Finding Invariant Tori in the Problem of a Periodically Corrugated Waveguide

Adriano Fábio Rabelo* and Edson D. Leonel
Departamento de Estatística, Matemática Aplicada e Computação,
Universidade Estadual Paulista, CEP: 13506-900, Rio Claro, SP, Brazil

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Some dynamic properties for a light ray suffering specular reflections inside a periodically corrugated waveguide are studied. The dynamics of the model is described in terms of a two dimensional nonlinear area preserving map. We show that the phase space is mixed in the sense that there are KAM islands surrounded by a large chaotic sea that is confined by two invariant spanning curves. We have used a connection with the Standard Mapping near a transition from local to global chaos and found the position of these two invariant spanning curves limiting the size of the chaotic sea as function of the control parameter.

Keywords: Chaos; Invariant Spanning Curves; Nonlinear Mappings

I. INTRODUCTION

In recent years the interest in the problem of guiding a light ray has increased. Particularly, such interest is mainly because the subject can be applied in many different fields of science including that in the dynamics of ray chaos in underwater acoustic [1–3], quantum transport in ballistic cavities [4], quantized ballistic conductance in a periodically modulated quantum channel [5] and scattering of a quantum particle in a rippled waveguide [6].

The approach we are considering in the present paper consists basically in describing the dynamics of a periodically corrugated waveguide by using the formalism of discrete mappings. Such a formalism is particularly used in the description of classical billiard problems. There are three different classes concerning on billiard problems, namely: (i) mixed; (ii) ergodic and (iii) integrable. For class (i), the phase space of the model exhibit invariant spanning curves (invariant tori) and chaotic seas that generally surround KAM islands [7–10]. For case (ii), the time evolution of a single initial condition is enough to fills up entirely and ergodically all the accessible phase space [11]. Finally case (iii) shows only periodic and quasi periodic behaviour in the phase space.

In this papper we revisit the problem of a periodic corrugated waveguide seeking to understand and describe the position of the two lower invariant spanning curves in the phase space. We consider a connection with the Standard Mapping near a transition from local to global chaos and obtain an effective control parameter as well as an effective angle that describe the position of the first invariant tori in the phase space.

The organization of the paper is as follows. In section II we describe the model and the mapping. Section III describes the connection with the Standard Mapping. Our conclusion and final remarks are drawn in section IV.

*Electronic address: adrifara@rc.unesp.br

II. THE MODEL AND THE NONLINEAR MAPPING

Let us describe the model and construct the equations of the mapping. The model consists in considering a light ray wich is specularly reflected by two reflective surfaces. One of them is assumed to be a parallel plate at $y=0$ while the other one is described by $y=a+d\cos(kx)$. The geometry of the model is shown in Fig. 1 where a is the average distance of the

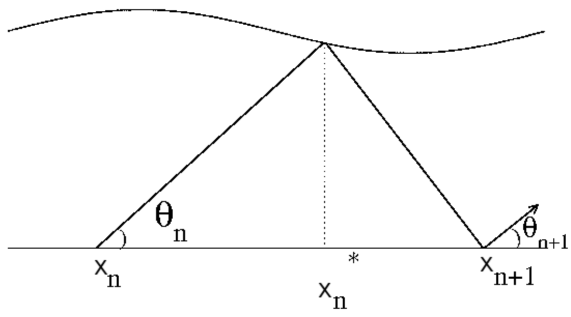

FIG. 1: Reflection from a corrugated surface of a light ray coming from the flat surface at $y=0$. The dynamic variables x_n and θ_n are defined along the text.

surfaces, d is the corrugation and k is the wavenumber. The dynamics of the model is given by a mapping T that gives the reflection angle of the light with respect to the X-axis θ_n and the location of the reflection x_n, i.e. $T(\theta_n,x_n)=(\theta_{n+1},x_{n+1})$ where the index n denotes the nth reflection of the light with the flat plate. Thus, given an initial condition, (θ_n,x_n), we can obtain the new reflection angle and the position (θ_{n+1},x_{n+1}) using purely geometrical arguments as they are shown in Fig. 2.

Considering the first part of the light ray trajectory we obtain that

$$x_n^*-x_n=a+d\cos(kx_n^*)/\theta_n\ . \quad (1)$$

On the other hand, the equation for the second part of the light

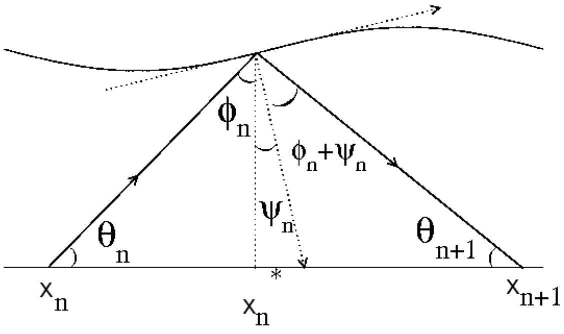

FIG. 2: Details of the trajectory before and after a reflection with the corrugated surface. We stress the dynamic variables x_n and θ_n are defined along the text.

trajectory is given by,

$$x_{n+1} - x_n^* = a + d\cos(kx_n^*)/\theta_{n+1} \ , \tag{2}$$

where the variable x_n^* gives the exact position of the light at the reflection with the corrugated surface. The angle θ_n is given by,

$$\theta_{n+1} = \theta_n - 2\psi_n \ , \tag{3}$$

where ψ_n is the slope of the corrugated surface at $x = x_n^*$. It is obtained from $\tan(\psi_n(x)) = dy(x)/dx = -dk\sin(kx_n^*)$. It must be emphasized that x_n^* can only be numerically obtained from solution of the transcendental equation (see Eq. (1)). Thus the solution of Eq. (1) and evaluation of Eqs. (2) and (3) gives the evolution of a light ray in the complete version of the model.

In this paper however, we will consider only small values for the corrugation d so that we can propose a simplified version of the model and avoid solving equation (1) numerically. Before writing the equations let us propose two approaches for the model. Firstly, we shall consider that the relative corrugation of the upper surface is very small. This implies that, $a + d\cos(kx_n^*) \cong a$. For this limit of corrugation, we assume that $\tan(\psi_n) \cong \psi_n$.

Considering these initial approaches, it is easy to see that for $d = 0$, the system is integrable and for this case only parallel lines are observed in the phase space. However, for $d \neq 0$, the system is non-integrable and one can observe a mixed structure in the phase space including KAM islands surrounded by a chaotic sea that is limited by a set of invariant spanning curves. Thus, we can conclude that the transition from integrability to non-integrability [12] depends on the control parameter d. Before writing the equations of the mapping, it is convenient to define dimensionless variables. They are defined as $\delta = d/a$, $\gamma_n = \theta_n/k$ and $X_n = kx_n/a$. With these new variables, we can write de mapping that describes the dynamics of the simplified version as

$$T : \begin{cases} X_{n+1} = X_n + \left[\frac{1}{\gamma_n} + \frac{1}{\gamma_{n+1}}\right] \bmod 2\pi \\ \gamma_{n+1} = \gamma_n + 2\delta\sin\left[X_n + \frac{1}{\gamma_n}\right] \end{cases} . \tag{4}$$

We can see that there is only a single and relevant control parameter δ.

It is shown in Fig (3) the corresponding phase space gener-

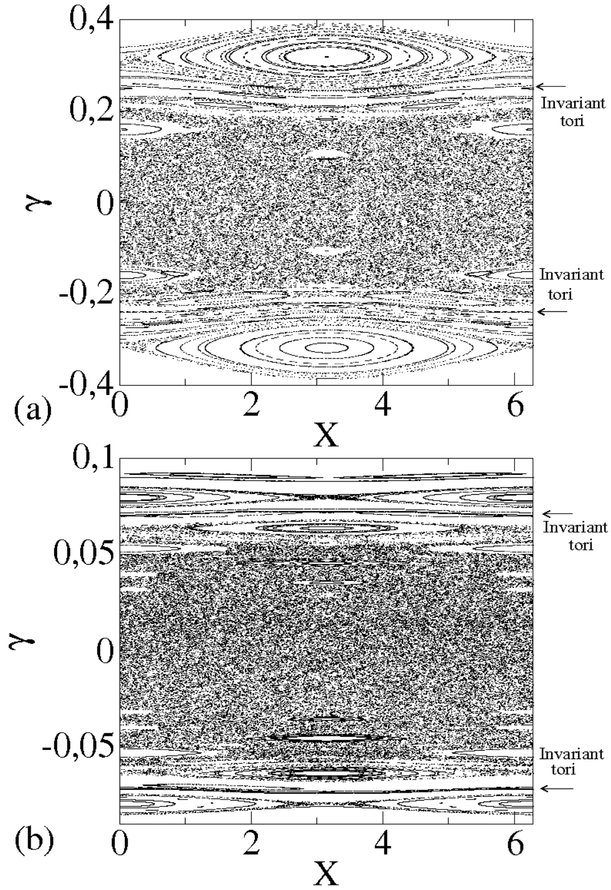

FIG. 3: Phase space for mapping (4) for the control parameters: (a) $\delta = 10^{-2}$ and (b) $\delta = 10^{-3}$.

ated from the iteration of mapping (4). It is easy to see that the mixed phase space structure is evident and it includes a large chaotic sea that is confined by two invariant tori at both positive and negatives sides and a set of KAM islands. The control parameters used in the construction of Fig 3 were: (a) $\delta = 10^{-2}$ and (b) $\delta = 10^{-3}$. It is also easy to see that the location of the invariant tori in both figures is different, so that the strength of the control parameter δ plays an important role in the dynamics.

III. THE STANDARD MODEL

In this section we will briefly discuss some properties of the Standard Mapping. We also make a connection between the Standard Mapping and the waveguide models in order to localize the position of the invariant spanning curves in the phase space. The Standard Mapping is useful to describe the dynamics of a single kicked rotor [13] and it is given by

$$S : \begin{cases} I_{n+1} = I_n + K\sin(\Theta_n) \mod 2\pi \\ \Theta_{n+1} = \Theta_n + I_{n+1} \mod 2\pi \end{cases} , \quad (5)$$

where K is a control parameter and both variables are defined as mod(2π). It is well known in the literature [7] that such a model exhibits a transition from local to global chaos as the control parameter K raises. The critical point that marks this change of behavior is $K_c = 0.971\ldots$ (see [7] for more details). Thus for $K < K_c$ the phase space shows invariant spanning curves separating different portions of the phase space. On the other hand, for $K > K_c$ all the invariant spanning curves were destroyed and the chaotic sea can spread over the phase space. It is shown in Fig. 4 the phase space generated from

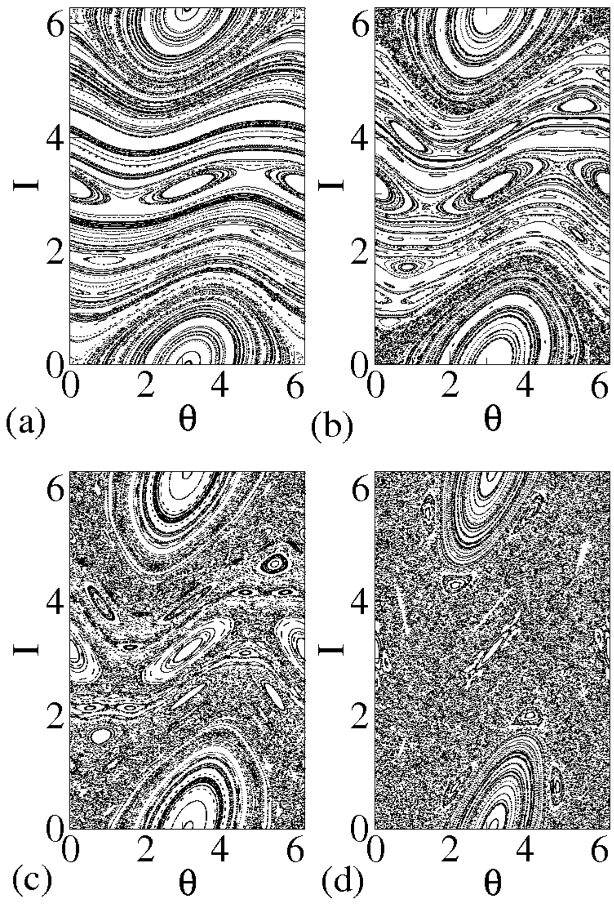

FIG. 4: Phase space generated from iteration of Eq. (5). The control parameters used were: (a) $K = 0.5$; (b) $K = 0.75$; (c) $K = 0.9716$ and (d)$K = 1.5$.

the iteration of Eq. (5) considering four different control parameters namely: (a) $K = 0.5$; (b) $K = 0.75$; (c) $K = 0.9716$ and (d)$K = 1.5$. It is easy to see that for $K > K_c$, as it is the case of Fig. 4(d), the invariant tori no longer exists in the phase space.

The connection of this result with the periodically corrugated waveguide consits basically in suppose that near the invariants spanning curves, which limit the size of the chaotic sea, the reflection angle can be written as

$$\gamma^*_{n+1} \cong \gamma^* + \Delta\gamma_{n+1} , \quad (6)$$

where γ^* is a typical value of the reflection angle along the invariant spanning curve and $\Delta\gamma_{n+1}$ is small perturbation of the angle. After defining $Z_n = X_n + \frac{1}{\gamma_n}$, the first equation of mapping (4) might be written as

$$Z_{n+1} = Z_n + \frac{2}{\gamma_{n+1}} . \quad (7)$$

Using Eq. (6), we can rewrite Eq. (7) as

$$Z_{n+1} = Z_n + \frac{2}{\gamma^*}\left[\frac{1+\Delta\gamma_{n+1}}{\gamma_n}\right]^{-1} . \quad (8)$$

Expanding Eq. (8) in Taylor series, we obtain that

$$Z_{n+1} = Z_n + \frac{2}{\gamma^*}\left[1 - \frac{\Delta\gamma_{n+1}}{\gamma^*} + O\left(\frac{\Delta\gamma_{n+1}}{\gamma^*}\right)^2\right] . \quad (9)$$

Taking into account only terms of first order, we can rewrite Eq. (9) as

$$Z_{n+1} = Z_n + \frac{2}{\gamma^*}\left[1 - \frac{\Delta\gamma_{n+1}}{\gamma^*}\right] . \quad (10)$$

The second equation of the mapping (4) can also be written as

$$\gamma^* + \Delta\gamma_{n+1} = \gamma^* + \Delta\gamma_n + 2\delta\sin(Z_n) . \quad (11)$$

Multiplying both sides of Eq. (11) by $-2/\gamma^{*2}$ and adding $2/\gamma^*$ in both sides, we obtain the following term

$$I_{n+1} = -\frac{2\Delta\gamma_{n+1}}{\gamma^{*2}} + \frac{2}{\gamma^*} . \quad (12)$$

Introducing now $\phi_n = Z_n + \pi$, we rewrite the mapping (4) as

$$T : \begin{cases} I_{n+1} = I_n + \left(\frac{4\delta}{\gamma^{*2}}\right)\sin(\phi_n) \\ \phi_{n+1} = \phi_n + I_{n+1} \end{cases} . \quad (13)$$

Comparing the mapping of the Standard Model (see Eq. (5)) and the mapping (13) it is easy to see that there is an effective control parameter K_{eff} which is given by

$$K_{eff} \cong \frac{4\delta}{\gamma^{*2}} . \quad (14)$$

Therefore, since the transition from local to global chaos occurs at $K_{eff} \cong 0.971\ldots$, the localization of the two invariant spanning curves are given by

$$\gamma^* \cong \pm 2\sqrt{\frac{\delta}{0.971...}} . \quad (15)$$

Thus, we conclude that the size of the chaotic sea is defined to be, both the positive and negative sides, proportional to $\sqrt{\delta}$.

IV. CONCLUSIONS

We confirm through analytical arguments that near the invariant tori and locally, the dynamics of the corrugated waveguide can be described by the standard mapping. Such approach was usefull to obtain the localization of the two first (either positive and negative) invariant spanning curves as a function of the control parameter δ.

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