Casimir Effect for Differential Forms in Real Compact Hyperbolic Spaces

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We calculate the Casimir energy associated with abelian gauge fields in real compact hyperbolic spaces. The cosmological applications of the vacuum energies are briefly considered.

I. INTRODUCTION

In this note we present some results concerning abelian gauge fields in locally symmetric spaces. In particular, we calculate the topological Casimir energy for abelian gauge fields ($p-$forms) in compact hyperbolic spaces $X_\Gamma = \Gamma\backslash X = \Gamma\backslash G/K$, where $G = SO_1(N,1)$, $K = SO(N)$ is a maximal compact subgroup of G, and $\Gamma \subset G$ is a discrete group of isometries. Locally symmetric spaces in general and real hyperbolic spaces in particular play important role in supergravity [8], superstring theory [9], and cosmology [1–13].

Casimir effect in spaces with non-trivial topology is an important issue in different areas of quantum field theory and cosmology (see for example [2]). Calculations involving quantum fields in $X = \mathbb{R}^N, \mathbb{S}^N$ have been actively investigated, but not much has been done for hyperbolic spaces. Calculations involving scalar and spinor fields in $X = \mathbb{H}^N$ have been considered in [1–7]. Here we present some results for the case of gauge fields.

II. SPECTRAL FUNCTIONS OF HYPERBOLIC GEOMETRY

A formal expression for the Casimir energy can be written as follows: $E_v = \frac{1}{2}\sum\lambda_j^{1/2}$, where $\left\{\lambda_j\right\}_{j\geq 0}^{\infty}$ is the set of eigenvalues of a Laplace-Beltrami operator $\mathcal{L}_p^{(CE)}$, acting on coexact part of $p-$forms (see [15]). We can use the zeta function method and get

$$E_v = \frac{1}{2}\zeta\left(s;\mathcal{L}_p^{(CE)}\right)|_{s=-1/2}. \quad (1)$$

Calculation of the zeta function $\zeta\left(s;\mathcal{L}_p^{(CE)}\right)$ may start from representation

$$\zeta\left(s;\mathcal{L}_p^{(CE)}\right) = \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\,\mathrm{Tr}\, e^{-t\mathcal{L}_p^{(CE)}}. \quad (2)$$

The trace of the heat kernel $\mathcal{K}_t = e^{-t\mathcal{L}_p^{(CE)}}$ can be calculated using the Fried formula [11]. Taking into account the physical degrees of freedom we get

$$\mathrm{Tr}\, e^{-t\mathcal{L}_p^{(CE)}} = I_\Gamma^{(p)}(\mathcal{K}_t) + H_\Gamma^{(p)}(\mathcal{K}_t). \quad (3)$$

The first term is the contribution from the identity element of the isometry group Γ, and it is given by

$$I_\Gamma^{(p)}(\mathcal{K}_t) = \frac{\chi(1)\,Vol(\Gamma/G)}{4\pi}\int_{\mathbb{R}} dr\mu_{\sigma_p}(r)\,e^{-t\left(r^2+p+\rho_0^2\right)}. \quad (4)$$

The second term is the contribution from the remaining elements of the isometry group,

$$H_\Gamma^{(p)}(\mathcal{K}_t) = \frac{1}{\sqrt{4\pi t}}\sum_{\gamma\in C_\Gamma-\{1\}}\frac{\chi(\gamma)}{j(\gamma)}t_\gamma C(\gamma) \times\chi_{\sigma_p}\left(m_\gamma\right)e^{-t\left(\rho_0^2+p\right)-t_\gamma^2/4t}. \quad (5)$$

Here χ is a homomorphism $\Gamma\to S^1$. For more details and notation see [3]. Spectral properties of Laplace-Beltrami operator is controled by Harish-Chandra-Plancherel measure $\mu_{\sigma_p}(r)$ wich is given by

$$\mu_{\sigma_p}(r) = C_p^{N-1}C_G\pi \times\begin{cases}\sum_{\ell=0}^{n-1}a_{2\ell}\tanh(\pi r)\,r^{2\ell+1}, & \text{for } N=2n\\ \sum_{\ell=0}^{n}a_{2\ell}r^{2\ell}, & \text{for } N=2n+1\end{cases},$$

where $C_G^{-1} = 2^{2N-4}\Gamma^2(N/2)$ and $a_{2\ell}$ are the Miatello coefficients [14].

III. THE CASIMIR ENERGY

The zeta function related to the identity integral $I_\Gamma^{(p)}(\mathcal{K}_t)$ is calculated to be

$$\zeta^{(2n)}\left(s;\mathcal{L}_p^{(CE)}\right) = \frac{\pi}{8}\chi(1)\,Vol(\Gamma\backslash G)\,C_\Gamma C_p^{N-1}\sum_{\ell=0}^{n-1}a_{2\ell}\ell! \times\sum_{j=0}^{\ell}\frac{k_{\ell-1}\left(s-j-1;(\rho_0-p)^2,\pi\right)}{(\ell-j)!\,(s-1)(s-2)\ldots(s-(j+1))} +\frac{1}{\Gamma(s)\Gamma(1-s)}\int_0^\infty\frac{\psi_\Gamma(\rho_0+t+\rho_0-p)}{\left(2t(\rho_0-p)+t^2\right)^s}, \quad (6)$$

where $k_m(s;\delta,a) \stackrel{\text{def}}{=} \int_{\mathbb{R}} dr\, r^{2m}\left(\delta+r^2\right)^{-s}\mathrm{sech}^2(ar)$, and $\psi_\Gamma(z)$ is the logarithmic derivative of the Selberg zeta function. For odd dimensions we have

$$\zeta^{(2n+1)}\left(s;\mathcal{L}_p^{(CE)}\right)=\frac{\chi(1)\,Vol\,(\Gamma\backslash G)}{4\Gamma(s)}C_\Gamma C_p^{2n}$$
$$\times\sum_{\ell=0}^{n}\frac{a_{2\ell}\Gamma\left(j+\frac{1}{2}\right)}{(\rho_0-p)^{2s-2j-1}}\Gamma\left(s-\ell-\frac{1}{2}\right)$$
$$+\frac{1}{\Gamma(s)\Gamma(1-s)}\int_0^\infty\frac{\psi_\Gamma(\rho_0+t+\rho_0-p)}{(2t(\rho_0-p)+t^2)^s}. \quad (7)$$

For odd N there are poles at $s=-1/2$, therefore Casimir energy $E_v^{(2n+1)}$ cannot be obtained by means of zeta function regularization.

The regularized Casimir energy related to co-exact forms on real compact even-dimensional hyperbolic manifolds is

$$E_v^{(2n)}=\frac{\pi}{8}\chi(1)\,Vol\,(\Gamma\backslash G)\,C_\Gamma C_p^{2n-1}\sum_{\ell=0}^{n-1}a_{2\ell}\ell!$$
$$\times\sum_{j=0}^{\ell}\frac{k_{\ell-1}\left(-j-\frac{3}{2};(\rho_0-p)^2,\pi\right)}{(\ell-j)!\,(s-1)(s-2)\ldots(s-(j+1))}$$
$$-\frac{1}{\pi}\int_0^\infty\frac{\psi_\Gamma(\rho_0+t+\rho_0-p)}{(2t(\rho_0-p)+t^2)^{-1/2}}. \quad (8)$$

IV. CONCLUDING REMARKS

Cosmological predictions, such as the microwave background anisotropies and the current acceleration expansion of the universe [17], depend pretty much on details of theoretical model under consideration. Recent data obtained by the Wilkinson Microwave Anisotropy Probe (WMAP) [18] satellite confirmed, and set new standards of accuracy to previous COBE's measurements wich are in agreement with the assumption that the topology of the universe migtht be non-trivial, with particular enphasis of a compact hyperbolic space. Combined with this observation, the WMAP satellite also indicates that around 60% of the critical energy density of the universe is contributed by a smoothly distribucted vaccum energy or dark energy, whose net effect is repulsive (leading thus to an accelerated expansion of the universe). Note also that topological component of the Casimir energy for co-exact forms on even-dimensional manifolds, associated with the trivial character is always negative.

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