{ds}, \quad (39)$$

since $ds=dt'=\gamma^{-1}dt$ is the element of proper time for an observer at rest in the commoving frame $\boldsymbol{S}'$ (with standard coordinates $\left(x'^0=t',x'^i\right)$. The integral $\int_{\Gamma_t}\mathbf{E}'\cdot d\mathbf{l}'$ is interpreted as the difference of potential measured by a voltmeter carried by an observer in $\boldsymbol{S}'$ (this is obviously clear when the field $\mathbf{v}$ is constant and the loop Γ_t is not deforming). So, we see that $\int_{\Gamma_t}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}$ differs by terms of second order in $\mathbf{v}^2$ from the differential potential measured by an observer in $\boldsymbol{S}'$.

Finally, we show an important result.[8] Let $F=\frac{1}{2}F_{\mu\nu}dx^\mu\wedge dx^\nu$ a 2-form field, be the so called *electromagnetic field* [1,24,25], where $\left(x^0=t,x^i\right)$ are standard

[8] Which may be intelligible for readers with working knowledge of the mathematical methods of modern field theory [1,2,24].

coordinates in a inertial reference frame in a Minkowski spacetime.[9] Then the antisymmetric matrix with entries $F_{\mu\nu}$ is

$$\begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}. \tag{40}$$

We now show that Φ is an invariant relativistic quantity. Indeed, it can be written as

$$\Phi = \int_{\mathcal{S}_t} F. \tag{41}$$

To show that, recall that if $A = \phi dx^0 + A_i dx^i$ is the *electromagnetic potential* (ϕ being the the *scalar potential* and $\mathbf{A} = (A_1, A_2, A_2)$ the *vector potential*), then $F = dA$. Then by Stokes theorem we can write (taking into account that $\mathcal{S}_t$ is, for any t, a 2-dimensional open spacelike surface in Minkowski spacetime and that $\Gamma_t = \partial\mathcal{S}_t$ is the boundary of $\mathcal{S}_t$)

$$\begin{aligned} \int_{\mathcal{S}_t} F &= \int_{\mathcal{S}_t} dA = \int_{\partial\mathcal{S}_t} A \\ &= \int_{\Gamma_t} A = \int_{\Gamma_t} \phi dx^0 + \int_{\Gamma_t} A_i dx^i \\ &= \int_{\Gamma_t} A_i dx^i = \int_{\Gamma_t} \mathbf{A} \cdot d\mathbf{l} \\ &= \int_{\mathcal{S}_t} (\nabla \times \mathbf{A}) \cdot \mathbf{n} da = \int_{\mathcal{S}_t} \mathbf{B} \cdot \mathbf{n} da. \end{aligned} \tag{42}$$

This shows that Eq. (3) (and its equivalent Eq. (34)) is a relativistic invariant law.

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[9] Minkowski spacetime [2, 24] is a manifold diffeomorphic to $\mathbb{R} \times \mathbb{R}^3$ equipped with a metric field η and its Levi-Civita connection. In a standard coordinates of an inertial reference frame $\eta = \eta_{\mu\nu} dx^\mu \otimes dx^\nu$, where the matrix with entries $\eta_{\mu\nu}$ is $\mathrm{diag}(1, -1, -1, -1)$.