Energy Momentum Complex

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We show that the definition of the energy-momentum complex given by Møller using Weitzenböck space-time in the calculations of gravitational energy gives results which are different from those obtained from other definitions given in the framework of general relativity.

Keywords: Møller's energy-momentum complex, Weitzenböck space-time, energy-momentum complexes in general relativity.

Since the advent of Einstein's energy-momentum complex, used for calculating energy and spatial momentum in a general relativistic system, many definitions of the energy-momentum complexes have been found for instance Landau and Lifshitz, Papapetrou and Weinberg [1]. All these definitions of the energy-momentum complexes mainly depend on the metric tensor $g_{\mu\nu}$. The major difficulty of these definitions is that they are coordinate dependent. Møller [2] has constructed an energy-momentum complex which enables one to evaluate energy and spatial momentum in any coordinate system. However, Møller's [3] definition of this energy-momentum complex suffer some criticism. Teleparallel theories of gravity have been considered long time ago in connection with attempts to define the energy of gravitational field [4, 5]. It is clear from the properties of the solutions of Einstein field equation of an isolated system that a consistent expression for the energy density of the gravitational field would be given in terms of second order derivatives of the metric tensor. It is well known that there exists no covariant, nontrivial expression constructed out of the metric tensor, both in three and four dimensions that contain such derivatives. However, covariant expressions that contain second order derivatives of the tetrad fields are feasible. Thus it is legitimate to conjecture that the difficulties regarding the problem of defining the gravitational energy-momentum is related to the geometrical description of the gravitational field rather than being an intrinsic drawback of the theory [6]. Møller later showed that Weitzenböck space-time [7] description of gravitation allows a more satisfactory treatment of the energy-momentum complex than does general relativity [8]. It is the aim of this brief report to show that the energy-momentum complex given by Møller in 1978 gives results which are different from the energy-momentum complexes given in the framework of general relativity.

Now let us consider the solution given by Mikhail et al. [9]. They obtained a spherically symmetric solution in vacuum in Møller tetrad theory of gravitation for a tetrad which has three unknown functions of the radial coordinate r and having spherical symmetry. This solution is given by

$$
\left(\lambda_i{}^{\mu}\right) = \begin{pmatrix} \dfrac{i\mathcal{A}}{X} & iX_1\mathcal{D}R & 0 & 0 \\[2ex] \dfrac{\mathcal{A}\mathcal{D}R\sin\theta\cos\phi}{X} & X_1\sin\theta\cos\phi & \dfrac{\cos\theta\cos\phi}{R} & -\dfrac{\sin\phi}{R\sin\theta} \\[2ex] \dfrac{\mathcal{A}\mathcal{D}R\sin\theta\sin\phi}{X} & X_1\sin\theta\sin\phi & \dfrac{\cos\theta\sin\phi}{R} & \dfrac{\cos\phi}{R\sin\theta} \\[2ex] \dfrac{\mathcal{A}\mathcal{D}R\cos\theta}{X} & X_1\cos\theta & \dfrac{-\sin\theta}{R} & 0 \end{pmatrix}, \tag{1}
$$

where $i=\sqrt{-1}$ is a factor to preserve Lorentz signature and $\left(\lambda_i{}^{\mu}\right)$ are the contravariant components of the tetrad field. Here $\mathcal{A}$ and $\mathcal{D}$ are given in terms of the unknown function $\mathcal{B}(\mathcal{R})$ as

$$
\mathcal{A}(\mathcal{R}) = \frac{1}{1 - R\mathcal{B}'}, \quad \mathcal{D}(\mathcal{R}) = \frac{1}{X_1}\sqrt{\frac{2M}{R^3} - \frac{\mathcal{B}'}{R}(1 - X_1)}, \tag{2}
$$

$$
\mathcal{B}' = \frac{d\mathcal{B}(\mathcal{R})}{dR}, \quad R = \frac{r}{\mathcal{B}}, \quad X = 1 - \mathcal{D}^2R^2, \quad X_1 = (1 - R\mathcal{B}'). \tag{3}
$$

As is clear from (1) that the solution contains one arbitrary function of the radial coordinate, i.e., $\mathcal{B}(\mathcal{R})$. In spite of this all previous solutions can be obtained from it [9]. For example when $\mathcal{B}(\mathcal{R}) = 1$ or $\mathcal{B}(\mathcal{R}) = \sqrt{2M/R}$ we discover the solutions given in [13] which reproduce the Schwarzschild metric. The associated metric, which is defined by $g_{\mu\nu} \overset{\text{def.}}{=} \left(\lambda_i{}_{\mu}\right)\left(\lambda_i{}_{\nu}\right)$, with solution (1) gives Schwarzschild

metric in the form

$$ds^2 = -\eta_1 dT^2 + \frac{dR^2}{\eta_1} + R^2 d\Omega^2, \qquad \eta_1(R) = \left(1 - \frac{2M}{R}\right),$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \qquad\qquad (4)$$

regardless the value of $\mathcal{B}(\mathcal{R})$.

Now we are going to calculate the energy content associated with solution (1) using different definitions of the energy-momentum complex given in general relativity and that given by Møller in 1978.

The Einstein energy-momentum complex is [2]

$$\Theta_i{}^k = \frac{1}{16\pi} H_i{}^{kl}{}_{,\,l},$$

$$H_i{}^{kl} = -H_i{}^{lk} = \frac{g_{in}}{\sqrt{-g}} \left[-g(g^{kn}g^{lm} - g^{ln}g^{km}) \right]_{,\,m},$$

$$(5)$$

where comma denotes the differentiation with respect to the coordinates. The energy and spatial momentum are given by

$$P_i = \frac{1}{16\pi} \int\int H_i{}^{0\alpha} n_\alpha dS, \qquad (6)$$

where n_α is the outward unit normal vector over the infinitesimal surface element dS, P_0 and P_α stand for the energy and spatial momentum.

The symmetric energy-momentum complex of Landau

and Lifshitz is [10]

$$L^{ik} = \frac{1}{16\pi} \lambda^{iklm}{}_{,\,lm}, \qquad \lambda^{iklm} = -g(g^{ik}g^{lm} - g^{il}g^{km}), \quad (7)$$

L^{00} and $L^{\alpha 0}$ are the energy and spatial momentum density components. The energy and spatial momentum are given by

$$P^i = \frac{1}{16\pi} \int\int \lambda^{i0\alpha m}{}_{,\,m} n_\alpha dS, \qquad (8)$$

The symmetric energy-momentum complex of Papapetrou is [11]

$$\sum{}^{ik} = \frac{1}{16\pi} N^{iklm}{}_{,lm},$$

$$N^{iklm} = \sqrt{-g}(g^{ik}\eta^{lm} - g^{il}\eta^{km} + g^{lm}\eta^{ik} - g^{lk}\eta^{im}),$$

$$(9)$$

$\eta^{ik} = diag(-1,1,1,1)$, $\sum{}^{00}$ and $\sum{}^{\alpha 0}$ are the energy and spatial momentum density components. The energy and spatial momentum are given by

$$P^i = \frac{1}{16\pi} \int\int N^{i0\alpha\beta}{}_{,\,\beta} n_\alpha dS, \qquad (10)$$

The symmetric energy-momentum complex of Weinberg is [12]

$$W^{ik} = \frac{1}{16\pi} D^{lik}{}_{,\,l}, \qquad D^{lik} = \frac{\partial h^a{}_a}{\partial x_l}\eta^{ik} - \frac{\partial h^a{}_a}{\partial x_i}\eta^{lk} - \frac{\partial h^{al}}{\partial x_a}\eta^{ik} + \frac{\partial h^{ai}}{\partial x_a}\eta^{lk} + \frac{\partial h^{lk}}{\partial x_i} - \frac{\partial h^{ik}}{\partial x_l}, \qquad (11)$$

$h_{ik} = g_{ik} - \eta_{ik}$ and the indices on h_{ik} or $\partial/\partial x_i$ are raised or lowered by $\eta's$. W^{00} and $W^{\alpha 0}$ are the energy and spatial momentum density components. The energy and spatial momentum are given by

$$P^i = \frac{1}{16\pi} \int\int D^{\alpha 0i} n_\alpha dS, \qquad (12)$$

The energy-momentum complex given by Møller in 1978 is [13]

$$\mathcal{M}_\mu{}^\nu = \mathcal{U}_\mu{}^{\nu\lambda}{}_{,\lambda}, \qquad \mathcal{U}_\mu{}^{\nu\lambda} = \frac{(-g)^{1/2}}{16\pi} P_{\chi\rho\sigma}{}^{\tau\nu\lambda} [\Phi^\rho g^{\sigma\chi} g_{\mu\tau} - \lambda g_{\tau\mu}\gamma^{\chi\rho\sigma} - (1-2\lambda)g_{\tau\mu}\gamma^{\sigma\rho\chi}], \qquad (13)$$

where

$$P_{\chi\rho\sigma}{}^{\tau\nu\lambda} \stackrel{\text{def.}}{=} \delta_\chi{}^\tau g_{\rho\sigma}{}^{\nu\lambda} + \delta_\rho{}^\tau g_{\sigma\chi}{}^{\nu\lambda} - \delta_\sigma{}^\tau g_{\chi\rho}{}^{\nu\lambda}. \quad g_{\rho\sigma}{}^{\nu\lambda} \stackrel{\text{def.}}{=} \delta_\rho{}^\nu \delta_\sigma{}^\lambda - \delta_\sigma{}^\nu \delta_\rho{}^\lambda,$$

$$\gamma_{\mu\nu\rho} \stackrel{\text{def.}}{=} \underset{i}{\lambda}_\mu \underset{i}{\lambda}_{\nu;\,\rho}, \qquad\qquad\qquad\qquad \Phi_\mu \stackrel{\text{def.}}{=} \gamma^\rho{}_{\mu\rho}, \qquad\qquad (14)$$

and the semicolon denotes covariant differentiation with respect to Christoffel symbols. The energy and spatial momentum are given by

$$P_i = \int\int \mathcal{U}_i{}^{0\alpha} n_\alpha dS, \qquad (15)$$

Now we proceed to calculate the energy associated with solution (1) using the definitions (6,8,10,12) which are given in general relativity

$$E_E = E_{LL} = E_P = E_W = M, \qquad (16)$$

while if we calculate the energy of the same solution using the definition (15) given by Møller in the Weitzenböck space-time we get

$$E_M = 2M - \lim_{R\to\infty}(R^2 \mathcal{B}'), \qquad (17)$$

where E_E, E_{LL}, E_P, E_W *and* E_M stand for the energy (P_0) using the definition of the energy-momentum complex given by Einstein, Landau and Lifshitz, Papapetrou, Weinberg and Møller.

The arbitrary function $\mathcal{B}$ must be non vanishing so that the area of a sphere of constant R is finite [14]. We also assume that $\mathcal{A}(\mathcal{R})$ and $\mathcal{B}(\mathcal{R})$ satisfy the asymptotic condition

$$\lim_{R\to\infty}\mathcal{A}(\mathcal{R}) = \lim_{R\to\infty}\mathcal{B}(\mathcal{R}) = 1 \quad and \quad \lim_{R\to\infty}(R\mathcal{B}') = 0, \qquad (18)$$

using (18) in (17) we get

$$E = 2M. \qquad (19)$$

As is clear from (16) and (19) that there is a difference between the calculations done using the definitions (6,8,10,12) constructed within general relativity and that constructed within Weitzenböck space-time. This is due to the fact that the definitions given in general relativity mainly depend on the metric tensor $g_{\mu\nu}$ and as is clear from (4) that the metric tensor of this solution depends only on the gravitational mass M and the radial coordinate R. However, the definition (15) depends mainly on the covariant components of tetrad field $\underset{i}{\lambda}_\mu$ or the contravariant components.

The difference between the two results (16) and (19) does not means that the calculations using the definitions within the framework of general relativity theory is more accurate than that used in the framework of Weitzenböck space-time. But may be related to some reasons:

i) Dividing the tetrad into two classes, the one in which the components $(\underset{a}{\lambda}_0)$ and $(\underset{0}{\lambda}_\alpha)$ of the parallel vector fields $(\underset{a}{\lambda}_\mu)$ tend to zero faster than $1/\sqrt{R}$ for large R and the other, in which those components go to zero as $1/\sqrt{R}$. It was found that the equality of the energy and the gravitational mass holds only in the first class. It is of interest to note that the tetrad structures (1) has such property, i.e., the components $(\underset{a}{\lambda}_0)$ and $(\underset{0}{\lambda}_\alpha)$ go to zero as $1/\sqrt{R}$. So its energy content is different from the energy content given by (16) [14].

ii) Many authors believe that a tetrad theory should describe more than a pure gravitation field. In fact; Møller himself [15] considered this possibility in his earlier trials to modify general relativity. In these theories, the most successful candidates for the description of the other physical phenomenon are the skew-symmetric tensors of the tetrad space, e.g., $\Phi_{\mu;\nu} - \Phi_{\nu;\mu}$. Some authors; e.g; [16, 17], believe that these tensors are related to the presence of an electromagnetic field. Others; e.g.; [18] believe that these tensors are closely connected to the spin phenomenon. There are a lot of difficulties to claim that Møller's theory deserves such a wider interpretation. This needs a lot of investigations before arriving at a concrete conclusion.

The result given by (19) shows that the calculations of energy depends only on the mass of the system and this is consistence with the principle of energy that the total energy of the system is independent of the coordinate. Also this result shows that the arbitrary function $\mathcal{B}$) has no effect on the calculations of energy in spite that it plays a central role in obtaining the previous known solution by given it a specific form.

The same problem (i.e. the definitions within the framework of general relativity and the definition given by Møller using Weitzenböck space-time will lead to different results) will appear for the solutions obtained before by Shirafuji et al. [14] and [19].

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