ζ-Function Method for Repulsive Casimir Forces

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Received 04 June, 1998; Revised 16 October, 1998

We compute the Casimir pressure between an unusual pair of parallel plates, namely, a perfectly conducting plate ($\epsilon \to \infty$) and an infinitely permeable one ($\mu \to \infty$) with the generalized ζ-function method. The result for this problem, which has been rarely discussed in the literature, is a repulsive Casimir force. The ζ-function method provides a very compact and economic way of obtaining the final result.

Since Casimir's paper [1] on the attraction of two neutral parallel (perfectly) conducting plates due to the vacuum fluctuations of the electromagnetic field, a considerable amount of work has been done on this subject, varying from the application of alternative techniques to the exploration of new geometries and theories. Excellent reviews as well as a large list of relevant references on this subject can be found in the work of Plunien et al[2] and in the book by Mostepanenko and Trunov[3].

The first experimental test of Casimir's prediction was made by Sparnaay[4] only in 1958. However, the accuracy of the results was far from being reasonable: approximately one hundred percent of error. Since then, experiments involving dielectrics have been done, but only four decades after Sparnaay's work an experiment of the Casimir force between metals was repeated: in 1997, Lamoreaux[5] made an experiment with a slightly different geometry (a spherical metallic lens and a flat metal plate) where theory and the experimental data agree within a few percents. More recently, another experiment with metals was performed by U. Mohideen and A. Roy [6]. Using an atomic force microscope they measured the Casimir force between a metallized sphere of diameter equal to 196μm and a flat metal plate for separations from $0,1\mu$m to $0,9\mu$m. They showed that the experimental data are in very good agreement with the theoretical predictions. May be this is one of the most striking results of quantum field theory once it is a macroscopic (measurable) manifestation of the quantum vacuum.

Instead of paying attention to the sources (the perfect conducting plates), Casimir's approach for this problem consisted basically in computing the interacting energy between the plates as the (regularized) difference between the zero point energies with and without the boundary conditions dictated by the physical situation (perfectly conductor character of the plates). In fact, the great novelty of Casimir's paper of 1948 was not the fact that two neutral objects attracted each other[1], but the simplicity of the method to do it in the context of quantum field theory. However, since Casimir's work, many other techniques were developed which may be more appropriate depending on the physical situation under study. In particular, methods of computing effective actions are in general very powerful for our purposes.

We shall be concerned here with one of these methods, namely, the so called generalized ζ-function method. In this communication we shall apply it to the unusual case of a pair of parallel plates, where one of them is perfectly conducting ($\epsilon \to \infty$), while the other is infinitely permeable ($\mu \to \infty$). This problem was solved by T. Boyer [8] two decades ago in the context of random electrodynamics (a kind of classical electrodynamics which includes classical electromagnetic zero-point radiation) and has appeared again recently in the literature[9]. The relevance of this problem is that it seems at first sight to contradict the heuristic interpre-

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[1] Recall that Van der Waals dissertation was presented in 1873 (although it was published only almost two decades after that) and that London's [7] explanation for the non-retarded long range dispersive forces was published in 1930.

tation for the direction of the Casimir force: naively, the Casimir force between two parallel plates should point always inwards independently of the nature of the plates, since there would be more field modes outside the plates exerting pressure than inside (for the usual case, a computation of the Casimir pressure based on this argument can be found in the excellent book by Milonni[10]). Things are not exactly like this (see ref.[9] for the details). In this paper we clarify those issues keeping our formalism as simple as possible. In this way we consider the geometry of parallel plates and the fields at zero temperature; a much more involved calculation can be done at finite temperature [11].

In order to apply the generalized ζ-function method, let us introduce the (pure electromagnetic) vacuum persistence amplitude $\langle 0_+|0_-\rangle$ in the presence of external agents, as for example external fields, external sources, boundary conditions, etc..

In the case at hand, we are interested only in the energy shift in the electromagnetic vacuum induced by a pair of parallel plates whose distance from each other is d, but there are no other external sources or external fields. Hence, we can write

$$< 0_+|0_- >= e^{iW^{(1)}(d)}, \qquad (1)$$

where the one-loop effective action $W^{(1)}(d)$ can be computed as if the electromagnetic field were a massless scalar field. All we have to do is to put by hand an extra factor 2 to take into account for the two possible polarizations of the electromagnetic field modes (electromagnetic results are not generally just twice those for a massless scalar field, but for the particular case of plane geometry as it is the case at hand this can be done).

Therefore, in a euclidian spacetime we have

$$\begin{aligned} W^{(1)}(d) &= \ln \int_{F_d} D\phi e^{-\frac{1}{2}\int d\tau d\vec{x}\, \phi(-\Box_E)\phi} \\ &= \left(-\frac{1}{2}\right) \ln \det(-\Box_E|F_d), \end{aligned} \qquad (2)$$

where the symbol F_d means a set of functions which satisfy some boundary conditions on the plates.

The electromagnetic vacuum energy (the Casimir energy) is then given by

$$\mathcal{E}(d) = -(2)\frac{W^{(1)}}{T_E}, \qquad (3)$$

where $T_E = \int d\tau$ must be taken as infinite after the calculations are made and we have already included the polarization factor 2 mentioned before.

The generalized ζ-function method, introduced two decades ago[12], consists basically in the following three steps: **(i)** first, we compute the eigenvalues of $-\Box_E$ and write $\zeta(s; -\Box_E) = Tr(-\Box_E)^{-s}$; **(ii)** second, we make an analytical extension of $\zeta(s; -\Box_E)$ to a meromorphic function of the whole complex s-plane; **(iii)** finally, we compute $\det(-\Box_E|F_d) = \exp\left\{-\frac{\partial\zeta}{\partial s}(s=0; -\Box_E)\right\}$. Combining the previous equations, we get

$$\mathcal{E}(d) = -\frac{\zeta'(s=0; -\Box_E)}{T_E}. \qquad (4)$$

Choosing the cartesian axes such that the axis OZ is perpendicular to both plates with the perfectly conducting plate at $z=0$ and the infinitely permeable one at $z=d$, the boundary conditions are the following: tangential components of the electric field as well as the normal component of the magnetic field must vanish at $z=0$, while the tangential components of the magnetic field must vanish at $z=d$. It can be shown that the allowed frequencies for the electromagnetic vacuum field modes between these plates can be simulated by a massless scalar field which satisfies mixed boundary conditions (provided we do not forget an extra factor of 2 due to the two possible photon polarizations). Hence, we must find out the eigenvalues of $-\Box_E$ whose eigenfunctions $\phi(\tau, \vec{x})$ satisfy:

$$\phi(\tau, x, y, z=0) = 0 \quad ; \quad \frac{\partial}{\partial z}\phi(\tau, x, y, z=d) = 0. \qquad (5)$$

This leads to the following eigenvalues

$$\left\{ k_0^2 + k_1^2 + k_2^2 + (n+\frac{1}{2})^2\frac{\pi^2}{d^2} | k_0,\ k_1,\ k_2\, \epsilon R;\ n = 0, 1, 2, ... \right\} \qquad (6)$$

The ζ-function then reads

$$\begin{aligned}\zeta(s;-\Box_E) &= T_E L^2 \sum_{n=0}^{\infty} \int \frac{dk_0 dk_1 dk_2}{(2\pi)^3} \left[k_0^2 + k_1^2 + k_2^2 + (n+\frac{1}{2})^2 \frac{\pi^2}{d^2}\right]^{-s} \\ &= \frac{T_E L^2}{2\pi^2} \sum_{n=odd} \int K^2 dK \left[K^2 + \left(\frac{n\pi}{2d}\right)^2\right]^{-s}, \end{aligned} \tag{7}$$

where L^2 is the area of the plates, $K^2 = k_0^2 + k_1^2 + k_2^2$ and the angular integration was already made.

Using the following integral representation for the Euler beta function

$$\int_0^{\infty} dx\, x^{\mu-1}(x^2+a^2)^{\nu-1} = \frac{1}{2} a^{\mu+2\nu-2} B\left(\frac{\mu}{2}, 1-\nu-\frac{\mu}{2}\right), \tag{8}$$

which is valid for $Re(\nu+\mu/2) < 1$ and $Re\mu > 0$, where $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, we get

$$\zeta(s;-\Box_E) = \frac{T_E L^2}{4\pi^2} \frac{,\,(3/2),\,(s-3/2)}{,\,(s)} \left(\frac{\pi}{2d}\right)^{3-2s} \sum_{n=odd} \frac{1}{n^{2s-3}}. \tag{9}$$

In order to connect the summatin on the r.h.s. of the above equation to the Riemann ζ-function, we use the following trick: we sum and subtract the same expression with even n, namely

$$\begin{aligned}\sum_{n=odd} \frac{1}{n^z} &= \sum_{n=1}^{\infty} \frac{1}{n^z} - \sum_{n=even} \frac{1}{n^z} \\ &= \left(1 - \frac{1}{2^z}\right) \zeta_R(z). \end{aligned} \tag{10}$$

The three series that appear in (10) are absolutely convergent in the domain $Re\ z > 1$ and so the trick we used leads to a series that is rigorously identical with the original sum on the odd integers. Substituting this result into (9) and using that $,\,(3/2) = \sqrt{\pi}$, we have

$$\zeta(s;-\Box_E) = \frac{T_E L^2}{8\pi^{3/2}} \frac{,\,(s-\frac{3}{2})}{,\,(s)} \left(\frac{\pi}{2d}\right)^{3-2s} \left(1 - \frac{2^3}{2^{2s}}\right) \zeta_R(2s-3). \tag{11}$$

Recalling that $,\,(s) \sim 1/s$ for small s and having in mind that an analytic extension is tacitly assumed for $\zeta(s,\Box_E)$, it is straightforward to show that

$$\begin{aligned}\zeta'(s=0;-\Box_E) &= \frac{T_E L^2 \sqrt{\pi}}{8\pi^2},\,(-3/2) \left(\frac{\pi}{2d}\right)^3 (-7)\zeta_R(-3) \\ &= \left(-\frac{7}{8}\right) \frac{T_E L^2 \pi^2}{720} \left(\frac{1}{d^3}\right), \end{aligned} \tag{12}$$

where we used that $,\,(-3/2) = 4/3\sqrt{\pi}$ and $\zeta_R(-3) = 1/120$.

From equations (4) and (12), the Casimir energy per unit area is simply given by

$$\frac{\mathcal{E}(d)}{L^2} = \left(\frac{7}{8}\right) \frac{\pi^2}{720} \frac{1}{d^3}. \tag{13}$$

Bringing back the universal constants $\hbar$ and c, the Casimir pressure is then given by

$$\frac{\mathcal{F}(d)}{L^2} = -\frac{\partial}{\partial d}\left(\frac{\mathcal{E}(d)}{L^2}\right) = \left(+\frac{7}{8}\right) \frac{\pi^2 \hbar c}{240} \left(\frac{1}{d^4}\right), \tag{14}$$

Observe that the above result is $-\frac{7}{8}$ times the standard result (two perfectly conducting plates), which means that for the case at hand the Casimir pressure between the plates is **repulsive** instead of **attractive** (although slightly weaker) in perfect agreement with references[8, 9]. In other words, the generalized ζ-function method supports the (very) few methods that have been applied to this unusual problem. Besides, this method has shown to be once more a very economic regularization prescription for this kind of problem. As a final remark, we would like to stress that this kind of unusual boundary conditions (plates with different nature) should be used in more interesting problems as for instance in the so called dynamical Casimir effect, where real photons can be created due to the movement of the boundaries.

The authors would like to thank A. Tort, F. Santos and H. Boschi-Filho for helpful discussions and CNPq (The National Research Council of Brazil) for a partial financial support.

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