Remarks on Topological Models and Fractional Statistics

C.A.S. Almeida*

Universidade Federal do Ceará - Departamento de Física

C.P. 6030, 60470-455, Fortaleza, CE, Brazil

One of the most intriguing aspects of Chern-Simons-type topological models is the fractional statistics of point particles which has been shown essential for our understanding of the fractional quantum Hall effects. Furthermore these ideas are applied to the study of high Tc superconductivity. We present here an recently proposed model for fractional spin with the Pauli term. On the other hand, in D=4 space-time, a Schwarz-type topological gauge theory with antisymmetric tensor gauge field, namely $B \wedge F$ model, is reviewed. Antisymmetric tensor fields are conjectured as mediator of string interaction. A dimensional reduction of the previous model provides a (2+1) dimensional topological theory, which involves a 2-form B and a 0-form ϕ. Some recent results on this model are reported. Recently, there have been thoughts of generalizing unusual statistics to extended objects in others space-time dimensions, and in particular to the case of strings in four dimensions. In this context, discussions about fractional spin and antisymmetric tensor field are presented.

I $B \wedge F$ Models

Schwarz-type theories are purely topological in the sense that their partition functions are independent of the metric and that the only observables in these theories are topological invariants of the underlying space-time manifold $\mathcal{M}$. Other observables describe linking and intersection number of manifolds of any dimension.

Commonly called BF systems, they are characterized by a BRST-gauge fixed quantum action which differ from the classical action only by a BRST-commutator which contains the whole metric dependence of the quantum action. On the other hand, since the vaccum expectation value of a BRST-commutator vanishes, these field theories may be obtained from the classical actions [1]. Furthemore, if we denote as $\mathcal{Q}$ the BRST-operator which is nilpotent, in these theories the energy-momentum tensor is $\mathcal{Q}$ trivial, i.e.,

$$T_{\mu\nu} = \{\mathcal{Q}, \Phi_{\mu\nu}\} \qquad (1)$$

where $\Phi_{\mu\nu}$ represents fields and the metric.

Connected to BF systems, it is worth mentioning that antisymmetric tensor fields theories have been studied during the past years. They play an important role in the realization of the various strong-weak coupling dualities among string theories. An antisym-

metric tensor of rank $p - 1$ couples naturally to an elementary extended object of dimension $p - 2$, namely a $(p - 2)$ brane.

As an example of an abelian BF system consider the following metric independent action on an D-dimensional manifold $\mathcal{M}$.

$$S(D, p) = \int_{\mathcal{M}} B_p \wedge dA_{D-p-1}, \qquad (2)$$

where A and B are forms , p denoting their rank, $\wedge$ denoting their wedge product and d is the exterior derivative.

In particular the abelian $B \wedge F$ four-dimensional action is

$$S_{BF} = \int_{M_4} \{B \wedge F\}. \qquad (3)$$

$$B = B_{\mu\nu}dx^\mu \wedge dx^\nu, F = dA, A = A_\mu dx^\mu. \qquad (4)$$

This action is formulated in terms of the two-form potential B while $F = dA$ is the field-strength of a one-form gauge potential A.

Applications:

• Field theories describing the low-energy limit of fundamental string theories typically contain higher-rank tensor fields.

• The topological contribution coming from BF theories appear even in those physical theories with non trivial physical Hamiltonian where the BF term appears as an interaction term.

• Color confinement models.

• Axionic cosmic strings.

• QCD strings.

• Topologically massive models.

II   Gauge invariant massive $B \wedge F$ model in $D = 4$.

Our starting point is an abelian gauge theory which contains the vector field A and the antisymmetric field B, and incorporated the topological term $B \wedge F$ in the four-dimensional action [2]

$$S_{BF} = \int_{M_4} \left\{ \frac{1}{2} H \wedge {}^*H - \frac{1}{2} F \wedge {}^*F + kB \wedge F \right\}. \quad (5)$$

Here $H = dB$ is the field-strength of a two-form gauge potential B, k is a mass parameter, and $*$ is the Hodge star (duality) operator. The action above is invariant under the following transformations:

$$\delta A = d\theta, \delta B = d\Lambda, \quad (6)$$

where θ and Λ are zero and one-form transformation parameters respectively, and gives the equations of motion

$$d\,{}^*H = \kappa F \quad (7)$$

and

$$d\,{}^*F = \kappa H. \quad (8)$$

Applying $d\,{}^*$ on both sides of eq. (8) and using the eq. (7), we get

$$(d^*d^* + \kappa^2)F = 0. \quad (9)$$

Repeating the procedure above in reverse order, we obtain the equation of motion for H

$$(d^*d^* + \kappa^2)H = 0. \quad (10)$$

These equations can be rewritten as

$$(\Box + \kappa^2)F = 0 \quad (11)$$

and

$$(\Box + \kappa^2)H = 0. \quad (12)$$

III   Abelian gauge invariant massive models in $D = 3$

• Dimensional reduction $\rightarrow B \wedge \varphi$ models.

Dimensional reduction is usually done by expanding the fields in normal modes corresponding to the compactified extra dimensions, and integrating out the extra dimensions. This approach is very useful in dual models and superstrings. Here, however, we only consider the fields in higher dimensions to be independent of the extra dimensions.

In this case, we assume that our fields are independent of the extra coordinate x_3. From (3), on performing dimensional reduction as described above, we get in three dimensions

$$S = \int_{M_3} \left\{ B \wedge d\phi + V \wedge F \right\}, \quad (13)$$

where V and ϕ are a 1-form and a 0-form fields respectively.

We recognize that $B \wedge d\phi$ is topological in the sense that there is no explicit dependence on the space-time metric. One has to stress that this term may not be confused with the two-dimensional version of the $B \wedge F$, which involves a scalar and a one-form fields. Moreover, a term that is equivalent to the four-dimensional $B \wedge F$ term is present in action (13) (the so-called mixed Chern-Simons term, $V \wedge F$).

• Non-Chern-Simons gauge invariant massive models in $D = 3$.

Now, in order to show the topological mass generation for the vector and tensor fields, we consider the model with the topological term $B \wedge d\phi$, and with propagation for the two-form gauge potential B and for the zero-form field, represented by the action

$$S = \int_{M_3} \left\{ \frac{1}{2} H \wedge {}^*H + \frac{1}{2} d\phi \wedge {}^*d\phi + \kappa B \wedge d\phi \right\}, \quad (14)$$

where the second term is a Klein-Gordon term, κ is a mass parameter and $H = dB$ is a three-form field-strength of B.

The action above is invariant under the following transformations:

$$\delta A = -d\theta, \delta \varphi = \theta, \delta B = d\Lambda, \quad (15)$$

where θ and Λ are zero and one-form transformation parameters respectively.

We follow here the same steps that has been used by Allen *et al.* [2] in order to show the topological mass generation in the context of $B \wedge F$ model. Thus, we find the equations of motion for scalar and tensor fields, which are respectively $d\,^*H = \kappa d\phi$ and $d\,^*d\phi = -\kappa H$. Consequently, we obtain the equations $(d^*d^* + \kappa^2)d\phi = 0$ and $(d^*d^* + \kappa^2)H = 0$.

These equations can be rewritten as

$$(\Box + \kappa^2)\partial_\mu \phi = 0 \tag{16}$$

and

$$(\Box + \kappa^2)H = 0. \tag{17}$$

Therefore, the fluctuations of ϕ and H are massive. Obviously, these two possibilities can not occurs simultaneously. Indeed, in the most interesting case, the degree of freedom of the massless ϕ field is "eaten up" by the gauge field B to become massive and the ϕ field completely decouples from the theory [3].

IV $N = 1 - D = 4$ massive $B \wedge F$ $\to N = 2 - D = 3$ massive $B \wedge \varphi$ models

- $N = 1 - D = 4$ massive $B \wedge F$ model.

Let us begin by introducing the $N = 1 - D = 4$ supersymmetric BF extended model. For extended we mean that we include mass terms for the Kalb-Ramond field. This mass term will be introduced here for later comparison to the tridimensional case. Actually, this construction can be seen as a superspace and abelian version of the so called BF-Yang-Mills models. These models are described by the action

$$S_{BF-YM} = \int_{M_4} Tr\left\{kB \wedge F + \frac{g^2}{4}B \wedge {}^*B\right\}. \tag{18}$$

Note that, on-shell, (18) is equivalent to the standard YM action. This formalism was used by Fucito *et al.* [4] in order to study quark confinement.

As our basic superfield action we take [5]

$$
\begin{aligned}
S_{BF}^{SS} &= \frac{1}{8}\int d^4x\{-i\kappa[\int d^2\theta B^\alpha W_\alpha - \int d^2\overline{\theta}\overline{B}_{\dot\alpha}\overline{W}^{\dot\alpha}]\\
&\quad +\frac{g^2}{2}[\int d^2\theta B^\alpha B_\alpha + \int d^2\overline{\theta}\,\overline{B}_{\dot\alpha}\overline{B}^{\dot\alpha}]\}. \tag{19}
\end{aligned}
$$

where W_α is a spinor superfield-strenght, B_α is a chiral spinor superfield, $\bar{D}_{\dot\alpha}B_\alpha = 0$, κ and g are massive

parameters. Their corresponding θ-expansions are:

$$
\begin{aligned}
W_\alpha(x,\theta,\bar\theta) &= 4i\lambda_\alpha(x) - [4\delta_\alpha^\beta D(x) + 2i(\sigma^\mu\bar\sigma^\nu)_\alpha^\beta F_{\mu\nu}(x)]\theta_\beta\\
&\quad +4\theta^2\sigma_{\alpha\dot\alpha}^\mu \partial_\mu\bar\lambda^{\dot\alpha} \tag{20}
\end{aligned}
$$

$$B_\alpha(x,\theta,\overline{\theta}) = e^{i\theta\sigma^\mu\overline{\theta}\partial_\mu}[i\psi_\alpha(x) + \theta^\beta T_{\alpha\beta}(x) + \theta\theta\xi_\alpha(x)], \tag{21}$$

where

$$T_{\alpha\beta} = T_{(\alpha\beta)} + T_{[\alpha\beta]} = -4i(\sigma^{\mu\nu})_{\alpha\beta}B_{\mu\nu} + 2\varepsilon_{\alpha\beta}(M + iN). \tag{22}$$

Our conventions for supersymmetric covariant derivatives are

$$
\begin{aligned}
D_\alpha &\equiv \frac{\partial}{\partial\theta^\alpha} + i\sigma_{\alpha\dot\alpha}^\mu \bar\theta^{\dot\alpha}\partial_\mu\\
\bar{D}_{\dot\alpha} &\equiv -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\sigma_{\alpha\dot\alpha}^\mu\partial_\mu. \tag{23}
\end{aligned}
$$

We call attention for the electromagnetic field-strenght and the antisymmetric gauge field which are contained in W_α and B_α, respectively. In terms of the components fields, the action (19) can be read as

$$
\begin{aligned}
S &= \int d^4x\{[-\frac{i\kappa}{2}\left(\xi\lambda - \bar\xi\bar\lambda\right) + \frac{\kappa}{2}B^{\mu\nu}\widetilde{F}_{\mu\nu} - \kappa DN]\\
&\quad +\frac{\kappa}{2}\left(\psi^\alpha\sigma_{\alpha\dot\alpha}^\mu \partial_\mu\bar\lambda^{\dot\alpha} + \bar\psi_{\dot\alpha}\left(\bar\sigma^\mu\right)^{\dot\alpha\alpha}\partial_\mu\lambda_\alpha\right)\\
&\quad +g^2[\frac{1}{8}\left(\psi\xi + \bar\psi\bar\xi\right) + \frac{1}{2}B^{\mu\nu}B_{\mu\nu} - \frac{1}{2}\left(M^2 + N^2\right)]\}\\
&= \int d^4x[(\frac{i\kappa}{2}\Xi\gamma^5\Lambda + \frac{\kappa}{2}\bar\Psi\gamma^\mu\partial_\mu\Lambda + \frac{\kappa}{2}B^{\mu\nu}\widetilde{F}_{\mu\nu} - \kappa DN)\\
&\quad +g^2(\frac{1}{8}\bar\Psi\Xi + \frac{1}{2}B^{\mu\nu}B_{\mu\nu} - \frac{1}{2}\left(M^2 + N^2\right))]. \tag{24}
\end{aligned}
$$

In the last equality above, the fermionic fields have been organized as four-component Majorana spinors as follows

$$\Xi = \begin{pmatrix}\xi_\alpha\\\bar\xi^{\dot\alpha}\end{pmatrix} \ ;\ \Lambda = \begin{pmatrix}\lambda_\alpha\\\bar\lambda^{\dot\alpha}\end{pmatrix} \ ;\ \Psi = \begin{pmatrix}\psi_\alpha\\\bar\psi^{\dot\alpha}\end{pmatrix}, \tag{25}$$

and we denote the dual field-strenght defining $\widetilde{F}_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$. Furthermore, we use the following identities

$$
\begin{aligned}
\bar\Psi\Lambda &= \bar\psi\bar\lambda + \psi\lambda\\
\bar\Psi\gamma^5\Lambda &= \bar\psi\bar\lambda - \psi\lambda\\
\bar\Psi\gamma^\mu\Lambda &= \psi\sigma^\mu\bar\lambda + \bar\psi\bar\sigma^\mu\lambda. \tag{26}
\end{aligned}
$$

We have not considered coupling with matter fields and a propagation term for the gauge fields. On the other hand, our supespace BF term was constructed in a very simple way. A quite similar construction was introduced by Clark *et al.* [6].

The off-diagonal mass term $\xi\lambda$ (or $\bar{\Xi}\gamma^5\Lambda$) has been shown by Brooks and Gates, Jr. [7] in the context of super-Yang-Mills theory. Note that the identity

$$\gamma_5\sigma^{\mu\nu} = \frac{i}{2}\varepsilon_{\mu\nu\alpha\beta}\sigma^{\alpha\beta} \qquad (27)$$

reveals a connection between the topological behaviour denoted by the Levi-Civita tensor $\varepsilon_{\mu\nu\alpha\beta}$, and the pseudo-escalar γ_5.

So, it is worthwhile to mention that this term has topological origin and it can be seen as a fermionic counterpart of the BF term. In our opinion, this fermionic mass term deserves more attention.

- **The $N = 2 - D = 3$ massive $B \wedge \varphi$ model**

We will now carry out a dimensional reduction in the bosonic sector of (24). Hence, after dimensional reduction, the bosonic sector of (24) can be written as [5]

$$S_{bos.} = \int d^3x\{[\kappa\varepsilon_{\mu\alpha\beta}V^\mu F^{\alpha\beta} + \kappa\varepsilon_{\mu\nu\alpha}B^{\mu\nu}\partial^\alpha\varphi - \kappa DN]$$
$$+ g^2[\frac{1}{2}B^{\mu\nu}B_{\mu\nu} - V^\mu V_\mu - \frac{1}{2}\left(M^2 + N^2\right)]\} , (28)$$

where V^μ is a vectorial field and φ represents a real scalar field. Notice that the first term in r.h.s. of (28) can be transformed in the Chern-Simons term if we identify $V^\mu \equiv A^\mu$. The second one is the so called $B \wedge \varphi$ term.

Now let us proceed to the dimensional reduction of the fermionic sector of the model. First, note that the Lorentz group in three dimensions is $SL(2,R)$ rather than $SL(2,C)$ in $D = 4$. Therefore, Weyl spinors with four degrees of freedom will be mapped into Dirac spinors. So the correct associations keeping the degrees of freedom are sketched as

$$\Xi = \begin{pmatrix} \xi_\alpha \\ \bar{\xi}^{\dot\alpha} \end{pmatrix} \rightarrow \Xi_\pm = \xi_\alpha \pm i\tau_\alpha$$

$$\Lambda = \begin{pmatrix} \lambda_\alpha \\ \bar{\lambda}^{\dot\alpha} \end{pmatrix} \rightarrow \Lambda_\pm = \lambda_\alpha \pm i\rho_\alpha$$

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot\alpha} \end{pmatrix} \rightarrow \Psi_\pm = \psi_\alpha \pm i\chi_\alpha . \qquad (29)$$

From (29), we find that

$$\Psi\bar{\Xi} \rightarrow \frac{1}{2}(\Psi_+\Xi_- + \Psi_-\Xi_+)$$

$$\bar{\Psi}\gamma^\mu\partial_\mu\Lambda \rightarrow \frac{1}{2}(\Psi_+\gamma^{\widehat{\mu}}\partial_{\widehat{\mu}}\Lambda_- + \Psi_-\gamma^{\widehat{\mu}}\partial_{\widehat{\mu}}\Lambda_+)$$

$$\Xi\gamma^5\Lambda \rightarrow \frac{1}{2}(\Xi_+\Lambda_+ + \Xi_-\Lambda_-) . \qquad (30)$$

where *hatted* index means three-dimensional space-time.

Thus, the dimensionally reduced fermionic sector of (24) may be written

$$S_{ferm.} = \int d^3x\{\frac{i\kappa}{4}\left(\Xi_+\Lambda_+ + \Xi_-\Lambda_-\right) + \frac{\kappa}{4}(\Psi_+\gamma^{\widehat{\mu}}\partial_{\widehat{\mu}}\Lambda_-$$
$$+ \Psi_-\gamma^{\widehat{\mu}}\partial_{\widehat{\mu}}\Lambda_+) + \frac{g^2}{16}\left(\Psi_+\Xi_- + \Psi_-\Xi_+\right)\} . (31)$$

The action $S = S_{bos.} + S_{ferm.}$ is invariant under the following supersymmetry transformations (from now on, greek indices mean three-dimensional space-time):

$$\delta\lambda_\alpha = -iD\eta_\alpha - (\sigma^\mu\sigma^\nu)_\alpha^\beta\,\eta_\beta F_{\mu\nu}$$
$$\delta\rho_\alpha = iD\zeta_\alpha - (\sigma^\mu\sigma^\nu)_\alpha^\beta\,\zeta_\beta F_{\mu\nu}$$
$$\delta F^{\mu\nu} = i\partial^\mu\left(\eta\sigma^\nu\rho - \lambda\sigma^\nu\zeta\right) - i\partial^\nu\left(\eta\sigma^\mu\rho - \lambda\sigma^\mu\zeta\right)$$
$$\delta D = \partial_\mu\left(-\eta\sigma^\mu\rho + \lambda\sigma^\mu\zeta\right) \qquad (32)$$

$$\delta\left(\psi_\alpha \pm i\chi_\alpha\right) = \delta\Psi_\pm = i\eta^\beta\widetilde{T}_{\beta\alpha} \pm \zeta^\beta\widetilde{T}_{\beta\alpha},$$
$$\delta\widetilde{T}_{\beta\alpha} = -\eta_\beta\xi_\alpha + \zeta^\lambda\sigma_{\beta\lambda}^\mu\partial_\mu\psi_\alpha,$$
$$\delta\left(\xi_\alpha \pm i\tau_\alpha\right) = \delta\Xi_\pm = -i\zeta_\lambda\left(\sigma^\mu\right)^{\lambda\beta}T_{\beta\alpha}$$
$$\mp\eta_\lambda\left(\bar{\sigma}^\mu\right)^{\beta\lambda}T_{\beta\alpha} , \qquad (33)$$

where η and ζ are supersymmetric parameters, which indicates that we have two supersymmetries in the aforementioned action.

V Fractional statistics - anyons

The fractional statistics [8] with its theoretical and applicable consequences plays an interesting interplay role between quantum field theory and condensed matter physics. Previous speculations [9] that the fractional quantum Hall effect could be explained by quasiparticles (anyons) obeying fractional statistics were confirmed and the behaviour of two-dimensional materials such as vortices in superfluid helium films may be explained by fractional statistics. As it is known, the presence of Chern-Simons terms in (2+1) dimensional gauge theories induce fractional statistics. In such theories, it has been known that there exist excitations, called anyons, which continuously interpolate between bosons and fermions. In the well-known physical realization, anyons are composite quasi-particles where magnetic flux-tubes are attached to charged particles.

Recently, there have been thoughts of generalizing exotic statistics to extended objects to the case of

strings in four dimensions [10]. Abelian BF models in four dimensions has also been exploited in dual models of cosmic strings, and axionic black hole theories where the axion charge is physically detectable only by external cosmic strings in a four dimensional Aharanov-Bohm type process [11].

Aneziris *et al.* [12] showed that more general statistics can exist in (3+1) dimensions. Statistical phases of BF theory can be seen to arise from certain cosmic string and superstring phenomena, as well as in the Nambu-Goto string theory modified with the inclusion of the Kalb-Ramond field (B field) [13].

VI Linking number - intersection number

In a recent interesting work, Ashtekar and Corichi [14] showed that there is a precise in which the Heisenberg uncertainty between fluxes of electric and magnetic fields through finite surfaces is given by the Gauss linking number of the loops that bound these surfaces.

Topological field theories presents observables other than the partition function. Witten has argued that in these theories Wilson loops are appropriate metric independent and gauge invariant objects. Polyakov has related the vacuum expectation values of Wilson loops in the abelian Chern-Simons theory to the classical Gauss linking number of two loops.

In the case of BF systems, we can reinterpreting the linking number as the intersection number of one loop with a disc bounded by the other loop. So, this observable has a natural generalization to other dimensions. Considering the action (2), the fields B_p and A_{D-p-1} allow us to form the following metric independent and gauge invariant expressions ("Wilson surfaces"):

$$W[L] = \exp\left(\int_L A\right), W[\Sigma] = \exp\left(\int_\Sigma B\right) \quad (34)$$

where Σ and L are disjoint compact and oriented $p-$ and $(D - p - 1)$-dimensional boundaries of two oriented submanifold of an D-dimensional oriented manifold $\mathcal{M}$. This formalism was presented by Blau and Thompson [15], who proved that the expectation value $W(\Sigma, L) = \langle W_B(\Sigma) W_A(L) \rangle$ is equal to the linking number of the "surfaces ".

VII Fractional statistics in D=3 from $B \wedge \varphi$ term?

Consider the following action

$$S = S_0 + \int d^3x \left(\frac{\kappa}{2}\varepsilon^{\mu\nu\alpha}B_{\mu\nu}\partial_\alpha\varphi + \frac{g}{2}J^{\mu\nu}B_{\mu\nu} + hj\varphi\right), \quad (35)$$

where g, h are coupling constants, $J^{\mu\nu}$ and j are currents and sources. S_0 depends only on fields that originate currents and sources. Integrating out the fields $B_{\mu\nu}$ and φ, we arrive at

$$S_{eff} = S_0 - \frac{hg}{4\kappa} \int \int d^3x d^3y J^{\mu\nu}(x) \langle B_{\mu\nu}(x)\varphi(y)\rangle j(y). \quad (36)$$

From (35) and using the Landau gauge, is easy to see that

$$\langle B_{\mu\nu}(x)\varphi(x)\rangle = \varepsilon_{\mu\nu\alpha}\partial_x^\alpha G(x - y), \quad (37)$$

where

$$G(x - y) = -\frac{1}{4\pi}\frac{1}{|x - y|}, \quad (38)$$

Therefore

$$\langle B_{\mu\nu}(x)\varphi(x)\rangle = \frac{\varepsilon_{\mu\nu\alpha}}{4\pi}\frac{(x - y)^\alpha}{|x - y|^3}, \quad (39)$$

The correlation function $\langle B_{\mu\nu}(x)\varphi(y)\rangle$ is tantamount to the correlation function $\langle A_\mu(x)A_\nu(y)\rangle$ of the pure Chern-Simons theory in the Landau gauge (transverse propagator). The effective action (36) can be rewritten as

$$S_{eff} = S_0 - \frac{hg}{4\kappa}\frac{1}{4\pi}\varepsilon_{\mu\nu\alpha} \int \int d^3x d^3y J^{\mu\nu}(x)\frac{(x - y)^\alpha}{|x - y|^3}j(y) \quad (40)$$

and

$$S_{eff} = S_0 - \frac{hg}{4\kappa}(linking \quad number) \quad (41)$$

On the other hand, Blau and Thompson [15] suggest application of their formalism to the case where B is a zero-form and A is a one-form, involving a linking number of a point P and a circle γ, through the expression

$$W_B(P)W_A(d) = \exp(B(P) + \oint_\gamma A) \quad (42)$$

where a disc d is bounded by γ.

These results support our speculation that we can define a linking number from the $B \wedge \varphi$ term, and that it can exist a fractional statistics even in this case.

VIII   Pauli's term and fractional statistics in D=3.

As it is known, the presence of Chern-Simons terms in (2+1) dimensional gauge theories induce fractional statistics[16, 17]. Stern [18] was the first, as far as we know, to suggest a nonminimal term in the context of the Maxwell-Chern-Simons electrodynamics with the intention of mimicking an anyonic behavior without a pure Chern-Simons limit. This term can be interpreted as a tree level Pauli-type coupling, i. e., an anomalous magnetic moment. It is a specific feature of (2+1) dimensions that the Pauli coupling exists, not only for spinning particles, but also for scalar ones [19].

We consider here an Abelian Chern-Simons-Higgs theory where the complex scalar fields couples directly to the electromagnetic field strength (Pauli-type coupling). The Lagrangian of the model under investigation is

$$L = |\nabla_\mu \phi|^2 + \frac{\kappa}{2} \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda - A_\mu \partial^\mu b + \frac{\alpha}{2} b^2 \quad (43)$$

where $\nabla_\mu \phi \equiv (\partial_\mu - ieA_\mu - i\frac{g}{4}\varepsilon_{\mu\lambda\sigma}F^{\lambda\sigma})\phi$. Note that this covariant derivative includes both the usual minimal coupling and the contribution due to Pauli's term. Here A_μ is the gauge field and the Levi-Civita symbol $\varepsilon_{\mu\nu\lambda}$ is fixed by $\varepsilon_{012} = 1$ and $g_{\mu\nu} = diag(1, -1, -1)$. The multiplier field b has been introduced to implement the covariant gauge-fixing condition.

Before quantizing the theory, we analyze the above Lagrangian in terms of Hamiltonian methods. Here we follow the approach used by Shin *et al.* [20]. We carry out the constraint analysis of this model, in order to obtain a consistent formulation of the theory.

The canonical momenta of the Lagrangian (43), which can be easily seen by considering its temporal and spatial components separately, are given by

$$\pi_0 = 0, \quad (44)$$

$$\pi_b = -A_0, \quad (45)$$

$$\pi^j = -\frac{\kappa}{2}\varepsilon^{ij}A_i - i\frac{g}{2}\varepsilon^{ij}[\phi^*(D_i\phi) - \phi(D_i\phi)^*]$$
$$- \frac{g^2}{2}\partial^j A_0 |\phi|^2 + \frac{g^2}{4}(\partial_0 A_0)|\phi|^2, \quad (46)$$

$$\pi = (\partial_0 \phi^*) + ieA_0\phi^* + i\frac{g}{4}\phi^*\varepsilon^{ij}F_{ij}, \quad (47)$$

$$\pi^* = (\partial_0 \phi) - ieA_0\phi - i\frac{g}{4}\phi\varepsilon^{ij}F_{ij}, \quad (48)$$

where π_0, π^j, π_b, π and π^* are the canonical momenta conjugate to A_0, A_j, b, ϕ and ϕ^* respectively. Also we have used $\varepsilon_{ij} = \varepsilon_{0ij}$, $D_i = \partial_i - ieA_i$ and $i, j = 1, 2$.

The canonical momenta (44) and (45) do not involve explicit time dependence and hence are primary constraints. Performing the Legendre transformation, the canonical Hamiltonian can be written as

$$H_c = \pi^*\pi + |D\phi|^2 + ieA_0(\pi\phi - \pi^*\phi^*) + \kappa\varepsilon^{ij}A_0\partial_i A_j$$
$$+ A_i\partial^i b - \frac{\alpha}{2}b^2 - i\frac{g}{2}\varepsilon^{ij}\partial_j A_0 [\phi^*(D_i\phi) - \phi(D_i\phi)^*]$$
$$- \frac{g^2}{4}\partial_i A_0 \partial^i A_0 |\phi|^2 - \frac{g}{4}\varepsilon^{ij}F_{ij}[\phi^*(D_0\phi) - \phi(D_0\phi)^*]$$
$$- \frac{g^2}{8}F^{ij}F_{ij}|\phi|^2. \, (49)$$

Now, in order to implement the primary constraints in the theory, we construct the primary Hamiltonian as

$$H_p = H_c + \lambda_0 \pi + \lambda_1 (\pi_b + A_0), \quad (50)$$

where λ_0 and λ_1 are Lagrange multiplier fields. Conserving in time the primary constraints yields the secondary constraints

$$\psi_1 = \pi_0 \approx 0, \quad (51)$$

$$\psi_2 = \pi_b + A_0 \approx 0, \quad (52)$$

which are also conserved in time and where the symbol $\approx$ indicates weak equality, i. e., the constraints can be identically set equal to zero only after computing the relevant Poisson brackets. Thus there is no more constraint and the above equations are the set of fully second-class constraints. On the other hand, there is no first-class conditions and so, no gauge conditions to be determined in theory. This is an effect of the gauge fixing condition imposed previously. As it is known, the lack of physical significance allows that the second-class constraints can be eliminated by means of Dirac brackets (DB's).

Following the standard Dirac brackets formalism and quantizing the system, we obtain the following set of non-vanishing equal-time commutators:

$$[A_0(x), b(y)] = i\delta^2(x - y) \quad (53)$$

$$[A_i(x), \pi_j(y)] = i\delta_{ij}\delta^2(x - y) \quad (54)$$

$$[\phi(x), \pi(y)] = [\phi^*(x), \pi^*(y)] = i\delta^2(x - y) \quad (55)$$

After achieving the quantization we proceed to construct the angular momentum operator and compute the angular momentum of the matter field ϕ.

The symmetric energy-momentum tensor can be obtained by coupling the fields to gravity and then varying the action with respect to $g^{\mu\nu}$:

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = (\nabla_\mu \phi)^*(\nabla_\nu \phi) + (\nabla_\nu \phi)^*(\nabla_\mu \phi)$$
$$-A_\mu \partial_\nu b - A_\nu \partial_\mu b$$
$$-g_{\mu\nu}(|\nabla_\alpha \phi|^2 - A_\alpha \partial^\alpha b). \quad (56)$$

The angular momentum operator in (2+1) dimensions is given by

$$L = \int d^2 x \varepsilon^{ij} x_i T_{0j}.$$

Hence

$$L = \int d^2 x \varepsilon^{ij} x_i \{ (\pi \partial_j \phi + \pi^* \partial_j \phi^*) - ieA_j J_0$$
$$-i\frac{g}{2}\varepsilon_{jl}F^{l0}(\pi\phi - \pi^*\phi^*) - A_0 \partial_j b$$
$$+A_j \partial_0 b - i\frac{g}{2}A_j \varepsilon^{kl}\partial_k[\phi^*(D_l\phi) - \phi(D_l\phi)^*]$$
$$+i\frac{g^2}{2}A_j \partial_k(|\phi|^2 F^{0k})\}, \quad (57)$$

where

$$J_0 = i\{\pi\phi - \pi^*\phi^* - \frac{g}{2e}\varepsilon^{ij}\partial_i[\phi^*(D_j\phi) - \phi(D_j\phi)^*]$$
$$+i\frac{g^2}{2e}\partial_i(|\phi|^2 F^{0i})\} \quad (58)$$

is the temporal component of the conserved matter current.

The key point here is that Gauss' law is no more a constraint, while J_0 and $T_{\mu\nu}$ contain derivatives of A_μ . Note that, due to its topological character, the Chern-Simons term does not contribute to the energy-momentum tensor. These aspects are attributed to the non-linearity introduced by Pauli's term.

The rotational property of the ϕ field is obtained by computing the commutator $[L, \phi(y)]$. Using equations (53-55) and (57), it is easy to see that

$$[L, \phi(y)] = \varepsilon^{ij}y_i\partial_j\phi - [e\int d^2 x \varepsilon^{ij}x_i A_j J_0, \phi]$$
$$+i\frac{g}{2}\varepsilon^{ij}\varepsilon_{jk}y_i F^{k0}\phi. \quad (59)$$

This commutator can be rewritten by means of the electromagnetic charge operator

$$Q = \int d^2 x J_0(x)$$

and becomes

$$[L, \phi(y)] = \varepsilon^{ij}y_i\partial_j\phi - \frac{e^2}{4\pi\kappa}[Q^2, \phi(y)] + i\frac{g}{2}\varepsilon^{ij}\varepsilon_{jk}y_i F^{k0}\phi \quad (60)$$

or, in more familiar notation

$$[L, \phi(y)] = i(\mathbf{y} \times \nabla)\phi(y) - \frac{e^2}{2\pi\kappa}Q\phi(y) + i\frac{g}{2}\mathbf{y} \cdot \mathbf{E}\phi(y). \quad (61)$$

The first term in the right hand side of eq. (61) represents the intrinsic spin and the second is the so-called rotational anomaly, which is responsible for the fractional spin. Unlike the Chern-Simons term (whose contribution is related with magnetic field), the Pauli term induces an anomalous contribution for the spin of the system, which depends on electric field [21]. We stress that, here the nonminimal coupling constant is a free parameter.

It is worth mentioning that all the procedure above can be carried out even if there is no Chern-Simons term in the Lagrangian (43). In this case the anomalous contribution to spin would just come from the Pauli term.

Now we will discuss the above result in connection with theories in the broken-symmetry phase. Boyanovsky [22] has found that the low-lying excitations of a U(1) Chern-Simons theory in interaction with a complex scalar field in a broken symmetry state are massive bosons with *canonical statistics*. He explained his result as due to the screening of long-range forces in a broken symmetry phase. In this phase localized charge distributions cannot be supported, which is supposed to be essential for fractional spin. On the other hand, if we consider a non-minimally coupled Abelian-Higgs model, the long-distance damping effect by the "photon" mass κ no longer exists. This is an indication that Pauli's term, which induces an anomalous spin, can be relevant for the study of broken symmetry states (superfluid) in the context of effective theories in condensed matter.

In nonrelativistic limit, Carrington and Kunstatter [23] have shown that anomalous magnetic moment interactions gives rise to both the Aharonov-Bohm and Aharonov-Casher effects. They have speculated possible anomalous statistics without the CS term. As a matter of fact, we believe that this (in a relativistic theory) was proved here. On the other hand, the Abelian Chern-Simons term can be generated by means of a spontaneous symmetry breaking of a nonminimal theory. This connection between Chern-Simons and Pauli-type coupling was pointed out by Stern. So the Pauli

term at tree-level (with the nonminimal coupling constant g as a free parameter) can constitute an effective theory which bring us information about physical models in broken symmetry phase.

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