Solution for the electric potential distribution produced by sphere-plane electrodes using the method of images

(Solução para a distribuição do potencial elétrico produzido por eletrodos esfera-plano usando o método das imagens)

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Recebido em 26/9/2008; Revisado em 11/3/2009; Aceito em 24/4/2009; Publicado em 23/9/2009

The solution for the potential distribution between a spherical and planar electrode is briefly reviewed and derived in a simple way using the method of images. An analysis of the convergence of the potential, as a function of the number of image charges accounted, is also performed. Once the potential is obtained, the evaluation of the electric field, capacitance, energy of the system and the force between the electrodes is straightforward. The solution is illustrated with pictures, schematic representations and numerical examples.
Keywords: sphere-plane potential, electric potential, method of images, sphere-plane electrodes.

A solução para a distribuição do potencial em uma configuração de eletrodos esfera-plano é brevemente revisada e é obtida em detalhes usando-se o método das imagens. Uma vez que o potencial é obtido, o campo elétrico, a capacitância, a energia do sistema e a força entre os eletrodos podem ser deduzidos prontamente. É feita uma análise da convergência do potencial, como função do número de imagens consideradas no cálculo. A solução é ilustrada com figuras, representações esquemáticas e exemplos numéricos.
Palavras-chave: potencial esfera-plano, potencial elétrico, método das imagens, eletrodos esfera-plano.

1. Introduction

The solution of the potential distribution between a spherical and planar electrode constitutes an academic problem, which was already described by physicists almost 150 years ago. Nevertheless, it is not easy to find a reference with a solution that is immediately applicable. Although many authors, dealing with electrostatics, refer to the classical textbook by Smythe, unfortunately, this book does not present the potential distribution in a simple way [1].

The idea of image charge for field problems is due to Lord Kelvin, but Maxwell [2], Lodge [3], and Searle [?] extended the scope of the method. An excellent discussion on the distribution of the potential as a function of the system's coordinates was given by A. Foster in his PhD thesis [4] for a point charge between a sphere-plane capacitor. Foster uses the method of images to calculate the contribution of a point charge to the electric potential of the system.

Theoretical models for the electric potential distribution in the space between electrodes are useful for the calculation of the trajectories of charged particles and the prediction of flashover voltages over a given range of field conditions. Experiments based on high-voltage breakdown tests play an important rule in electrical engineering education. Real laboratory experiments designed to determine the voltage at breakdown have been combined with computer-based simulations and have produced stimulating teaching experiences. The experience gained with such a combination of teaching procedures is presented in the work of Lowther and Freeman [5]. The work of J.H. Cloete and J. van der Merwe [6] also describes an experiment for the determination of the voltage at breakdown by slowly decreasing the spacing between two conducting spherical electrodes. Their paper gives a detailed explanation on how to apply the method of images to model this practical problem. For many electrostatic systems, the method of images provides a simple solution, when solving Laplace equation would be very complicated, as is the case of the sphere-plane system of electrodes.

In this paper we also use the method of images to obtain an analytical solution for the potential distribution in the space between a conducting sphere and a plane electrode. Our goals are to familiarize students with this method by solving the sphere-plane electrostatic problem and to present easy to use equations for the potential and the electric field. Once the potential distribution is obtained, the electric field distribution and

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system parameters like capacitance, stored energy and force between the electrodes can be deduced straightforwardly.

2. Method of image charge

Almost without exception, old and new textbooks on applied electromagnetism discuss the method of images, and this clearly shows the importance of the method for science and engineering. According to Binns, Lawrenson and Trowbridge [7], the essence of the method of images consists in replacing the effects of a boundary related to an applied field by distributions of charges "behind" the boundary line as illustrated in Fig. 1 for two configurations that will be combined to deduce the potential for sphere-plane electrodes. The first configuration consists of a point charge and a grounded conducting infinite plane and this is shown in Fig. 1(a). The second configuration consists of a point charge and a grounded conducting sphere shown in Fig. 1(b). The solid and dashed lines represent the real and virtual fields respectively. The point from where the dashed lines diverge is the image charge. The field pattern, and consequently, the potential distribution for the real electrodes system is equivalent as the one generated by two point charges. The advantage of the image method is that the evaluation of the potential with point charges is simple and straightforward.

A schematic representation of the sphere-plane electrode system is presented in Fig. 2. The two electrodes are subject to a potential difference V and the deriva-

tion is presented in cylindrical coordinates. According to the chosen reference frame, the conducting plane is placed at the coordinate $z = 0$ and its electric potential ϕ_p is set to zero. The electric potential ϕ_s at the spherical electrode is V, being the potential difference between the spherical and the planar electrode. The sphere has radius a and its center is placed at $z = z_0$. The minimum distance d between the sphere and the grounded plane is $d = z_0 - a$.

To find the potential for this system using image charges, we start considering an isolated sphere with charge q_0, which generates the potential V at the surface. Then, q_0 can be expressed in terms of its resulting potential V as

$$q_0 = \frac{aV}{k},\tag{1}$$

where $k = 9 \times 10^9$ Vm/C is the electrostatic constant. In the presence of the plane at $z = 0$, charge q_0 generates an image of same magnitude and opposite sign $-q_0$ at position $-z_0$. The image charge $-q_0$ also generates an image in the sphere with position and magnitude given by

$$z_1 = z_0 - \frac{a^2}{2z_0},\tag{2}$$

$$q_1 = \frac{a}{2z_0}q_0.\tag{3}$$

(a)   (b)
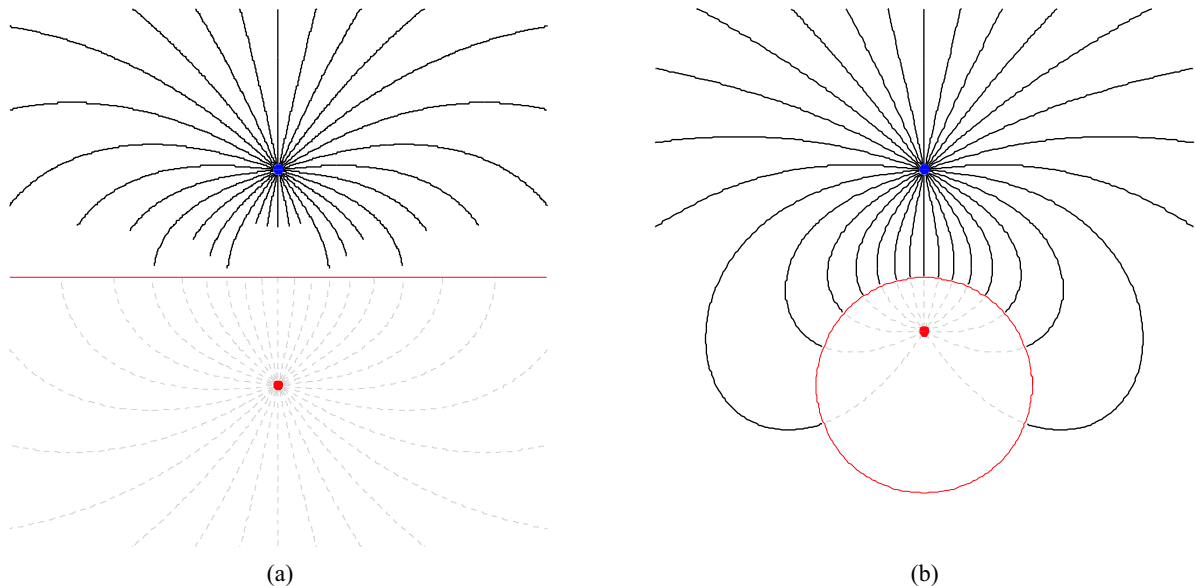
Figure 1 - The method of images for (a) a point charge and a grounded conducting plane and (b) a point charge and a grounded conducting sphere. The pattern of the electric field between point charge and electrode is identical to the pattern for two point charges.
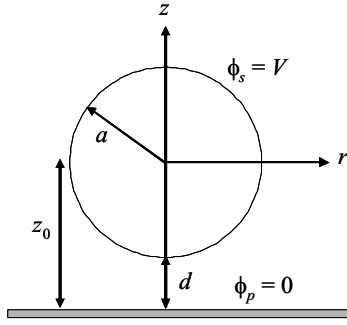
Figure 2 - Schematic representation of the sphere-plane electrodes in cylindrical coordinate system.

Table 1 - Schematic illustrations for the image charges and their first 6 values of ξ_i and z_i for 3 distances and $a = 1$ mm.

Image charges in the sphere	$\xi_i = q_i/q_0$	z_i (mm)
	$\xi_0 = 1$	$z_0 = 1.01$
	$\xi_1 = 0.495$	$z_1 = 0.515$
	$\xi_2 = 0.325$	$z_2 = 0.354$
	$\xi_3 = 0.238$	$z_3 = 0.277$
	$\xi_4 = 0.185$	$z_4 = 0.233$
	$\xi_5 = 0.149$	$z_5 = 0.206$
	$\xi_6 = 0.122$	$z_6 = 0.187$
$d = 10\ \mu m$		$z_\infty = 0.142$
	$\xi_0 = 1$	$z_0 = 1.1$
	$\xi_1 = 0.455$	$z_1 = 0.645$
	$\xi_2 = 0.260$	$z_2 = 0.527$
	$\xi_3 = 0.160$	$z_3 = 0.485$
	$\xi_4 = 0.101$	$z_4 = 0.469$
	$\xi_5 = 0.064$	$z_5 = 0.463$
	$\xi_6 = 0.041$	$z_6 = 0.460$
$d = 100\ \mu m$		$z_\infty = 0.458$
	$\xi_0 = 1$	$z_0 = 2.0$
	$\xi_1 = 0.25$	$z_1 = 1.750$
	$\xi_2 = 0.0667$	$z_2 = 1.733$
	$\xi_3 = 0.0178$	$z_3 = 1.732$
	$\xi_4 = 0.0047$	$z_4 = 1.732$
	$\xi_5 = 0.0013$	$z_5 = 1.732$
	$\xi_6 = 0.0003$	$z_6 = 1.732$
$d = 1000\ \mu m$		$z_\infty = 1.732$

This situation is depicted in Fig. 3(a). The image charge q_1, in turn, generates $-q_1$ at the plane, which generates q_2 in the sphere and so on. Fig. 3(b) represents the final charge distribution. The position and magnitude of the ith image charge are given by the recurrent relations

$$z_i = z_0 - \frac{a^2}{z_0 + z_{i-1}}, \qquad (4)$$

$$q_i = \frac{a}{z_0 + z_{i-1}} q_{i-1}. \qquad (5)$$

The derivation of Eqs. (2) to (5), which can be found in many textbooks, will not be repeated here.
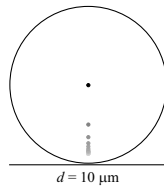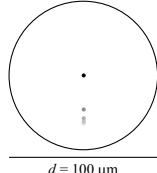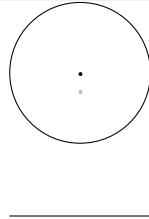
It is convenient to define a normalized charge $\xi_i = q_i/q_0$, that will be used later. Dividing Eq. (5) by q_0 we get

$$\xi_i = \frac{a}{z_0 + z_{i-1}} \xi_{i-1}, \qquad (6)$$

with $\xi_0 = 1$.

Charge distributions for three configurations of the problem are shown in Table 1. The results show the effect of the minimum distance d between the electrodes on the charge distribution in the spherical domain. The first six values of z_i and ξ_i for each d are also presented. It can be noticed from this table that $\xi_i \to 0$ and $z_i \to z_\infty$ (constant) as $i \to \infty$. In these pictures the relative magnitude of the charges is represented in a gray scale from black, for ξ_0 (unity), to white, for ξ_∞ (zero). In fact, these charges have images beneath the plane boundary, but they are not shown for the sake of simplicity. The position and magnitude of these image points are $-z_i$ and $-q_i$. The potential for the sphere-plane is equivalent of the potential of these two groups of charges $\{q_i, -q_i\}$.

3. Solution for the potential, the electric field and other physical quantities

The potential due to a charge of index i in the sphere and its image in the plane is given by

$$\phi_i = k \left(\frac{q_i}{[(z - z_i)^2 + r^2]^{1/2}} - \frac{q_i}{[(z + z_i)^2 + r^2]^{1/2}} \right). \qquad (7)$$
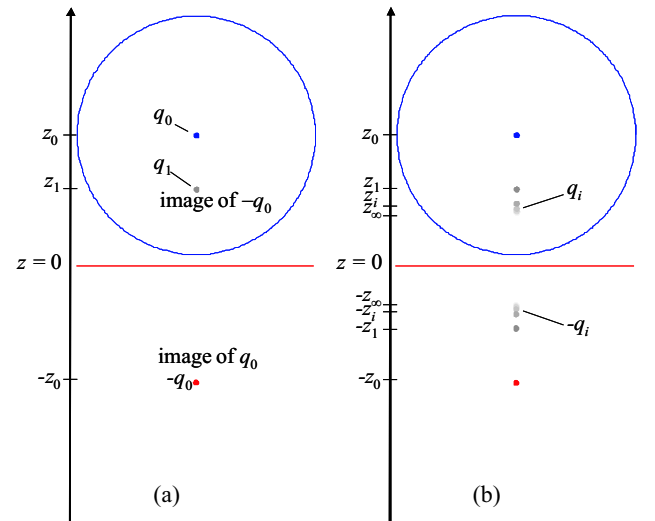

Figure 3 - Composition of the two systems of image points. (a) Representation of the first two images generated by the real charge and (b) the final distribution of image charges.

The potential due to all charges is completely determined by summing ϕ_i and using Eqs. (1) and (5) in Eq. (7), results in

$$\phi(r, z) = aV \sum_{i=0}^{\infty} \frac{\xi_i}{[(z - z_i)^2 + r^2]^{1/2}} - \frac{\xi_i}{[(z + z_i)^2 + r^2]^{1/2}}. \tag{8}$$

It is worth to point out that Eq. (8) is a solution for point charges, which is not the actual system of a sphere-plane. The solution in Eq. (8) is equivalent to the solution of the sphere-plane system only for $(z - z_0)^2 + r^2 \geq a^2$ (outside the sphere) and for $z \geq 0$ (above the plane). For the region inside the sphere $\phi = V$ at any point and beneath the plane $\phi = 0$ at any point. To account for these regions Eq. (8) has to be redefined as

$$\phi(r, z) = \begin{cases} 0 & z \leq 0 \\ V & r^2 + (z - z_0)^2 \leq a^2 \\ aV \sum_{i=0}^{\infty} \frac{\xi_i}{[(z - z_i)^2 + r^2]^{1/2}} - \frac{\xi_i}{[(z + z_i)^2 + r^2]^{1/2}} & \text{otherwise} \end{cases} \tag{9}$$

Once ϕ is given, electric field, capacitance, electrostatic energy and force on the electrodes can be obtained promptly. The electric field can be obtained from $\mathbf{E}(r, z) = -\nabla\phi$ and can be written as

$$E_r(r, z) = aVr \sum_{i=0}^{\infty} \frac{\xi_i}{[(z - z_i)^2 + r^2]^{3/2}} - \frac{\xi_i}{[(z + z_i)^2 + r^2]^{3/2}}, \tag{10}$$

$$E_z(r, z) = aV \sum_{i=0}^{\infty} \frac{\xi_i(z - z_i)}{[(z - z_i)^2 + r^2]^{3/2}} - \frac{\xi_i(z + z_i)}{[(z + z_i)^2 + r^2]^{3/2}}, \tag{11}$$

where E_r and E_z are the components of the field in r and z directions respectively, so that $\mathbf{E}(r, z) = E_r \, \hat{\mathbf{r}} + E_z \, \hat{\mathbf{z}}$. Fig. 4 shows the potential distribution as a density plot and the electric field lines. Details on the procedure to obtain these figures are shown in Appendix B.
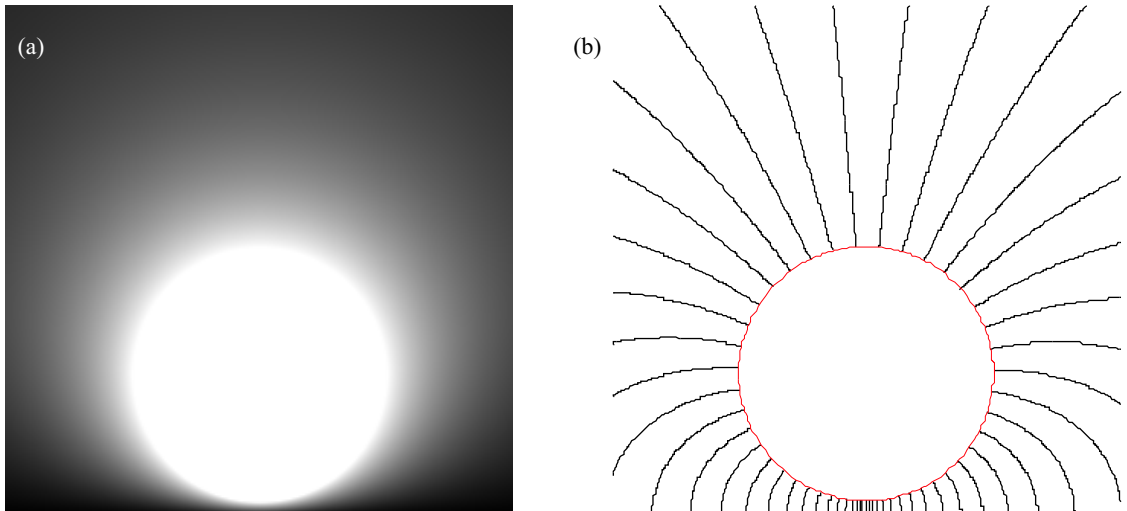

Figure 4 - (a) Density plot of the potential distribution from white, for $\phi = V$, to black, for $\phi = 0$; and (b) the field lines between the electrodes, where the field strength is represented by the density of the lines.

The sphere-plane electrode system is a capacitor of which the total charge q is the sum of all charges in the sphere, given by

$$q = \sum_{i=0}^{\infty} q_i = q_0 \sum_{i=0}^{\infty} \xi_i. \tag{12}$$

From the sequences of ξ_i in Table 1 it can be seen that the larger d is the faster ξ_i converges. So the number of terms to be summed for q in Eq. (12), to reach a specified accuracy, depends on d. A convergence analysis of q and of ϕ is presented in the next section.

The capacitance is obtained from $C = q/V$. Combining Eqs. (1) and (12) results in

$$C = \frac{a}{k} \sum_{i=0}^{\infty} \xi_i. \tag{13}$$

The electrostatic energy $U = \frac{1}{2}\, CV^2$ is given by

$$U = \frac{aV^2}{2k} \sum_{i=0}^{\infty} \xi_i, \tag{14}$$

and the force between sphere and plane $F = -\,\partial U/\partial z_0$ is

$$F = -\frac{aV^2}{2k} \sum_{i=0}^{\infty} \xi_i'. \tag{15}$$

The prime mark indicates a differentiation with respect to z_0. In this way, differentiation of (6) leads to the following recurrent relation to compute ξ_i'

$$\xi_i' = \frac{a}{z_0 + z_{i-1}} \left[-\xi_{i-1} \frac{1 + z_{i-1}'}{z_0 + z_{i-1}} + \xi_{i-1}' \right], \tag{16}$$

where z_i' is given by

$$z_i' = 1 + a^2 \frac{z_{i-1}'}{(z_0 + z_{i-1})^2}. \tag{17}$$

The starting conditions for the recurrent relations above are $\xi_0' = 0$ and $z_0' = 1$. All formulas are summarized in Appendix A. The treatment given above can also be applied to two spheres. If the spheres have the same size the formulas for the potential distribution are identical to Eq. (8). In this case the symmetry plane of the sphere-sphere problem becomes the plane of the sphere-plane system.

4. Convergence analysis of charge and potential

The number of terms needed for the total charge q in Eq. (12) to converge depends only on the ratio z_0/a. Table 2 shows the number of terms needed for q to converge to 99.9 % of its asymptotic value as a function of z_0/a.

Table 2 - Number of terms needed for the product $k.q$ to converge as a function of z_0/a.

z_0/a	$k.q$ ($\times\ 10^{-3}$ Vm)	Number of terms to converge to 99.9%
2	1.34	5
1.1	2.15	13
1.01	3.23	34
1.001	4.36	75

An analysis of the convergence of ϕ may be important as, for example, in fitting processes, where fitting parameters must be varied, then evaluated, and compared to experimental results repeatedly. A fitting procedure may take a lot of time and one must avoid computing unnecessary terms of ϕ. This analysis aims to give a good insight about the convergence behavior, i.e. where the convergence is faster/lower, how the convergence varies with z_0/a. In Fig. 5 we present the convergence behavior for two cases of z_0/a. In this figure the numbers represent the number of terms needed for ϕ in Eq. (8) to converge to 99.9 % of the asymptotic value at that point of the space. It can be seen that the region under the sphere needs more terms to converge. The number of terms needed is, at most, up to 50 even for z_0/a as small as 1.01. For much different cases than the examples used here, like: much higher accuracy, much wider range of the coordinates, ratio z_0/a much closer to unity, etc, requires an analysis for these specific cases. The asymptotic values of q and ϕ are taken with 100 terms in their sum.
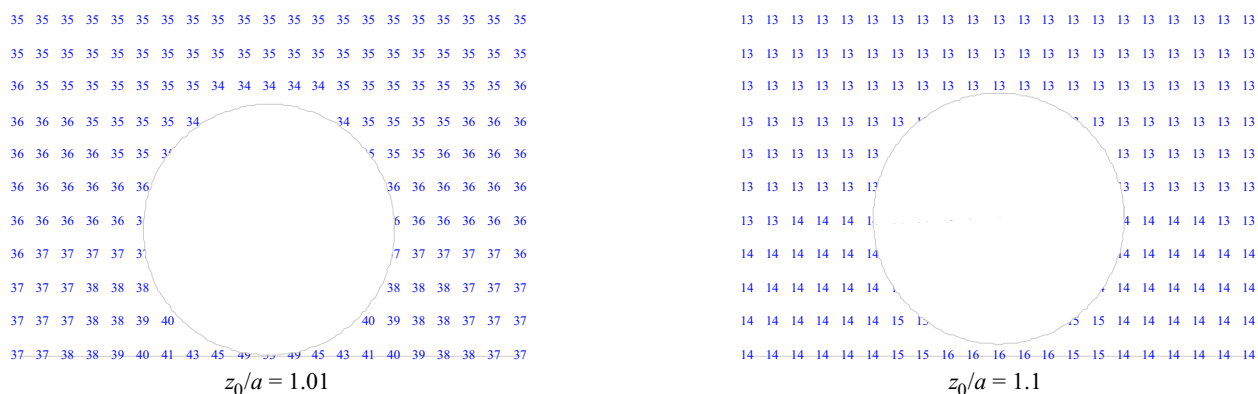
$z_0/a = 1.01$

$z_0/a = 1.1$

Figure 5 - Numbers indicate how many terms are necessary for the potential to converge to 99.9% of its asymptotic value at that point of the space.

5. Conclusion

We developed the solution for the potential, electric field, capacitance, energy and force for electrodes in a sphere-plane configuration using the method of image charges. Our equation, represented in a cylindrical coordinate system, can easily and conveniently be applied to systems with plane-sphere geometry. The methodology for solving this problem may be used in several other problems of electrostatics. We are confident that our treatment can conveniently be applied by students and engineers working on electrostatics.

Acknowledgments

Appendix A: Formulary

Image charge position	$z_i = z_0 - \frac{a^2}{z_0 + z_{i-1}}$ for $i > 0$
Initial (central) charge	$q_0 = \frac{aV}{k}$
Image charge magnitude	$q_i = \frac{a}{z_0 + z_{i-1}} q_{i-1}$ for $i > 0$
Relative charge magnitude q_i/q_0	$\xi_i = \frac{a}{z_0 + z_{i-1}} \xi_{i-1}$ for $i > 0$
$\partial z_i / \partial z_0$	$z_0' = 1;$ $z_i' = 1 + a^2 \frac{z_{i-1}'}{(z_0 + z_{i-1})^2}$ for $i > 0$
$\partial \xi_i / \partial z_0$	$\xi_0' = 0;$ $\xi_i' = \frac{a}{z_0 + z_{i-1}} \left[-\xi_{i-1} \frac{1 + z_{i-1}'}{z_0 + z_{i-1}} + \xi_{i-1}' \right]$ for $i > 0$
Total charge	$q = q_0 \sum_{i=0}^{\infty} \xi_i$
Potential	$\phi(r,z) = \begin{cases} 0 & z \le 0 \\ V & r^2 + (z - z_0)^2 \le a^2 \\ aV \sum_{i=0}^{\infty} \frac{\xi_i}{[(z-z_i)^2 + r^2]^{1/2}} - \frac{\xi_i}{[(z+z_i)^2 + r^2]^{1/2}} & \text{otherwise} \end{cases}$
Electric field: r component	$E_r(r,z) = aVr \sum_{i=0}^{\infty} \frac{\xi_i}{[(z-z_i)^2 + r^2]^{3/2}} - \frac{\xi_i}{[(z+z_i)^2 + r^2]^{3/2}}$
Electric field: z component	$E_z(r,z) = aV \sum_{i=0}^{\infty} \frac{\xi_i(z - z_i)}{[(z-z_i)^2 + r^2]^{3/2}} - \frac{\xi_i(z + z_i)}{[(z+z_i)^2 + r^2]^{3/2}}$
Capacitance	$C = \frac{a}{k} \sum_{i=0}^{\infty} \xi_i$
Energy	$U = \frac{aV^2}{2k} \sum_{i=0}^{\infty} \xi_i$
Force	$F = -\frac{aV^2}{2k} \sum_{i=0}^{\infty} \xi_i'$

Appendix B: Algorithms and programs for rendering graphics

Once the equations for the potential and the electric fields are established their corresponding graphics can be rendered for a visual presentation of the solution. This is not a trivial procedure. Many high level program languages are available for algebraic manipulation and graphics. In this section we will line out a procedure in *Mathematica*® [8]- a software with a very high level programming language- that we used to render the figures shown in this article. Next sections will provide an overall procedure to do the programs in *Mathematica* and will present the programs used.

B1. Image charges in the sphere

The figures shown in Fig. 3 and Table 1 was obtained as follows:

1. Numerical values are attributed to z_0, a.

2. Normalized charges ξ_i and their positions z_i are calculated for i from 0 to 100, which is more than sufficient.

3. The representation of the sphere, the plane and the points $(0, z_i)$ are drawn using built-in commands Circle[...], Line[...] and Point[...] respectively. The points are rendered in gray level using command GrayLevel[...].

4. Graphics are merged and rendered.

The program used is shown below. An animation can be found in the internet with URL given in Ref. [9].

```
Clear[i,a,z,zi,r,z0]; "Clear parameters attributions";
 a=10^-3; "radius of the sphere";
 d=10^-5; "distance sphere-plane";
 z0=a+d; "z coordinate of the sphere center";
 nter=100; "number of image charges to compute";
 zi[0]=z0; "position of first charge";
 xsi[0]=1; "magnitude of first charge";

Do[
    zi[i]=z0-a^2/(z0+zi[i-1]); "z positions of image charges";
    xsi[i]=a xsi[i-1]/(z0+zi[i]); "normalized magnitude of image charges";
    ,{i,nter}];

ImageCharges= Table[{PointSize[0.02], GrayLevel[1-xsi[i]], Point[{0,zi[i]}]}, {i,nter,0,-1}];
 "Image charges characteristics to be drawn as points";

 G=Graphics[{Circle[{0,z0},a], Line[{{-a,0},{a,0}}], ImageCharges}, AspectRatio-> 1,
 PlotRange->All]; "Draw of the circle and the line and the points";

 Show[G, AspectRatio->(z0+a)/(2a)];
 "Render the circle, the line and the points in the same graphic";
```

B2. Density plot of the potential

To render Fig. 4(a):

1. We attributed numbers to parameters a and z_0.

2. We computed z_i and ξ_i for i from 0 to 100.

3. We wrote the potential as in Eq. (9) like $\phi = $ If[*condition, then, else*], where *condition* is $(z - z_0)^2 + r^2 \geq a^2$, block *then* must be the expression for ϕ outside the sphere and block *else* is the expression for ϕ inside the sphere (V).

4. We rendered the potential with built-in command DensityPlot[$\phi(r,z)$].

The program used is shown below. See also URL in Ref. [10].

```
Clear[i,a,d,z0,nter,zi,xsi,Fi]; "Clear all variables that will be
used";
 a=0.001; "radius of the sphere";
 d=0.0001; "distance sphere-plane";
 z0=a+d; "z coordinate of the center of the sphere";
 nter=100; "number of terms to be computed in the potential";
```

```
V=1; "potential at the sphere";
zi[0]=z0; "position of the i(th) image charge";
xsi[0]=1; "relative magnitude of the i(th) image charge";

Do[
    zi[i]=z0-a^2/(z0+zi[i-1]); "z position of the image charges";
    xsi[i]=a xsi[i-1]/(z0+zi[i-1]); "normalized magnitude of the image charges";
    ,{i,nter}];

Fi=If[(z-z0)^2+r^2>=a^2,
    a V Sum[xsi[i]/Sqrt[(z-zi[i])^2+r^2]-xsi[i]/Sqrt[(z+zi[i])^2+r^2], {i,0,nter}],
    V]; "Attribution of a function to the potential outside and inside the sphere.";

DensityPlot[Fi, {r,-2a,2a}, {z,0,4a}, Mesh-> False, Frame-> False, PlotPoints-> 200,
AspectRatio-> 1];
  "Render the potential as a Density plot.";
```

B3. Field lines between the electrodes

To render Fig. 4(b):

    1. We attributed numerical values to a ,z_0 and few other parameters used.

    2. We computed z_i and ξ_i for i from 0 to 100.

    3. We attributed expressions for the electric field functions E_r and E_z as in Eqs. (10) and (11).

    4. Points on the circle that represents the sphere where selected as starting points of the electric field line. The starting point of each electric field line is not equally spaced from each other. Their spacing is proportional to the strength of the electric field, relative to the maximum field at the bottom of the sphere. This is done so the reader can visualize the field strength by the density of the field lines.

    5. The field line is built, point by point, using the coordinate of the previous point plus a constant step toward the field direction to calculate the next point of the field line. New points are calculated until the field line reaches the border of the region to be shown in the graphic. The points are stored in a list and plotted.

    The program is shown below. See also Ref. [11]. Fig. 1(a) and (b) are particular cases of this program and will not be shown here. Animations of Fig. 1(a) can be seen in Ref. [12].

```
"Clear parameters";
 Clear[a,d,z0,V,nter,zi,xsi,nlines,del,i,ii,j,k,r,z,rfl,zfl,fieldline,Er,Ez,Emod,tet,pretet];
 a=0.001;"Sphere radius";
 d=0.0001;"distance sphere-plane";
 z0=a+d;"z coordinate of the center of the sphere";
 V=1;"Voltage at the sphere";
 zi[0]=z0;"z positions of the initial charge";
 xsi[0]=1;"normalized magnitude of the initial charge";
 nter=100;"number of terms to compute the electric field";
 nlines=50;"number of field lines to draw";
 del=a/100.;"step distance between two points in a field line";

 Do[fieldline[j]={},{j,nlines}];"list to store the points of the k(th) field line ";
 Do[
    zi[i]=z0-a^2/(z0+zi[i-1]);"z positions of the image charges";
    xsi[i]=a xsi[i-1]/ (z0+zi[i-1]);
    "normalized magnitude of the image charges";
    ,{i,nter}];
Er=a V r Sum[xsi[i]/((z-zi[i])^2+r^2)^(3/2)-xsi[i]/((z+zi[i])^2+r^2)^(3/2),{i,0,nter}];
"r component of the electric field";
Ez=a V Sum[xsi[i](z-zi[i])/((z-zi[i])^2+r^2)^(3/2)- xsi[i](z+zi[i])/
((z+zi[i])^2+r^2)^(3/2),{i,0,nter}];
```

```
"z component of the electric field";
Emod=Sqrt[Er^2+Ez^2]; "module of the electric field";
Do[
  pretet=Abs[1/Emod]/.{r->a Sin[2 Pi k/nlines],z->z0-a Cos[2 Pi k/nlines]};
  "pretet is auxiliary to determine the points on the sphere at which a field line will depart";
  tet=2Pi Sum[pretet,{k,j}]/Sum[pretet,{k,nlines}];
  "tet is the angle between z-axes and the point at the sphere at which a field line will depart";
  rfl[0]=a Sin[tet];"initial r coordinate of the j(th) field line";
  zfl[0]=z0-a Cos[tet];"initial z coordinate of the j(th) field line";

  "-------------Block to build the j(th) field line-----------------";
  ii=0;"couter";
  While[rfl[ii]\[GreaterEqual]-2a && rfl[ii]<=2a && zfl[ii]>=0 && zfl[ii]<=4a,
    ii++;"Increment ii";
    rfl[ii]=(r+Er/Emod del)/.{r->rfl[ii-1],z->zfl[ii-1]};
    "r coordinate of the ii(th) point of the j(th) field line";
    zfl[ii]=(z+Ez/Emod del)/.{r->rfl[ii-1],z->zfl[ii-1]};
    "z coordinate of the ii(th) point of the j(th) field line";
    AppendTo[fieldline[j],{rfl[ii],zfl[ii]}];
    "Append last calculated point (rfl,zfl) to the j(th) field line";
    ];
  "----------------------------------------------------------------";
  P[j]=ListPlot[fieldline[j],PlotJoined->True,PlotRange->{{-2a,2a},{0,4a}},
      DisplayFunction->Identity];
  ,{j,nlines}];"Store the graphics of the j(th) field line in P[j]";
Plist=Table[P[j],{j,nlines}];"Does a list with all field lines";
G=Graphics[{RGBColor[1,0,0], Circle[{0,z0},a]}];
"Store in G the graphics of the field lines and the sphere";
Show[{Plist,G},Ticks->None,Axes->{True,False},AspectRatio->1, DisplayFunction-> $DisplayFunction];
"Render the field lines and the sphere together";
"-----------------------------------------------";
```

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