

# A Dynamic Model for the Analysis of Surge in a Tropical Storm

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Received 12 November, 1998; Revised 22 March, 1999

We obtain an analytic representation for the profile of the surge wave in the sea, due to a tropical storm at a low latitude. The results reproduce the known experimental data adequately. In the calculations we take into account the influence of Rossby's second parameter and we analyse its contribution to the surge wave, which is proved to be sufficiently small to be neglected in higher latitudes. The obtained analytical solution allows us to study in detail the surge wave formation in the presence of such a storm. To solve the problem, Laplace method and Green functions are used. This allows us to improve previous results obtained by Evsa in 1989.

## I Introduction

In the last years the interest of studying non-periodic oscillations of sea level has grown. In particular, it is of interest the study of those oscillations excited by pressure fields and strong winds in the tropical storm. It is possible to find previous information about this theme in the work of Jelesniansky [1] and more recently, Johns et al. [2], Fandry et al. [3], Jerome et al. [4], Signorini [5], Davies et al. [6] and Lardner et al. [7] and [8]. The majority of these papers employ numerical methods or qualitative analysis of the solution of the mathematical model. In the present paper we obtain an analytic solution of the problem, similar to the one formulated by Evsa [9]. In this work, there is an incorrect expression for the Green function of the problem. Therefore, for the analytic solution, Evsa does not take account of the multi-valuated character of the complex variable function in the path integrations. In the present paper we make correct use of the Green function method and the integral transform techniques. We also analyze the influence of Rossby's second parameter, which is usually

neglected at high latitudes by other authors because of its smallness with respect to Rossby's first parameter and because it allows to simplify the calculation. We check that Rossby's second parameter is small enough to be neglected at low latitudes.

## II Formulation of the problem

Let us consider a tropical cyclone (TC) which causes a perturbation of the sea surface in an ocean region distant from any coast. The interaction region has a characteristic deepness less than the storm horizontal size by two orders of magnitude. Therefore, we can use the shallow water approximation. We also consider a quasistationary regime for the storm, in order to keep a fixed coordinate center. Due to the axial symmetry of this problem, we use a cylindrical system of coordinates with center on the axis of the cyclone.

The Navier-Stokes equations in the cylindrical system of coordinates including the Coriolis terms are given by

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \varphi} + w \frac{\partial u}{\partial z} - \frac{v^2}{r} = f v - f^* w - \frac{1}{\rho} \frac{\partial P_a}{\partial r} + \frac{1}{\rho} \frac{\partial \sigma_{rz}}{\partial z}, \quad (1)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \varphi} + w \frac{\partial v}{\partial z} - \frac{uv}{r} = -f u - \frac{1}{\rho r} \frac{\partial P_a}{\partial \varphi} + \frac{1}{\rho} \frac{\partial \sigma_{\varphi z}}{\partial z}, \quad (2)$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + \frac{v}{r} \frac{\partial w}{\partial \varphi} + w \frac{\partial w}{\partial z} = f^* u - \frac{1}{\rho} \frac{\partial P_a}{\partial z} - g, \quad (3)$$

with  $u = v_r$ ,  $v = v_\varphi$ ,  $w = v_z$ ,  $f = 2\Omega \sin \phi$ ,  $f^* = 2\Omega \cos \phi$ . Here  $\Omega$  is the angular Earth velocity and  $\phi$  is the latitude of the region under consideration.

In these equations the term of the divergence of the stress tensor contains only the vertical stress forces because they are much larger than the rest of the internal friction effects. In equation (3), all the stress effects are neglected in comparison with the vertical pressure gradient and with the gravity acceleration.

The equation of continuity in cylindrical coordinates is

$$\frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \varphi} + \frac{\partial w}{\partial z} + \frac{u}{r} = 0. \quad (4)$$

The boundary conditions are: in the bottom, the adherence condition; at the sea surface, the kinematic condition ( $w|_{z=\xi} = \frac{d\xi}{dt} \equiv u \frac{\partial \xi}{\partial r} + \frac{v}{r} \frac{\partial \xi}{\partial \varphi} + \frac{\partial \xi}{\partial t}$ ).

The Oz axis coincides with the external normal vector to the Earth surface. We also suppose the integration time in these equations is much smaller than the cyclone displacement characteristic time. Therefore, we take into account the time derivatives, although we consider the TC static.

By integration of equation (1)-(3) along the z direction, we obtain the following system, for the shallow water approximation, in dimensionless form,

$$St^{-1} \frac{\partial u}{\partial t} + (\xi + h) \frac{\partial P_a}{\partial r} + Fr^{-1} (\xi + h) \frac{\partial \xi}{\partial r} - Ro^{-1} v + Ro^{*-1} \int_{-h}^{\xi} w dz = \tau_r^s - \tau_r^f, \quad (5)$$

$$St^{-1} \frac{\partial v}{\partial t} + \frac{(\xi + h)}{r} \frac{\partial P_a}{\partial \varphi} + Fr^{-1} \frac{(\xi + h)}{r} \frac{\partial \xi}{\partial \varphi} + Ro^{-1} u = \tau_\varphi^s - \tau_\varphi^f, \quad (6)$$

$$St^{-1} \frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \varphi} + \frac{u}{r} = 0. \quad (7)$$

We neglect the convective acceleration terms, because the absence of coasts implies that the velocity gradients are very small in the considered region. In the above equations  $\xi$  is the sea perturbation average level;  $h$  is the non-perturbed sea deepness;  $P_a$  is the atmospheric pressure field originated by the TC;  $\tau_i^s$  are the surface stresses produced by the TC winds and  $\tau_i^f$  are the bottom stresses ( $i = r, \varphi$ );  $\rho_0$  is the water density;  $H$  is the characteristic deepness of the region;  $St = (U_0 T)/L$  is the Struhal's number;  $Fr = U_0^2/(gH)$  is Froude's number;  $Ro = U_0/(fL)$  is Rossby's first parameter and  $Ro^* = U_0/(f^*H)$  is Rossby's second parameter;  $U_0$  is the characteristic velocity;  $L$  is the characteristic size on the horizontal ocean surface plane. We take  $\Phi \approx 5^\circ$  because the contribution of  $f^*$  is bigger at

very low latitudes. This is precisely the interest of our study, maintaining  $f$  not equal to zero in the model.

In the following calculation we neglect  $\tau_i^b$  because the processes are essentially superficial and therefore the bottom does not matter. Also, we consider  $\xi + h \approx h = 1$  because in the problem  $\xi \ll h$  and that the problem is axisymmetric so that  $\frac{\partial f}{\partial \varphi} \equiv 0$ . The presence of Rossby's second parameter in equation (5) makes it impossible to use the two-dimensional approach employed by other authors. To avoid this problem, we assume an exponential dependence of the horizontal velocities with respect to the vertical coordinate,

$$\begin{aligned} u(r, \varphi, z, t) &= u_0(r, \varphi, t) \exp[-a(\xi - z)], \\ v(r, \varphi, z, t) &= v_0(r, \varphi, t) \exp[-a(\xi - z)]. \end{aligned} \quad (8)$$

We take an exponential dependence because, in many fluid problems, that is the law of variation of the horizontal velocities with respect to the deepness and considering the gravity constant in the studied region. This assumption is consistent with our model, in which  $Fr$

is constant.

With this dependence we have that, at the surface  $z = \xi$ , velocities are  $u_0$  and  $v_0$ . The parameter  $a$  controls the decrease of the velocities with  $z$ . Expressions (8), using (4), allow us to obtain

$$\int_{-h}^{\xi} w dz = H \left( u \frac{\partial \xi}{\partial r} + \frac{v}{r} \frac{\partial \xi}{\partial \varphi} \right) - \frac{H}{a} \left( \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \varphi} + \frac{u}{r} \right) \equiv \Pi . \tag{9}$$

We have neglected the convective derivative of  $\xi$  in the first term because the calculations are made in open sea [10], [11] and also because we only consider the long wave generated by TC. Hence ( $H = 1$ , the dimensionless formulation)

$$\Pi = \frac{1}{a} \left( \frac{\partial u}{\partial r} + \frac{u}{r} \right) . \tag{10}$$

Then, system (1) – (3) takes the form

$$St^{-1} \frac{\partial u}{\partial t} + \frac{\partial Pa}{\partial r} + Fr^{-1} \frac{\partial \xi}{\partial r} - Ro^{-1} v + Ro^{*-1} \Pi = \tau_r^s , \tag{11}$$

$$St^{-1} \frac{\partial v}{\partial t} + Ro^{-1} u = \tau_\varphi^s , \tag{12}$$

$$St^{-1} \frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial r} + \frac{u}{r} = 0 . \tag{13}$$

To solve this system we consider homogeneous initial conditions and the following boundary conditions,

$$u|_{r \rightarrow \infty} = v|_{r \rightarrow \infty} = \xi|_{r \rightarrow \infty} = 0, \quad \xi|_{r=0} < M . \tag{14}$$

To accomplish the calculations we take, following Johns et al. [2],  $\tau_r^s = \tau_0 \sin \alpha \beta^2(r)$ ,  $\tau_\varphi^s = \tau_0 \cos \alpha \beta^2(r)$ , where  $\tau_0 = \rho_1 \gamma_1^2 W_0^2$  and  $\beta(r) = 2(r/R)/[1 + (r/R)^2]$ . Here  $\rho_1$  is the density of the air,  $\gamma_1$  is the drag coefficient,  $W_0$  is the wind velocity in the radius  $R$  of the greatest winds, and  $\alpha$  is the fluency angle of the wind in the TC. We take the following dependence for the atmospheric pressure  $Pa(r) = P_\infty - \Delta P/[1 + (r/R)^2]^{1/2}$  in correspondence with the common assumption in the studies of surge in tropical storms (see, for instance, [1, 11]).

### III Calculation

In the system of equations (11)-(13),  $\varepsilon \equiv (Ro^*)^{-1} \ll 1$ . Hence, we assume an analytic dependence of the solu-

tion  $u, v, \xi$  (see e.g. [12]),

$$u(r, \varphi, t) = \sum_{n=0}^{\infty} u_n(r, \varphi, t) \varepsilon^n , \tag{15}$$

$$v(r, \varphi, t) = \sum_{n=0}^{\infty} v_n(r, \varphi, t) \varepsilon^n , \tag{16}$$

$$\xi(r, \varphi, t) = \sum_{n=0}^{\infty} \xi_n(r, \varphi, t) \varepsilon^n . \tag{17}$$

Substituting (15) – (16) into equations (11) – (13) and taking only the two first terms of the sequence  $u_0, v_0, \xi_0$ , and  $u_1, v_1, \xi_1$ , we obtain the following systems:

<sup>1</sup> If directly we shall take a linear variation of the vertical velocity, for instance,  $w = (1 + z/H)(d\xi/dt)$ , it simply shall give another numerical value of the coefficient in (10).

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-Zero-order system

$$St^{-1} \frac{\partial u_0}{\partial t} + \frac{\partial Pa}{\partial r} + Fr^{-1} \frac{\partial \xi_0}{\partial r} - Ro^{-1} v_0 = \tau_r^s, \quad (18)$$

$$St^{-1} \frac{\partial v_0}{\partial t} + Ro^{-1} u_0 = \tau_\varphi^s, \quad (19)$$

$$St^{-1} \frac{\partial \xi_0}{\partial t} + \frac{\partial u_0}{\partial r} + \frac{u_0}{r} = 0; \quad (20)$$

-First-order system.

$$St^{-1} \frac{\partial u_1}{\partial t} + \xi_1 \frac{\partial Pa}{\partial r} + Fr^{-1} \frac{\partial \xi_1}{\partial r} - Ro^{-1} v_1 + \Pi_0 = 0, \quad (21)$$

$$St^{-1} \frac{\partial v_1}{\partial t} + Ro^{-1} u_1 = 0, \quad (22)$$

$$St^{-1} \frac{\partial \xi_1}{\partial t} + \frac{\partial u_1}{\partial r} + \frac{u_1}{r} = 0. \quad (23)$$

In our case, it is sufficient to take the solution at first order in perturbation theory, because the small parameter  $\varepsilon$  in (11) is multiplying a term which is of a very small order, due to the smoothness of the used model. This fact leads to a non-singular perturbation problem. Hence, (15) – (17) are, not only asymptotic, but also monotonic expansions. We apply the Laplace transform to system (18) – (20). For  $E_0(r, p) = L\{\xi_0(r, t)\}$  we obtain the following equation

$$\frac{1}{r} \frac{d}{dr} \left[ r \frac{dE_0}{dr} \right] - R_p E_0 = Fr \left[ \frac{\partial F(r, p)}{\partial r} - \frac{F(r, p)}{r} \right], \quad (24)$$

where  $F(r, p) = F_1(r, p) - (St/pRo)F_2(r, p)$ ,  $F_1(r, p) = [\tau_0 \sin \alpha \beta^2(r) - \partial Pa / \partial r] / p$ ,  $F_2(r, p) = \tau_0 \cos \alpha \beta^2(r) / p$ ,  $R_p = \delta(p^2 + \theta^2)$ ,  $\delta = Fr / St^2$ ,  $\theta = St / Ro$  and  $p = s + i\sigma$ .

We can express the solution of equation (24) as

$$E_0(r, p) = Fr \int_0^\infty G(r, s) \left[ \frac{\partial F(s, p)}{\partial s} - \frac{F(s, p)}{s} \right] ds, \quad (25)$$

where

$$G(r, s) = \begin{cases} \frac{I_0(\sqrt{R_p r})}{I_0(\sqrt{R_p R_1})} [I_0(\sqrt{R_p R_1}) K_0(\sqrt{R_p s}) - I_0(\sqrt{R_p s}) K_0(\sqrt{R_p R_1})], & 0 \leq r \leq s < R_1 \\ \frac{I_0(\sqrt{R_p s})}{I_0(\sqrt{R_p R_1})} [I_0(\sqrt{R_p R_1}) K_0(\sqrt{R_p r}) - I_0(\sqrt{R_p r}) K_0(\sqrt{R_p R_1})], & 0 \leq s \leq r < R_1 \end{cases} \quad (26)$$

is the Green function of the operator of equation (24), which satisfies the boundary conditions  $G(r, s)|_{r=0} < M$ ;  $G(r, s)|_{r \rightarrow \infty} = 0$ ;  $R_1$ ;  $R_1$  is the TC external radius.

The inverse Laplace transform gives

$$\begin{aligned} \xi_0(r, t) &= \int_0^r \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} G(r, s) \left[ \frac{A_1(s)}{p} + A_2(s)p^2 \right] \exp(pt) dp ds \\ &+ \int_r^{R_1} \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} G(r, s) \left[ \frac{A_1(s)}{p} + A_2(s)p^2 \right] \exp(pt) dp ds, \end{aligned} \quad (27)$$

with

$$\begin{aligned} A_1(s) &= Fr\tau_0 \sin \alpha \beta(s) [2\partial\beta(s)/\partial s - \beta(s)/s] + [\partial^2 Pa/\partial s^2 - \partial Pa/\partial s] , \\ A_2(s) &= (FrSt/R_o)\tau_0 \cos \alpha \beta(s) [\beta(s)/s - 2\partial\beta(s)/\partial s] . \end{aligned} \tag{28}$$

Usual techniques of the theory of functions of complex variables allow us to calculate the contour integrals in (27). We obtain

$$\xi_0(r, t) = \frac{2}{\pi^i} \sum_{i=1}^4 I_i , \tag{29}$$

where

$$I_1 = \int_0^r \int_0^\theta G(r, s) \frac{A_1(s) \sin(\rho t)}{\rho} d\rho ds , \tag{30}$$

$$I_2 = \int_0^r \int_0^\theta G(r, s) \frac{A_2(s)(1 - \cos(\rho t))}{\rho^2} d\rho ds , \tag{31}$$

$$I_3 = \int_r^{R_1} \int_0^\theta G(r, s) \frac{A_1(s) \sin(\rho t)}{\rho} d\rho ds , \tag{32}$$

$$I_4 = \int_r^{R_1} \int_0^\theta G(r, s) \frac{A_2(s)(1 - \cos(\rho t))}{\rho^2} d\rho ds . \tag{33}$$

In similar form we apply the Laplace transform to system (21)-(23) and obtain for  $E_1(r, p) = L \{ \xi_1(r, t) \}$  the equation

$$\frac{1}{r} \frac{d}{dr} \left[ r \frac{dE_1}{dr} \right] - R_p E_1 = F_r \left[ \frac{\partial F(r, p)}{\partial r} - F(r, p) \right] , \tag{34}$$

where

$$F(r, p) = \frac{pE_0(r, p)}{a St} . \tag{35}$$

The solution of (34) is

$$E_1(r, p) = Fr \int_0^{R_1} G(r, s) \left[ \frac{\partial F(s, p)}{\partial s} - F(s, p) \right] ds . \tag{36}$$

We proceeded as in the above zero order case in order to obtain  $\xi_1(r, t)$ . The results are

$$\xi_1(r, t) = \frac{2}{\pi} \sum_{j=1}^4 \sum_{i=1}^2 (A_i^j + B_i^j) , \tag{37}$$

where  $A_i^j$  and  $B_i^j$  are given in the appendix.

## IV Results and Discussion

We have obtained an analytic representation for the profile of the surge wave in the sea due to a TC at low latitude, given by

$$\xi(r, t) = \xi_0(r, t) + \varepsilon \xi_1(r, t) , \tag{38}$$

where  $\xi_0$  and  $\xi_1$  are given by (29) and (37), respectively.

As it can be seen, these expresions present diffi-

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culties to analyse the main properties of the solution. However, they allow checking the analitical dependence with respect to model input parameters. The obtained dependence shows that the expression is stable with respect to any continued variation of the corresponding parameter.

We have numerically calculated the integrals involved in (38) in order to verify the correspondence

between the obtained solution and the experimental data. The values of the parameters are the following:  $t = 0.5, 1, \dots, 4$  hours,  $R = 50$  Km,  $W_0 = 38$  m/s,  $\gamma_1 = 0.0025$ ,  $\Delta_p = 55$  mBar,  $\alpha = 10^\circ$ ,  $\rho = 1.293$  Kg/m<sup>3</sup>,  $St = 0.01$ ,  $R_0 = 0.79$ ,  $\epsilon = 0.14$  and  $Fr = 0.0001$ . We

elaborate graphs of sea excitation vs. TC radius.

From Figs. 1, 2 and 3, it can be seen the dynamics of the surge wave formation. The plots show that the solution qualitatively maintains its form during time and that it smooth.

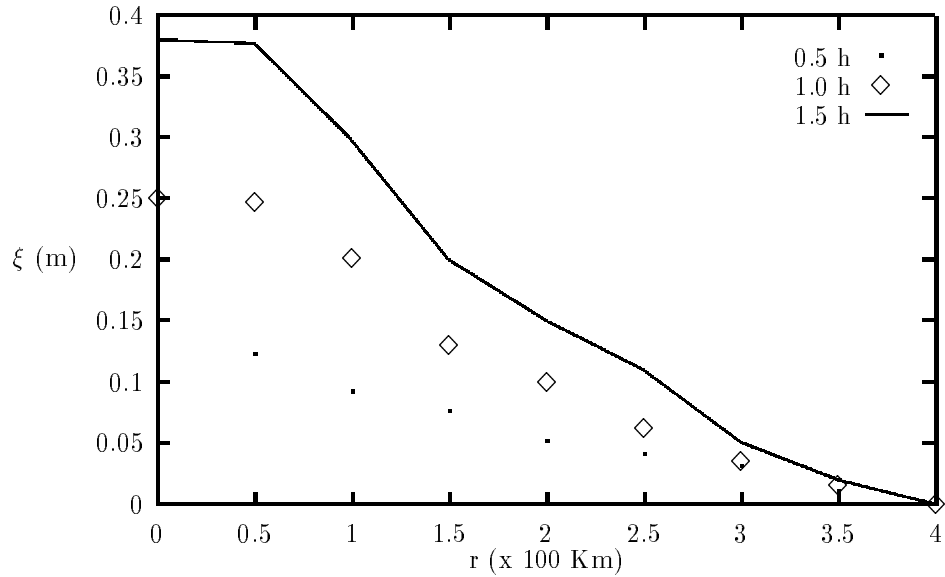


Figure 1. Profiles of the surge wave for  $t=0.5, 1.0$  and  $1.5$  hours.

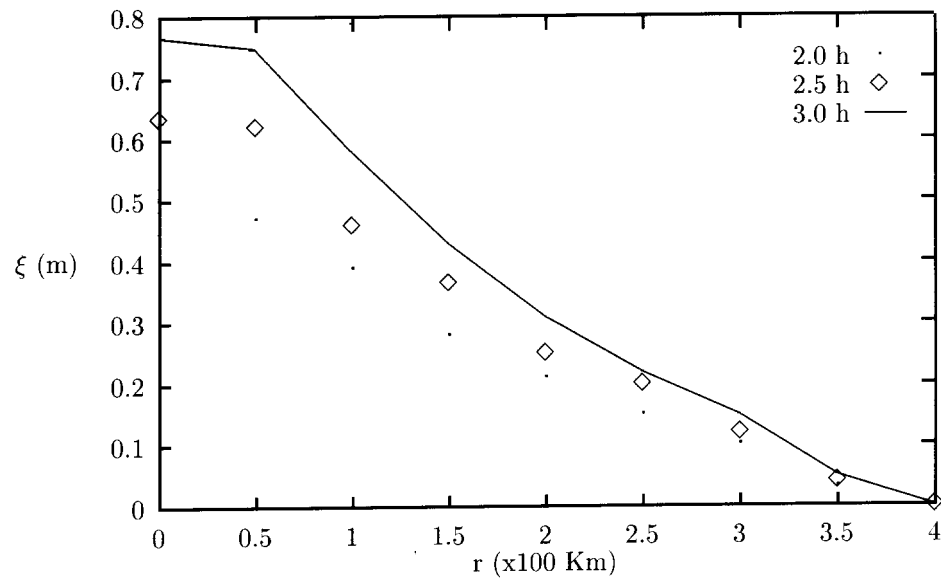


Figure 2. Profiles of the surge wave for  $t=2.0, 2.5$  and  $3.0$  hours.

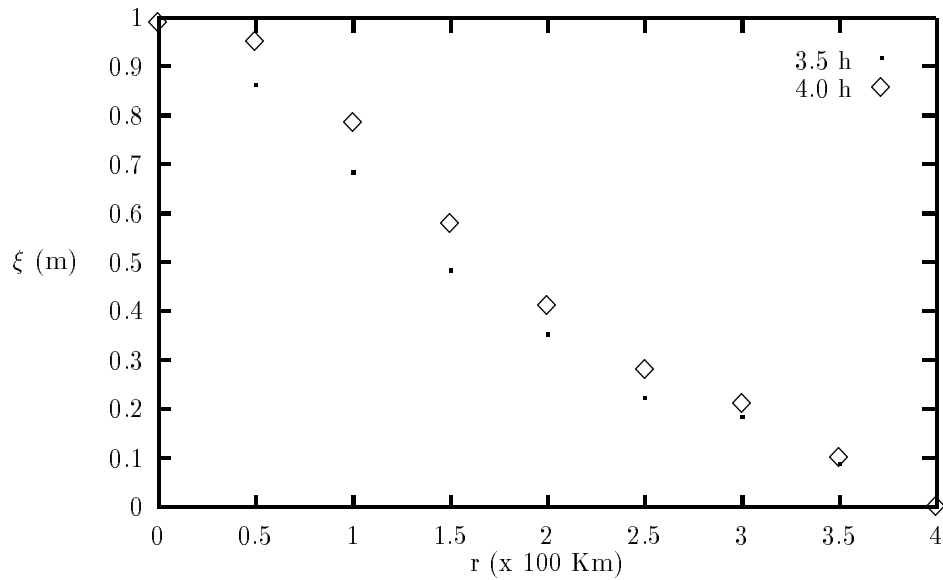


Figure 3. Profiles of the surge wave for t=3.5 and 4.0 hours.

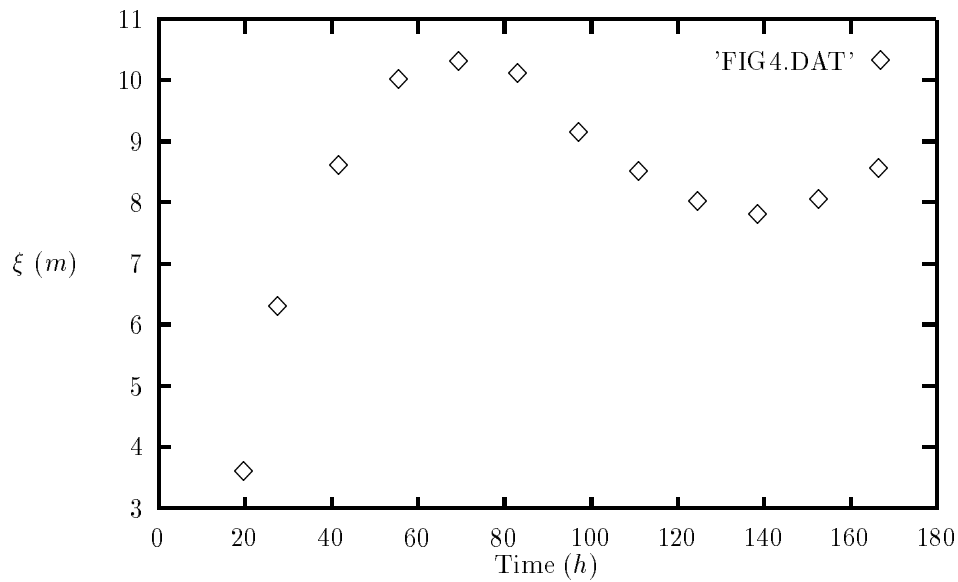


Figure 4. Evolution in time of the surface excitation in r=0.

Table 1. Contribution of the second Rossby parameter to the surface excitation in r=0 for several times.

Excitation ( $r = 0$ )	$t = 1$ h	$t = 2$ h	$t = 3$ h	$t = 4$ h
$\xi_0$ (m)	0.255	0.510	0.764	1.020
$\xi = \xi_0 + \varepsilon \xi_1$ (m)	0.257	0.512	0.766	1.023

In Fig. 4 a second experiment shows that the solution tends to a bounded oscillatory expression as the time  $t$  increases. It is necessary to say that the limit values of the excitation in this case do not correspond

to the real values, because, with the increasing of time, the validity of the quasistationary model does not hold. Nevertheless, this experiment assures that our formal solution does not diverge in time.

In Table 1, we show the result of another experiment in which it can be seen the real contribution of Rossby's second parameter, i.e. the contribution to the surface excitation, due to the horizontal streams caused by the deflection of the vertical internal streams into the sea as a result of the Earth rotation.

It can be seen that this contribution is small, due to the fact that the vertical streams are quite small in comparison with the horizontal streams induced by the wind of the TC. Moreover, the relative increase of the term in the equation containing Rossby's second parameter with respect to the term that contains Rossby's first parameter is due to the abrupt decrease of the latter. Hence, it is possible to neglect the contribution of

Rossby's streams even at low latitudes.

We have obtained an analytical expression of the excitation of the sea level in the presence of a tropical cyclone, whose results are similar to those obtained by numerical methods. The analytical solution allows us to study in detail the surge wave formation in the presence of a tropical cyclone. It is possible to see that a non-periodical wave is obtained which tends to some stationary state. It is interesting to highlight that the contribution of Rossby's second parameter is quite small even in low latitudes. This result justifies the direct use of the two-dimensional system of order zero to calculate the surge.

## Appendix

Expressions for the coefficients  $A_i^j$  and  $B_i^j$  in formula (37).

$$A_1^1 = \int_0^r \int_0^s \int_0^\theta G_1(s)G_2(r) \frac{G_2(s)}{s} G_1(y)A_1(y) \cos(\rho t) d\rho dy ds \quad (39)$$

$$A_1^2 = \int_0^r \int_0^s \int_0^\theta G_1(s)G_2(r) \frac{\partial G_2(s)}{\partial s} G_1(y)A_1(y) \cos(\rho t) d\rho dy ds \quad (40)$$

$$A_1^3 = \int_0^r \int_s^{R_1} \int_0^\theta G_2(r)G_1(s) \frac{G_1(s)}{s} G_2(y)A_1(y) \cos(\rho t) d\rho dy ds \quad (41)$$

$$A_1^4 = \int_0^r \int_s^{R_1} \int_0^\theta G_2(r)G_1(s) \frac{\partial G_1(s)}{\partial s} G_2(y)A_1(y) \cos(\rho t) d\rho dy ds \quad (42)$$

$$B_1^1 = \int_r^{R_1} \int_0^s \int_0^\theta G_1(r)G_2(s) \frac{G_2(s)}{s} G_1(y)A_1(y) \cos(\rho t) d\rho dy ds \quad (43)$$

$$B_1^2 = \int_r^{R_1} \int_0^s \int_0^\theta G_1(r)G_2(s) \frac{\partial G_2(s)}{\partial s} G_1(y)A_1(y) \cos(\rho t) d\rho dy ds \quad (44)$$

$$B_1^3 = \int_r^{R_1} \int_s^{R_1} \int_0^\theta G_1(r)G_2(s) \frac{G_1(s)}{s} G_2(y)A_1(y) \cos(\rho t) d\rho dy ds \quad (45)$$

$$B_1^4 = \int_r^{R_1} \int_s^{R_1} \int_0^\theta G_1(r)G_2(s) \frac{\partial G_1(s)}{\partial s} G_2(y)A_1(y) \cos(\rho t) d\rho dy ds \quad (46)$$

$$A_2^1 = \int_0^r \int_0^s \int_0^\theta G_1(s)G_2(r) \frac{G_2(s)}{s} G_1(y)A_2(y) \frac{\sin(\rho t)}{\rho} d\rho dy ds \quad (47)$$



$$A_2^2 = \int_0^r \int_0^s \int_0^\theta G_1(s)G_2(r) \frac{\partial G_2(s)}{\partial s} G_1(y)A_2(y) \frac{\sin(\rho t)}{\rho} d\rho dy ds \quad (48)$$

$$A_2^3 = \int_0^r \int_s^{R_1} \int_0^\theta G_1(s)G_2(r) \frac{G_1(s)}{s} G_2(y)A_2(y) \frac{\sin(\rho t)}{\rho} d\rho dy ds \quad (49)$$

$$A_2^4 = \int_0^r \int_s^{R_1} \int_0^\theta G_1(s)G_2(r)G_1(s)G_2(y)A_2(y) \frac{\sin(\rho t)}{\rho} d\rho dy ds \quad (50)$$

$$B_2^1 = \int_r^{R_1} \int_0^s \int_0^\theta G_1(r)G_2(s) \frac{G_2(s)}{s} G_1(y)A_2(y) \frac{\sin(\rho t)}{\rho} d\rho dy ds \quad (51)$$

$$B_2^2 = \int_r^{R_1} \int_0^s \int_0^\theta G_1(r)G_2(s) \frac{\partial G_2(s)}{\partial s} G_1(y)A_2(y) \frac{\sin(\rho t)}{\rho} d\rho dy ds \quad (52)$$

$$B_2^3 = \int_r^{R_1} \int_s^{R_1} \int_0^\theta G_1(r)G_2(s) \frac{G_1(s)}{s} G_2(y)A_2(y) \frac{\sin(\rho t)}{\rho} d\rho dy ds \quad (53)$$

$$B_2^4 = \int_r^{R_1} \int_s^{R_1} \int_0^\theta G_1(r)G_2(s) \frac{\partial G_1(s)}{\partial s} G_2(y)A_2(y) \frac{\sin(\rho t)}{\rho} d\rho dy ds \quad (54)$$

and

$$G_1(x) = I_0(x\sqrt{\theta^2 - \rho^2}) \quad (55)$$

$$G_2(x) = K_0(x\sqrt{\theta^2 - \rho^2}) - \frac{K_0(R_1\sqrt{\theta^2 - \rho^2})}{I_0(R_1\sqrt{\theta^2 - \rho^2})} I_0(x\sqrt{\theta^2 - \rho^2}) \quad (56)$$

$$A_1(y) = \left( H \frac{Fr^2}{aSt} \right) \left\{ \tau_0 \sin \alpha \beta(y) \left[ \frac{\beta(y)}{y} - 2 \frac{\partial \beta(y)}{\partial y} \right] + \left[ \frac{\partial^2 Pa}{\partial y^2} - \frac{1}{y} \frac{\partial Pa}{\partial y} \right] \right\} \quad (57)$$

$$A_2(y) = \left( \frac{HSt^2}{a} \right) \tau_0 \cos \alpha \beta(y) \left[ 2 \frac{\partial \beta(y)}{\partial y} - \frac{\beta(y)}{y} \right], \quad (58)$$

where  $a = 100$  and  $H = 1$ .

## Acknowledgments

We want to express our gratitude to the Department of Theoretical Physics of the University of Havana; to the colleagues of the IV Congress of Applied Mathematics in Vic, Barcelona, Spain, in September 1995; to colleagues of the University of Zaragoza and of UNED in Madrid, to Dr. P. Ripa from CICESE in Mexico, for a great interest in our work, and to Prof. A. Liñan from the Aeronautical Institute of Madrid, Spain, for fruitful discussions and comments. JMA wants to acknowledge the support of the Alma Mater grant to his research project.

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