Notas e Discussões

Gauss's law, infinite homogenous charge distributions and Helmholtz theorem

(A lei de Gauss, distribuições de carga homogêneas de extensão infinita e o teorema de Helmholtz)

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Rediscutimos a validade da lei de Gauss no caso de distribuições discretas e contínuas de carga que se estendem por todo o espaço.
Palavras-chave: eletromagnetismo clássico, eletrostática, lei de Gauss.

We rediscuss the validity of Gauss's law in the case of homogenous discrete and continuous charge distributions fulfilling all space.
Keywords: classical electromagnetism, electrostatics, Gauss's law.

In physics, some questions that we can formulate though frequently not realizable from a practical point of view, must as a matter of principle be satisfactorily answered. Moreover, they can be pedagogical useful tools and help the teacher to introduce new topics or discuss particularly hard ones. The following interesting question belongs to this type of question and can be found in [1]: Given a uniform charge distribution ρ that fulfills all the space, what is the value of the electric field at an arbitrary point of the distribution? The answer given in [1] though correct is nevertheless intriguing. By symmetry the electric field must be zero everywhere which makes perfect sense, but this means that the electric flux through an arbitrary surface $\mathcal{S}$ enclosing a certain charge $Q(\mathcal{S})$ is zero in conflict with the integral form of the Gauss's law that states that the flux must be equal to $Q(\mathcal{S})/\epsilon_0$. In order to keep the symmetrical solution for the field it is stated that Gauss's law is not valid in this situation because there are charges at the infinity. It is pedagogically worth to understand and rediscuss this problem and its answer and this is our aim in what follows.

Let us begin by taking an alternative route to Gauss's law. Consider a point charge q placed at a distance d from a point C and a hypothetical Gaussian sphere of radius R centered at C, see Fig. 1. Consider also a point P on the surface of the Gaussian sphere. If the distance of the point charge to P is r, then the electric field at P is given by
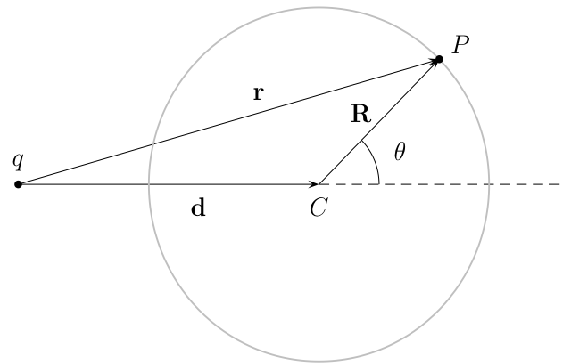
Figure 1 - Point charge and hypothetical Gaussian spherical surface.

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 \, r^2} \, \hat{\mathbf{e}}_r. \qquad (1)$$

It is convenient to treat d, R, and r as vectors, then $\mathbf{r} = \mathbf{d} + \mathbf{R}$ and we can write

$$\hat{\mathbf{e}}_r = \frac{\mathbf{r}}{r} = \frac{\mathbf{d} + \mathbf{R}}{r} = \frac{\mathbf{d} + \mathbf{R}}{(d^2 + R^2 + 2dR \cos\theta)^{1/2}}, \qquad (2)$$

where θ plays the role of a polar angle. The electric flux through an infinitesimal area centered around P is given by $d\Phi = \mathbf{E} \cdot \hat{\mathbf{n}} \, da$, where the outward normal vector $\hat{\mathbf{n}}$ at P can be set equal to $\hat{\mathbf{e}}_R = \mathbf{R}/R$. Therefore, the electric flux through the infinitesimal area $da = R^2 \sin\theta \, d\theta \, d\varphi$ is

$$d\Phi = \frac{q}{4\pi\epsilon_0} \frac{(\mathbf{d}+\mathbf{R})\cdot\hat{\mathbf{e}}_R\,da}{(d^2+R^2+2dR\cos\theta)^{3/2}}$$

$$= \frac{q}{4\pi\epsilon_0} \frac{(d\cos\theta+R)\,R^2\sin\theta\,d\theta\,d\varphi}{(d^2+R^2+2dR\cos\theta)^{3/2}}. \quad (3)$$

The total flux through the spherical surface is

$$\Phi = \frac{qR^2d}{2\epsilon_0} \int_0^\pi \frac{\cos\theta\,\sin\theta\,d\theta}{(d^2+R^2+2dR\cos\theta)^{3/2}}$$

$$+ \frac{qR^3}{2\epsilon_0} \int_0^\pi \frac{\sin\theta\,d\theta}{(d^2+R^2+2dR\cos\theta)^{3/2}}. \quad (4)$$

Introducing the standard variable transformation $x = \cos\theta$ and the real positive parameter $\lambda = d/R$, the total flux can be recast into the form

$$\Phi(\lambda) = \frac{q}{2\epsilon_0}\,F(\lambda), \quad 0\le\lambda<\infty, \quad (5)$$

where $F(\lambda)$ is defined by

$$F(\lambda) = \lambda \int_{-1}^{+1} \frac{x\,dx}{(1+\lambda^2+2\lambda x)^{3/2}} + \int_{-1}^{+1} \frac{dx}{(1+\lambda^2+2\lambda x)^{3/2}}. \quad (6)$$

This integral can be evaluated and the result is

$$F(\lambda) = 1 - \text{signum}\,(\lambda-1), \quad (7)$$

where for a real number p

$$\text{signum}\,(p) = \begin{cases} +1, & \text{if } p > 0\,, \\ -1, & \text{if } p < 0\,. \end{cases} \quad (8)$$

It folows that if $d > R$, that is $\lambda > 1$, then $F(\lambda) = 0$, and if $d < R$ or $0\le\lambda<1$, then $F(\lambda) = 2$. If $d = R$, or $\lambda = 1$, the integral is undefined. Therefore we conclude that if q is outside the Gaussian spherical surface, the electric flux is zero, if q is inside the flux is equal to q/ϵ_0. Because there is nothing special about the direction we have chosen in Fig. 1 to evaluate the flux we can extend this result to an arbitrary number of point charges. If we have N point charges outside the Gaussian sphere and M point charges inside of it we can make use of the superposition principle and write for the total flux

$$\Phi = \frac{1}{\epsilon_0} \sum_{j=1}^{N+M} q_j\,F(\lambda_j), \quad (9)$$

where $\lambda_j = d_j/R$. Making use of Eq. (8), we see that Eq. (9) simplifies to

$$\Phi = \frac{1}{\epsilon_0} \sum_{j=1}^{M} q_j. \quad (10)$$

All charges outside the region enclosed by Gaussian surface, no matter where they are, do not contribute to the electric flux through the Gaussian sphere. This result can be easily extend to continuous charge distributions. Of course we can obtain Gauss's law by making use of the concept of solid angle and arbitrarily shaped surfaces, but our Gaussian sphere can be made as large as we please and enclose any number of point charges (or a portion of the continuous distribution).

The main lesson that we can infer from the calculation above is that having charges at the infinity is not a good reason to discard Gauss's law in the context posed in [1], because these charges do not contribute to the flux whatsoever and hence seem not to be responsible for the failure of Gauss's law in this context. The flux depends entirely on the charges inside the Gaussian sphere and we still have a conundrum to solve.

One way out of it is to consider Maxwell's two equations for electrostatics

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}, \quad (11)$$

and

$$\nabla\times\mathbf{E} = 0, \quad (12)$$

and realize that the first one is not compatible with the answer dictated by the underlying symmetry of the distribution, that is, $\mathbf{E} = 0$ everywhere is not a valid solution of Maxwell's equations when we take ρ as a continuous uniform charge distribution fulfilling all the space. This solution is somewhat frustrating but seems to be the correct one. Is there some other solution for the electric field when ρ is a continuous uniform charge distribution fulfilling all the space? Yes, but there is a price we must pay, the homogeneity of the space is lost. The solution is

$$\mathbf{E}(\mathbf{r}) = \frac{\rho}{3\epsilon_0}\,(x\,\hat{\mathbf{x}}+y\,\hat{\mathbf{y}}+z\,\hat{\mathbf{z}}) = \frac{\rho}{3\epsilon_0}\,\mathbf{r}. \quad (13)$$

Isotropy is retained, but now there is a privileged point, the center of the distribution. Notice that given the nature of the charge distribution there are charges at the infinity (and the field increases indefinitely there). This solution is a mathematically valid solution of Eqs. (11) and (12) though it poses other kind of problems.

The same problem, but this time concerning the gravitational field in a cosmological context, was considered by Newton in his correspondence with Bentley [2]. What is the gravitational field in a uniform mass distribution that fulfills all the space? Newton's answer is analogous to the one given in [1] for the electrostatic case, namely: $\mathbf{g} = 0$ everywhere in space. Of course, Newton whose correspondence with Bentley is dated to 1692-93 was not aware of the existence of Gauss's law which was published in 1813 and his answer raised other type of questions. It must be also mentioned that a spheroidal shaped space filled with a uniform charge/mass distribution can be also considered, but in this case the field will not be purely radial.

We can ask ourselves under what conditions Eqs. (11) and (12) and their gravitational analogues have a finite, physical meaningful solution, in other words, under what conditions a vector field $\mathbf{F}$ is determined by its divergence and its curl? This question makes sense if we also specify certain conditions that the field must obey on the boundaries of the region of interest, the boundary conditions. If we impose the condition that the vector field must vanish very far away from its finite, localized sources, Helmholtz' theorem [3] assure us that the field is uniquely determined by its divergence and its curl. Specifically, given a vector field $\mathbf{F}(\mathbf{x})$, defined in some region $\mathcal{R}$, and the field equations

$$\nabla \cdot \mathbf{F}(\mathbf{x}) = D(\mathbf{x}), \tag{14}$$

$$\nabla \times \mathbf{F}(\mathbf{x}) = \mathbf{C}(\mathbf{x}), \tag{15}$$

with $\nabla \cdot \mathbf{C}(\mathbf{x}) = 0$, we can split $\mathbf{F}(\mathbf{x})$ into two parts, an irrotational part $\mathbf{F}_1(\mathbf{x})$ such that $\nabla \cdot \mathbf{F}_1 = D$ and $\nabla \times \mathbf{F}_1 = 0$, and a divergenceless one $\mathbf{F}_2(\mathbf{x})$ such that $\nabla \cdot \mathbf{F}_2 = 0$ and $\nabla \times \mathbf{F}_2 = \mathbf{C}$. Then we can write

$$\mathbf{F}_1(\mathbf{x}) = -\nabla\Phi, \tag{16}$$

and

$$\mathbf{F}_2(\mathbf{x}) = -\nabla \times \mathbf{A}(\mathbf{x}), \tag{17}$$

where Φ and $\mathbf{A}$ are auxiliary functions defined in $\mathcal{R}$. If we choose $\mathcal{R}$ conveniently and assume that the auxiliary functions behave at least as $1/r$ very far away from a field point labeled by $\mathbf{x}$, make use of Green's theorem, and finally perform the translation $\mathbf{x} \rightarrow \mathbf{x} - \mathbf{x}'$, we arrive at

$$\Phi(\mathbf{x}) = \frac{1}{4\pi} \int \frac{D(\mathbf{x}') \, d^3 x'}{\|\mathbf{x} - \mathbf{x}'\|}, \tag{18}$$

and

$$\mathbf{A}(\mathbf{x}) = \frac{1}{4\pi} \int \frac{\mathbf{C}(\mathbf{x}') \, d^3 x'}{\|\mathbf{x} - \mathbf{x}'\|}. \tag{19}$$

For more mathematical details and insights see [3–7]. This result is true for an extended localized source. For a pointlike source some extra care must be taken [3].

There are other examples of the trouble we can get into if we take too seriously unphysical situations and disregard appropriate boundary conditions. Consider,

for instance, a uniform but otherwise time-varying magnetic field pervading all space. Because there is not a single point that we can consider as the center, symmetry arguments lead us to conclude that the induced electric fields are null everywhere in conflict with Faraday's law.

The infinite homogeneous distributions that motivated this note do not obey the conditions under which Helmholtz' theorem holds. But how we should deal with these types of distributions that appear also in the form of plane charge and mass distributions, infinite line distributions, etc.? First of all notice that they are unphysical, though useful *approximations* if certain conditions are met. For those cases, symmetry arguments and not Eqs. (11) and (12) or their gravitational equivalents can be used to determine the electrostatic or gravitational field just as Newton and reference [1] did though their answers are not valid solutions of the corresponding field equations.

Acknowledgments

References

[1] J.M. Levy-LeBlond and A. Botuli, *La Physique en Questions Tome 2: Electricité et Magnetisme* (Vuibert, Paris, 1999), 2nd ed. Versão em português, *A Electricidade e o Magnetismo em Perguntas* (Gradiva, Lisboa, 1991).

[2] E. Harrison, *Cosmology: The Science of the Universe* (CUP, Cambridge, 2000), 2nd ed.

[3] R. Becker, *Electromagnetic Fields and Interactions* (Dover, New York, 1982).

[4] D.J. Griffiths, *Introduction to Electrodynamics* (Prentice Hall, Upper Saddle River, 1999), 3th ed.

[5] W.K.H. Panofsky and M. Philips, *Classical Electricity and Magnetism* (Addison-Wesley, Reading, 1962), 2nd ed.

[6] G.B. Arfken and H.J. Weber, *Mathematical Method for Physicists* (Academic, San Diego, 1995), 4th ed.

[7] H. Fleming, Revista Brasileira de Ensino de Física **2**, 155 (2001).