

Quantum Correlations in Inflationary Spectra and Violation of Bell Inequalities

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In spite of the macroscopic character of the fluctuation amplitudes, we show that the standard inflationary distribution of primordial density fluctuations still exhibits inherently quantum mechanical correlations (which cannot be mimicked by any classical stochastic ensemble). To this end, we propose a Gedanken experiment for which certain Bell inequalities are violated. We also compute the effect of decoherence and show that the violation persists provided that the decoherence lies below a certain non-vanishing threshold. Moreover, there exists a higher threshold above which no violation of any Bell inequalities can occur, so that the corresponding distributions can be interpreted as stochastic ensembles of classical fluctuations.

Inflation implies that the primordial density fluctuations arise from the amplification of vacuum fluctuations [1]. As a result, the initial vacuum state becomes a product of highly squeezed two-mode states [2] characterized by the wave vectors \mathbf{k} and $-\mathbf{k}$. This follows from the homogeneity of the state and the linearity of the mode equation. However the linear treatment is only approximate. Hence, when no longer neglecting the weak non-linearities of inflationary models [3], and when restricting attention to a given two-mode sector, decoherence will unavoidably develop. Since the interactions preserve the homogeneity and are weak (so as to justify the use of a Hartree approximation) inflationary distributions belong to the class of Gaussian homogeneous distributions obtained by slightly decohering the standard distribution obtained by neglecting non-linearities [4–6].

Given the macroscopic character of the fluctuation amplitudes at the end of inflation, one might ask whether some quantum properties have been preserved by these distributions, and if so, how to distinguish those which have kept some from those which have lost them. To show that the standard distribution has kept quantum properties and to sort out these two classes of distributions, we shall analyze the possibility of violating Bell inequalities [7].

I. THE INFLATIONARY SPECTRA

A. The 'standard' distribution

In inflationary models based on one inflaton field, one can express the dynamics of the linear perturbations around the background solution of Einstein equations in terms of a massless scalar field minimally coupled to gravity, ϕ . This field is the gauge-invariant linear combination of the gravitational potential and the inflaton fluctuations which satisfies canonical commutation relations [8]. We briefly outline how one obtains the two-mode squeezed states and the time-coherence of the modes [2, 5].

The background geometry is a Friedman-Robertson-Walker

space-time, with line element

$$ds^2 = a^2(\eta) [-d\eta^2 + \delta_{ij} dx^i dx^j]. \quad (1)$$

The spatial sections in the present Hubble volume are flat to an excellent precision if inflation has lasted enough to have increased the radius of curvature to super-horizon scales. The scale factor, as a function of the conformal time η is given by $a(\eta) = -1/H\eta$ in the inflationary period we approximate by a de Sitter space. At the reheating η_r , we paste this law to a radiation dominated phase characterized by $a(\eta) = (\eta - 2\eta_r)/(H\eta_r^2)$. We can then follow the time dependence of modes from the onset of inflation till horizon re-entry in the radiation dominated era.

After expanding the action of matter and metric fluctuations up to quadratic order around the background solution, the Hamiltonian for each complex Fourier mode $\phi_{\mathbf{k}}$ is

$$H_{\mathbf{k}} = \frac{1}{2} \left[|\partial_{\eta} \phi_{\mathbf{k}}|^2 + \left(k^2 - \frac{\partial_{\eta}^2 a}{a} \right) |\phi_{\mathbf{k}}|^2 \right]. \quad (2)$$

The state of $\phi_{\mathbf{k}}$ at the onset of inflation follows from the kinematics of the background [9]: if some inhomogeneities are present at an early stage of inflation, they are diluted by the quasi-exponential expansion over proper lengths bigger than our causal patch, i.e., to us, they are indistinguishable from the background. One is thus left with the vacuum. One can also reach this conclusion by considering the evolution of perturbations backward in time. Then, non-vacuum contributions (classical inhomogeneities as well as quantum excitations of the field) to the energy-momentum tensor blow-up thereby violating the consistency condition of no-large back-reaction. In conclusion, all the modes observable today were in their ground state about 70 e-folds before the end of inflation (the minimal duration of inflation to include today's horizon inside a causal patch).

This ground state is unambiguously defined for the relevant modes because, at that time, their wavelength was much shorter than the Hubble radius (i.e. $k^2 \gg \partial_{\eta}^2 a/a$ in Eq. (2)). One therefore works with the vacuum defined in the asymp-

otic past by positive frequency modes

$$\lim_{\eta \rightarrow -\infty} [i\partial_\eta - k] \phi_k^{in}(\eta) = 0, \quad (3)$$

where $\phi_k^{in}(\eta)$ are solution of the mode equation:

$$\begin{aligned} \partial_\eta^2 \hat{\phi}_{\mathbf{k}} + \left(k^2 - \frac{\partial_\eta^2 a}{a} \right) \hat{\phi}_{\mathbf{k}} &= 0, \\ \hat{\phi}_{\mathbf{k}}(\eta) &= \hat{a}_{\mathbf{k}}^{in} \phi_k^{in}(\eta) + \hat{a}_{-\mathbf{k}}^{in\dagger} \phi_k^{in*}(\eta). \end{aligned} \quad (4)$$

The time-dependent term $\partial_\eta^2 a/a$ mixes the positive \mathbf{k} and negative frequency $-\mathbf{k}$ modes. Therefore the vacuum unitarily evolves into an entangled state: a two-mode squeezed state. Confusion about the origin of this entanglement seems to persist: entanglement is *not* a consequence of the reality (self-adjointness) of $\hat{\phi}(\eta, \mathbf{x})$. Pair production of charged particles (in an electric field) also leads to entangled states. Two-mode entanglement is due to the mixing of positive and negative frequencies and to the homogeneity of the background.

At the end of inflation, $\partial_\eta^2 a$ vanishes and the amplification stops. During the radiation dominated era, in the absence of iso-curvature modes, the fluctuations of physical fields (those of the gravitational potential and of cold dark matter and radiation density fluctuations) are all linearity related to the values of ϕ and $\partial_\eta \phi$ evaluated at the end of inflation. This is true both classically and quantum mechanically because of the linearity of the transfer matrix.

Instead of working with these physical fields, it is simpler to keep working with $\phi_{\mathbf{k}}$, since its time evolution is simply governed by $e^{\pm ik\eta}$. We shall use this behavior to further simplify the writings by introducing creation and destruction operators $\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}}^\dagger$ defined in the adiabatic era by $\hat{\phi}_{\mathbf{k}} = (\hat{a}_{\mathbf{k}} e^{-ik\eta} + h.c.)/\sqrt{2k}$. In terms of these operators, the ‘in’ vacuum (i.e. the state annihilated by the $\hat{a}_{\mathbf{k}}^{in}$ of Eq. (4)) reads [2]

$$\begin{aligned} |0, in\rangle &= \prod_{\mathbf{k}} \left(1 - |z_{\mathbf{k}}|^2 \right)^{1/2} \\ &\times \exp \left(z_{\mathbf{k}} \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{-\mathbf{k}}^\dagger \right) |0, \mathbf{k}\rangle \otimes |0, -\mathbf{k}\rangle. \end{aligned} \quad (5)$$

The tilde tensorial product takes into account only half the modes. It must be introduced because the (squeezing) operator $\exp \left(z_{\mathbf{k}} \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{-\mathbf{k}}^\dagger \right)$ acts on the pairs of modes. The complex number $z_{\mathbf{k}}$ characterizes the two-mode states of wavenumber $k = |\mathbf{k}|$. It is related to the following expectation values

$$\langle \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}'} \rangle = n_{\mathbf{k}} \delta^3(\mathbf{k} - \mathbf{k}'), \quad \langle \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{k}'} \rangle = c_{\mathbf{k}} \delta^3(\mathbf{k} + \mathbf{k}'). \quad (6)$$

The mean occupation number is $n_{\mathbf{k}} = |z_{\mathbf{k}}|^2 / (|z_{\mathbf{k}}|^2 - 1)$ and the cross correlation term is $c_{\mathbf{k}} = z_{\mathbf{k}} / (|z_{\mathbf{k}}|^2 - 1)$. They obey $|c_{\mathbf{k}}|^2 = n_{\mathbf{k}}(n_{\mathbf{k}} + 1)$. This is the highest value of $|c_{\mathbf{k}}|$ for a given $n_{\mathbf{k}}$. It is obtained for the states which minimize Heisenberg uncertainty relations [6].

Asking that the spectrum describes fluctuations with r.m.s. amplitude 10^{-5} for wavelength observable in the CMB, one

gets the rough estimation $n_{\mathbf{k}} \sim 10^{100}$. The phase of $z_{\mathbf{k}}$ determines the temporal phase of the modes as they re-enter the Hubble radius [6]. Namely, the power spectrum of $\phi_{\mathbf{k}}$ at conformal time η is given by a product of two modes proportional to $e^{ik\eta} + z_{\mathbf{k}} e^{-ik\eta}$. In inflationary models, $z_{\mathbf{k}}$ is very close to -1 , giving rise to $\sin(k\eta)$ functions, i.e. to ‘growing modes’.

B. Partially decohered Gaussian distributions

The two-mode entanglement of squeezed states follows from the homogeneity of the vacuum and the linearity of the mode equation. However, as discussed in the introduction, the linear treatment is only approximate. When no longer neglecting the weak non-linearities of gravitation [3], and when restricting attention to a given two-mode sector, some decoherence develops. This is very similar to the description of an experiment of diffraction of a slightly anharmonic crystal. The calculation of the reduced density matrix of the two modes under scrutiny is very difficult. But the weakness of the interactions guarantees that the modifications of the power spectrum are extremely small. It also guarantees that the reduced density matrix can still be accurately described by a Gaussian distribution. Moreover, since homogeneity is preserved, these distributions always factorize into products of uncorrelated two-mode sectors. In conclusion, inflationary distributions of primordial fluctuations belong to the class of Gaussian homogeneous distributions obtained by (slightly) decohering the standard distribution $\hat{\rho}_{in} = |0, in\rangle\langle 0, in|$ [4, 6].

These are described by products of two-mode density matrices $\hat{\rho}_2$ which are characterized by the expectation values (6), where $n_{\mathbf{k}}$ and $|c_{\mathbf{k}}|$ are now *independent* parameters. We shall work at fixed $n_{\mathbf{k}}$ (fixed power spectrum), and fixed $\arg(c_{\mathbf{k}})$ (changing this phase does not change the entropy [6]), and let the norm $|c_{\mathbf{k}}|$ vary. A convenient parametrization is provided by $\delta_{\mathbf{k}}$:

$$|c_{\mathbf{k}}|^2 = (n_{\mathbf{k}} + 1)(n_{\mathbf{k}} - \delta_{\mathbf{k}}). \quad (7)$$

Squeezed states (5) are maximally coherent and correspond to $\delta_{\mathbf{k}} = 0$. The least coherent distribution, a product of two random density matrices, corresponds to $\delta_{\mathbf{k}} = n_{\mathbf{k}}$. Hence, the parameter $\delta_{\mathbf{k}}$ controls the level of decoherence of the distribution, or equivalently the strength of the correlations between the \mathbf{k} and the $-\mathbf{k}$ sectors. It also fixes the r.m.s. amplitude of the *decaying* mode, as seen by decomposing the field modes $\hat{\phi}_{\mathbf{k}}$ in terms of the growing and the decaying modes:

$$\hat{\phi}_{\mathbf{k}}(\eta) = \hat{g}_{\mathbf{k}} \frac{\sin(k\eta)}{\sqrt{k}} + \hat{d}_{\mathbf{k}} \frac{\cos(k\eta)}{\sqrt{k}}. \quad (8)$$

When $\delta \ll n$,

$$\begin{aligned} \langle \hat{g}_{\mathbf{k}} \hat{g}_{\mathbf{k}}^\dagger \rangle &= n + \frac{1}{2} - \text{Re}(c) = 2n \left(1 + O\left(\frac{\delta}{n}\right) \right), \\ \langle \hat{d}_{\mathbf{k}} \hat{g}_{\mathbf{k}}^\dagger \rangle &= n + \frac{1}{2} + \text{Re}(c) = n^{1/4} \left(1 + O\left(\frac{\delta}{n}\right) \right), \\ \langle \hat{d}_{\mathbf{k}} \hat{d}_{\mathbf{k}}^\dagger \rangle &= \text{Im}(c) = \frac{\delta}{2} + O(n^{-1/2}) \ll \langle \hat{g}_{\mathbf{k}} \hat{g}_{\mathbf{k}}^\dagger \rangle. \end{aligned} \quad (9)$$

We have dropped the indexes k since each two-mode can be analyzed separately (unless one is considering localized wave packets).

We emphasize that the increase of entropy from the initial pure vacuum state to a statistical mixture follows, as usual, from the neglect of correlations, see footnote p. 188 in [10]. In other words, ignoring the correlations between several two-modes is the coarse graining we adopt. We also notice that this Gaussian ansatz is a first order (Hartree) approximation. With the future experiments, deviations from Gaussianity will hopefully be measurable and will become a powerful tool to raise the degeneracy between various models for the primordial universe [11].

II. CLAUSER-HORNE INEQUALITIES FOR INFLATIONARY SPECTRA

A. Local Hidden Variable completions of Quantum Mechanics and the Clauser-Horne inequality

Bell inequalities and their generalizations form a set of constraints on the statistics of outcomes of measurements when one demands that these outcomes are compatible with the classical concept of locality [12].

Consider a system made of two subsystems I and II and let the two parts interact. In general they will become correlated. Suppose that after 'turning off' the interactions, the system is in a pure state of the form

$$|\chi\rangle = \sum_n A_n |\phi\rangle_n^I \otimes |\psi\rangle_n^{II}, \quad (10)$$

where there is more than one term in the sum, where $|\phi\rangle_n^I$ and $|\psi\rangle_n^{II}$ are two sets of orthonormal states in the Hilbert spaces \mathcal{H}^I and \mathcal{H}^{II} of I and II respectively, and where $P_n = |A_n|^2 < 1$ are the probabilities for the n -th outcome to be realized such that $\sum_n P_n = 1$. The states (10) are a special case of entangled states, see Section III.C for the definition of entanglement for statistical mixtures. (Notice that each two-mode squeezed state in Eq. (5) is an entangled state.) Being orthonormal, the $|\phi\rangle_n^I$ and $|\psi\rangle_n^{II}$ can be seen as eigenstates of observables $\hat{\Phi}^I \otimes \mathbf{1}^{II}$ and $\mathbf{1}^I \otimes \hat{\Psi}^{II}$ acting non-trivially on \mathcal{H}^I and \mathcal{H}^{II} respectively.

After having let the two subsystems interact, we separate them to two distant places where two measurement separated by a space-like interval will be performed. We suppose that we have an ensemble of copies of the same system all prepared in state $|\chi\rangle$ in order to perform different measurements and to accumulate statistical data. The space-like character of the interval guarantees that the state of one subsystem cannot be affected by any measurement performed on the other subsystem. Once the apparatus I has measured the observable $\hat{\Phi}^I$ on system I, we can predict with probability 1 (as checked by reproducing the experiment in the same conditions a large number of times and bringing the data together) the outcome of the measurement of $\mathbf{1}^I \otimes \hat{\Psi}^{II}$, irrespectively of the values of the probabilities P_n and despite the space-like separation of

the events of the measurements on subsystems I and II (supposing we have the *a priori* knowledge that the system is in the entangled state $|\chi\rangle$, as we do in inflationary cosmology, see eq. (6)).

In order to explain these correlations one might be tempted to consider that they result from unknown common properties shared by the two subsystems, and whose values were assigned while I and II were still interacting. These properties, to which we do not have access by the experiments, are called Hidden Variables. The Hidden Variable Programm consists in establishing the conditions which have to be met. In particular, Bell's theorem and its generalizations [12–14] show that Quantum Mechanics cannot be embedded in a *Local* Hidden Variable Theory. In this paper, we will consider a particular type of inequalities called the Clauser-Horne (CH) inequalities [12, 15].

1. Local Hidden Variable Theory

A Local Hidden Variable Theory (LHVT) is the specification of

- i) the conditional probability $p_\Phi[\phi; \lambda]$ (resp. $p_\Psi[\psi; \lambda]$) that the measurement of $\hat{\Phi}^I$ (resp. $\hat{\Psi}^{II}$) gives the eigenvalue ϕ (resp. ψ), given a value λ of the hidden variables,
- ii) a probability measure $p(\lambda)d\lambda$.

The requirement of locality forces that

- 1) the conditional probabilities p_Φ (resp. p_Ψ) is independent of $\hat{\Psi}^{II}$ (resp. $\hat{\Phi}^I$),
- 2) the probability distribution $p(\lambda)$ is independent of both $\hat{\Phi}^I$ and $\hat{\Psi}^{II}$,
- 3) given a value λ of the hidden variables, the outcome of the measurements of $\hat{\Phi}^I$ and $\hat{\Psi}^{II}$ are statistically independent, i.e.

$$p_{\Phi\Psi}(\phi, \psi; \lambda) = p_\Phi[\phi; \lambda] \times p_\Psi[\psi; \lambda]. \quad (11)$$

A given quantum state $\hat{\rho} \in \mathcal{H}^I \otimes \mathcal{H}^{II}$ admits a LHVT if, and only if, for any pair of observables with spectral decompositions $\hat{\Phi}^I = \sum_n \phi_n \hat{\Pi}_n^\Phi$ and $\hat{\Psi}^{II} = \sum_n \psi_n \hat{\Pi}_n^\Psi$, the statistical predictions of the model reproduce those of Quantum Mechanics, i.e. the joint probability is

$$\begin{aligned} P_{\Phi\Psi}(\phi, \psi) &= \int d\lambda p(\lambda) p_{\Phi\Psi}(\phi, \psi; \lambda) \\ &= \text{Tr} \left(\hat{\rho} \hat{\Pi}_n^\Phi \hat{\Pi}_n^\Psi \right). \end{aligned} \quad (12)$$

2. Clauser-Horne inequalities

The Clauser-Horne inequality is a condition verified by the joint probabilities $P_{\Phi\Psi}(\phi, \psi)$ if Quantum Mechanics can be embedded in a LHVT. They follow from the algebraic inequality satisfied by any set of four real numbers (x, x', y, y') , all lying in the interval $[0, 1]$, see [15] for a proof:

$$xy + xy' + x'y - x'y' - x - y \leq 0. \quad (13)$$

Identifying $x = p_\Psi[\Psi; \lambda]$, $x' = p_\Psi[\Psi; \lambda]$, $y = p_\Psi[\Psi; \lambda]$, $y' = p_\Psi[\Psi'; \lambda]$, and averaging over λ one gets

$$\begin{aligned} & [P_{\Phi\Psi}(\phi, \Psi) + P_{\Phi\Psi}(\phi, \Psi') + P_{\Phi\Psi}(\phi', \Psi) \\ & - P_{\Phi\Psi}(\phi', \Psi')] \times [P_\Phi(\phi) + P_\Psi(\Psi)]^{-1} \leq 1, \end{aligned} \quad (14)$$

where P_Φ, P_Ψ are the marginal probabilities.

B. A set of observables for Gaussian states

In searching for a violation of Bell inequalities in inflationary spectra, one encounters an unexpected difficulty because we are dealing with Gaussian states. Indeed, observables which are polynomials of the field amplitude and its conjugate momentum cannot violate Bell inequalities. The reason is that the Wigner representation of Gaussian density matrices is positive. Hence, it can be used as a probability distribution function to build an LHV description of the expectation values [13].

We must therefore consider observables which do not have a direct classical counterpart. Nonetheless the operators we shall consider have a clear and simple meaning. They are projectors on a given pair of coherent states $|v, \mathbf{k}\rangle$ and $|w, -\mathbf{k}\rangle$

$$\hat{\Pi}(v, w) = |v, \mathbf{k}\rangle \langle v, \mathbf{k}| \otimes |w, -\mathbf{k}\rangle \langle w, -\mathbf{k}|. \quad (15)$$

The coherent states obey, by definition, $\hat{a}_{\mathbf{k}}|v, \mathbf{k}\rangle = v|v, \mathbf{k}\rangle$ and $\hat{a}_{-\mathbf{k}}|w, -\mathbf{k}\rangle = w|w, -\mathbf{k}\rangle$.

We want to make a pose and advertise the pertinent role of coherent states in inflationary cosmology. Firstly, for quadratic Hamiltonians, they are minimal uncertainty Gaussian states which are stable in time. They are thus the quantum counterparts of classical points in phase space. More generally, one expects that they are still stable as long as the secular effects of non-linearities (changes in the width of coherent states, formation of small scale structures -whirls and tendrils- of the classical trajectories [17]) remain small. Such is the case in cosmology as long as the linear regime (and therefore the Gaussian approximation) are good approximations.

Secondly, for quadratic systems weakly interacting with a reservoir of modes (here the coupling is through the non-linear gravitational terms), there exists a finite time after which the system reaches a state which is a statistical mixture of coherent states (more generally, of minimal uncertainty Gaussian states) [18, 19][?].

Coherent states can therefore be thought as providing a particular realization of the Gaussian ensemble describing the primordial fluctuations. The probability that the semi-classical values v and w of the \mathbf{k} -th mode be realized is

$$\begin{aligned} Q(v, w; \delta) &= \text{Tr}[\hat{\rho}_2(\delta) \hat{\Pi}(v, w)] \\ &= Q_v(v) \times Q_w|_v(w; \delta), \\ &= \frac{1}{n+1} \exp\left[-\frac{|v|^2}{(n+1)}\right] \\ &\quad \times \frac{1}{1+\delta} \exp\left[-\frac{|w - \bar{w}(v)|^2}{1+\delta}\right]. \end{aligned} \quad (16)$$

We have written this probability in an asymmetric form to make explicit the power spectrum ($= n+1$), and the much smaller width ($= 1+\delta$) governing the dispersion of the values of w around $\bar{w}(v) = v^*c/(n+1)$. Hence, as long as $\delta \ll n$, the amplitude of the mode $-\mathbf{k}$ conditional to the detection of the mode \mathbf{k} in the coherent state $|v\rangle$ is fixed by that v . We see explicitly that it is δ , the decoherence level, which governs the strength of the correlations between the \mathbf{k} and the $-\mathbf{k}$ sectors.

Thirdly, given a realization, one can calculate conditional correlations. The latter have specific spatial properties best revealed by wave-packets. Although their Fourier content and their localization depends on the chosen wave-packet, their space-time structure is uniquely determined by the frequency mixing. More precisely, given the ensemble of realizations selected by $\hat{\Pi}_{w.p.}$ the projector on some wave packet, the conditional field amplitude is (see [5] for more details)

$$\langle \hat{\phi}(\eta, \mathbf{x}) \rangle = \frac{\langle \hat{\Pi}_{w.p.} \phi(\eta, \mathbf{x}) \hat{\Pi}_{w.p.} \rangle}{\langle \hat{\Pi}_{w.p.} \rangle} = \bar{\phi}_R + \bar{\phi}_L, \quad (17)$$

where

$$\begin{aligned} \bar{\phi}_R(\eta, \mathbf{x}) &= \int \frac{d^3k}{(2\pi)^{3/2}} (v_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}} \phi_{\mathbf{k}}^{out}(\eta) + c.c.), \\ \bar{\phi}_L(\eta, \mathbf{x}) &= \int \frac{d^3k}{(2\pi)^{3/2}} (\bar{w}_{\mathbf{k}}(v_{\mathbf{k}}) e^{-i\mathbf{k}\mathbf{x}} \phi_{\mathbf{k}}^{out}(\eta) + c.c.). \end{aligned}$$

One sees that the centers of these wave-packets propagate in opposite directions on the future light cone from the event of creation at the reheating. Hence, on a given time slice, e.g. the Last Scattering Surface (in the matter dominated era), they are separated by more than twice the Hubble radius at that time.

The important conclusion for Bell inequalities is that the measurement (15) (more precisely, its generalization which contains a product over \mathbf{k} so as to describe a localized wave packet) can be performed on subsystems which are causally disconnected. Violations of Bell inequalities with such observables were first considered by Wòdkiewicz *et al* in the context of Quantum Optics [16].

C. Separable states

A two-mode state is called separable, or classically correlated, if and only if it can be written as a convex sum of tensor products of one-mode density matrices [20]. Otherwise, the state is called entangled. Restricting to the class of homogeneous Gaussian states (6), separable states are of the form [21]

$$\hat{\rho}_2^{\text{sep.}}(\delta) = \int \frac{d^2v}{\pi} \frac{d^2w}{\pi} P(v, w; \delta) \hat{\Pi}(v, w), \quad (18)$$

where P is a Gaussian function. For homogeneous distributions, it is given by [6]

$$P(v, w; \delta) = \frac{1}{\Delta'} e^{-|v|^2/n} \times \exp\left[-\frac{|w - (cv^*/n)|^2}{\Delta'/n}\right],$$

where $\Delta' = n^2 - |c|^2$ must be positive. This implies $|c| \leq n$, or $\delta \geq n/(n+1)$. In other words, all the states such that $n < |c| \leq \sqrt{n(n+1)}$ are entangled.

The physical meaning of separable states is revealed by the fact that they can be prepared by a classical protocol: a random generator produces the four real numbers encoded in (v, w) with probability P . The result of each drawn is send by classical communication channels to two distant observers performing separate measurements on the subsystems \mathbf{k} and $-\mathbf{k}$ respectively so as to prepare them into the two-mode coherent state $|v, \mathbf{k}\rangle |w, -\mathbf{k}\rangle$. By construction, the statistical properties of separable states can be interpreted classically. In particular, separable states cannot violate Bell inequalities as demonstrated in [20]. We will use them to derive the CH inequalities.

When applied to separable Gaussian states (18), the probability (16) is

$$Q^{\text{sep.}}(v, w; \delta) = \int \frac{d^2\alpha}{\pi} \frac{d^2\beta}{\pi} P(\alpha, \beta; \delta) p(v|\alpha) p(w|\beta),$$

where $p(v|\alpha) = |\langle v|\alpha\rangle|^2 = e^{-|v-\alpha|^2} \leq 1$. This has the structure of Eqs. (11) and (12), where (α, β) play the role of the "hidden variable" λ , and $P(\alpha, \beta; \delta)$ the role of the probability density $\rho(\lambda)$. Then Eq. (14) gives

$$C(v, w; \delta) = [Q(0, 0; \delta) + Q(v, 0; \delta) + Q(0, w; \delta) - Q(v, w; \delta)] \times \frac{n+1}{2} \leq 1. \quad (19)$$

III. VIOLATION AS A FUNCTION OF THE LEVEL OF COHERENCE

We now have all the elements to search for combinations of v and w which maximize C of Eq. (19). The maximum is reached for $\arg(c^*vw) = \pi$ and $|v| = |w|$. To make contact with [16], we choose the arbitrary phase $2\arg(v) = \arg(c)$ to get $w = -v$. The maximum of this function is reached for

$$\begin{aligned} \frac{|v_M(\delta)|^2}{1+\delta} &= \frac{\ln \left[1 + \sqrt{\frac{n-\delta}{n+1}} \right]}{1 + 2\sqrt{\frac{n-\delta}{n+1}}} \\ &= \frac{\ln 2}{3} \left[1 + O\left(\frac{1+\delta}{n}\right) \right], \end{aligned} \quad (20)$$

and its value is

$$C_M(\delta) = \frac{1}{2(1+\delta)} \left[1 + \frac{3}{2^{4/3}} + O\left(\frac{1+\delta}{n}\right) \right]. \quad (21)$$

There is a violation even for macroscopic occupation numbers. This fact demonstrates that $n \gg 1$ is a necessary but *not* sufficient condition to have a classical (stochastic) distribution. The latter are only the separable ones for which $\delta \geq 1$.

More precisely, the inequality (19) is violated for

$$\delta < \frac{-1 + 3/2^{4/3}}{2} \simeq 0.095, \quad (22)$$

irrespective of the value of n when $n \gg 1$. Even though each probability in (19) decreases like $1/n$, the range of δ is asymptotically constant in the regime relevant for inflationary cosmology. One also verifies that, as expected, the maximal violation is obtained for $\delta = 0$, i.e. for pure states. In Fig. 1 we have plot $C(v)$ for three values of δ .

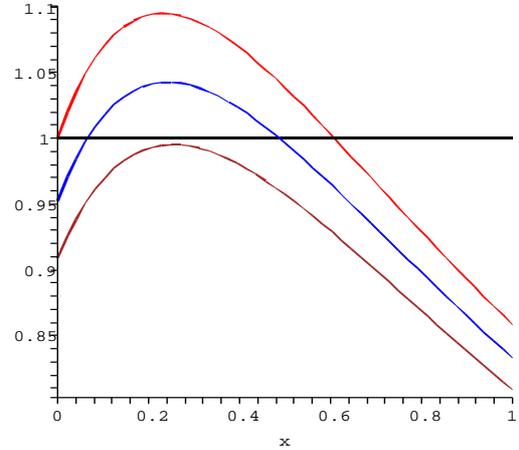


FIG. 1: *The loss of violation as decoherence increases.* The function is normalized to its maximal classical value: $C = 1$. The variable $x = |v|^2$ gives the square amplitude of the field configurations. The occupation number is $n = 100$, and the three values of δ are 0 (red, upper), the pure state, 0.05 (blue, middle), and 0.1 (brown, lower), the border regime. The function $C(v, -v, \delta)$ is asymptotically independent of n in the large n limit.

IV. FINAL REMARKS

First, there might be operators which violate Bell inequalities for $0.1 \leq \delta < 1$. However it is not guaranteed that these will have a simple interpretation in cosmological terms.

Second, an observational verification of the violation would require several steps. First, one should be able to distinguish \mathbf{k} from $-\mathbf{k}$ configurations, i.e. to have access to the primordial velocity field, see [5]. Second, one should isolate rare realizations, far from the r.m.s. values, and specified with a precision given by the spread of the coherent states ($= 1$) which is also much smaller than n . Given that $n \sim 10^{100}$, an observational verification seems therefore excluded.

It is nevertheless conceptually important to realize that if non-linearities, and hence decoherence, are weak enough so that $\delta < 1$, the distribution of primordial density fluctuations would have kept its quantum properties in spite of the macroscopic character of the amplitudes and even though we cannot observationally verify it.

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- [] A word of caution is in order. This result is demonstrated for finite temperatures only, and might not be valid for $T = 0$. Nevertheless this result should remain true since the amount of decoherence necessary to kill the quantum correlations corresponds to add incoherently 1 photon to each mode, see Section III.C. That this is realized in inflationary cosmology remains to be seen.