Non-Markovian Processes with Long-Range Correlations: Fractal Dimension Analysis

Manuel O. Cáceres*

Centro Atómico Bariloche and Instituto Balseiro

CNEA, and Universidad Nacional de Cuyo

Av. Ezequiel Bustillo Km 9.5

8400 San Carlos de Bariloche, Río Negro, Argentina

A particular class of strong non-Markovian stochastic processes have been studied by using a characteristic functional technique previously reported. Exact results for all moments and the whole Kolmogorov hierarchy are presented. The asymptotic scaling of the non-Markovian stochastic process has been characterized in terms of the long-range correlated noise appearing in the corresponding stochastic differential equation. A generalized Wiener process has therefore been completely characterized, its power spectrum and fractal dimensions have been studied and its possible connection with the $q-$statistics has been pointed out.

I Introduction

For a non-Markovian stochastic process (**s.p.**) $\mathbf{X}(t)$, it is well known that its complete characterization demands the knowledge of the whole Kolmogorov hierarchy, i.e.: the $m-$time joint probability distribution $P[X(t_1); X(t_2); \cdots; X(t_m)]$ for all m, or equivalently all the $m-$time moments $\langle X(t_1)X(t_2) \cdots X(t_m) \rangle$ or cumulants $\langle\langle X(t_1)X(t_2) \cdots X(t_m) \rangle\rangle$. Only when partial knowledge of the **s.p.** is required the 1-time probability distribution $P[X(t_1)]$ is enough, this is the case when $1-$time moments of the process $\langle X(t)^m \rangle$ are needed [1, 2, 3, 4]. This fact can easily be visualized using the functional calculus, i.e.: knowing $P[X(t_1)]$ is equivalent to the knowledge of the characteristic function $\langle \exp ikX(t_1) \rangle$. But in order to know the whole Kolmogorov hierarchy the knowledge of the characteristic functional $G_X([k(t)]) = \langle \exp \int ik(t)X(t)dt \rangle$ is required, which of course is a much more complex object [5]. The notation $G_X([k(t)])$ emphasizes that G

depends on the whole test function $k(t)$, not just on the value it takes at one particular time t_j. The convergence of the integral is accomplished because the functions $k(t)$ may be restricted to those that vanish for sufficiently large t. On the other hand the possibility of having a closed expression for the characteristic functional allows us to find the Kolmogorov hierarchy by quadrature. Very recently[6] a generalized Wiener process $\mathbf{X}(t)$—with natural boundary conditions—has been defined and its complete characterization has been given in terms of the arbitrary characteristic functional of the noise $G_\xi([k(t)])$.

In the present paper we are concerned with the asymptotic analysis of the non-Markovian effects—on $\mathbf{X}(t)$—induced by a long-range correlated noise, i.e.: when the **s.p.** $\mathbf{X}(t)$ is defined through the equation: $\dot{X} = \xi(t)$ being $\xi(t)$ a Gaussian non-white noise. Here we will characterize a stationary *short-range* noise correlation by an exponential function

---

*Also Investigador Independiente en el CONICET. Electronic mail: caceres@cab.cnea.edu.ar

$$\langle\langle \xi(t_1)\xi(t_2)\rangle\rangle = \frac{1}{2\theta}\exp\left(-\mid t_1 - t_2 \mid /\theta\right) \qquad , \qquad \theta \geq 0 \tag{1}$$

On the contrary a stationary *long-range* noise correlation will be characterize by a power-law function

$$\langle\langle \xi(t_1)\xi(t_2)\rangle\rangle = \frac{\tau^{-1}}{(1+\mid t_1 - t_2 \mid /\tau)^{\mu}} \qquad , \qquad \tau > 0 \; ; \; \mu \geq 0 \tag{2}$$

We remark that a *white-noise* force can be reobtained from (1) in the limit $\theta \to 0$; while a *constant force* can be obtained from (2) in the limit $\mu \to 0$. In order to study different scaling regimes, in this paper we will only be interested in τ and θ finites.

II   A strong non-Markovian process

The equation of motion of a 1-dimensional Brownian particle in a generalized (infinite) medium has a

Wiener-like stochastic differential equation (SDE)

$$\frac{dX}{dt} = \xi(t) \qquad , \qquad X \in (-\infty,\infty) \tag{3}$$

where $\xi(t) \in Re$ is an arbitrary time-dependent random force characterizing the medium (*the noise*). When $\xi(t)$ is a zero-mean Gaussian white noise, i.e.: $\langle\langle \xi(s_1)\xi(s_2)\rangle\rangle = \delta(s_1-s_2)$, Eq.(3) is the usual Wiener's SDE. In this case the **s.p.** $\mathbf{X}(t)$ characterizes the position of a Brownian particle[5]. Let us now generalize the noise $\xi(t)$, with $t \in [0,\infty)$, to be a zero-mean Gaussian noise characterized by some short or long-range correlation. Therefore its functional will be

$$G_\xi([k(t)]) = \exp\left(\frac{-,^2}{2}\int_0^\infty\int_0^\infty k(s_1)k(s_2)\;\langle\langle\xi(s_1)\xi(s_2)\rangle\rangle\;ds_1\;ds_2\right)$$

where the correlation function $\langle\langle\xi(s_1)\xi(s_2)\rangle\rangle$ is given by (1) or (2) and $,^2$ is proportional to the intensity of the noise. Non-Gaussian statistics can also be worked out in a similar manner [6].

Due to the fact that the **s.p.** $\mathbf{X}(t)$ is non-Markovian, there is not a *simple* partial differential equation available for its $1-$time probability distribution, and the problem is even more complicated if we want to know the $2-$times joint probability distribution[7], for example in order to calculate correlation functions. We remark that in this paper we do not make use of any partial differential equation, this is why our characteristic functional approach is of value.

By using proposition 3 of reference[6], it follows that the characteristic functional of the **s.p.** $\mathbf{X}(t)$ is

$$G_X([Z(t)]) = e^{+ik_0 X_o}\;G_\xi\left([\int_t^\infty Z(s)ds]\right)$$

$$= e^{+ik_0 X_o}\exp\left[\frac{-\Gamma_2}{2}\int_0^\infty\int_0^\infty\left(\int_{s_1}^\infty Z(s)ds\right)\left(\int_{s_2}^\infty Z(s')ds'\right)\langle\langle\xi(s_1)\xi(s_2)\rangle\rangle\,ds_1 ds_2\right] \tag{4}$$

here X_0 is the initial condition and k_0 is a functional of $Z(t)$ given by

$$k_0 = \int_0^\infty Z(s)ds \tag{5}$$

Therefore all $m-$time moments of the **s.p.** $\mathbf{X}(t)$ follow from $m-th$ order functional differentiation, i.e.:

$$\langle X(t_1)X(t_2)\cdots X(t_m)\rangle = i^{-m}\frac{\delta}{\delta Z(t_1)}\cdots\frac{\delta}{\delta Z(t_m)}G_X([Z(t)])\Big|_{Z=0} \tag{6}$$

see appendix for the explicit calculation of the second cumulant of $\mathbf{X}(t)$. On the other hand, the Kolmogorov hierarchy can immediately be obtained by quadrature from the functional (4). In general we can invert the characteristic functional by introducing the $n-$dimensional Fourier transform

$$P\left(X[t_1]; X[t_2]; \cdots ; X[t_n]\right) \equiv P\left(x_1, t_1; x_2, t_2; \cdots ; x_n, t_n\right)$$

$$= \tfrac{1}{(2\pi)^n} \int \cdots \int dk_1 \cdots dk_n \exp\left(-i \sum_{i=1}^{n} k_i \; x_i\right) \left[G_X\left([Z(t)]\right)\right]_{Z(t) = k_1 \delta(t - t_1) + \cdots + k_n \delta(t - t_n)} \tag{7}$$

We remark that (4) is an *exact result* which allows us to get a complete characterization of the non-Markovian **s.p.** $\mathbf{X}(t)$. For example the $1-$time probability distribution $P\left(X[t_1]\right)$ is given in terms of $G_X\left([Z(t)]\right)$ evaluated with the test function $Z(t) = k_1 \delta(t - t_1)$. From (4) and (7) we see that, to know the $1-$time probability distribution, we must calculate the integral

$$\sigma(t_1) \equiv \frac{,\,^2}{2} \int_0^{t_1} \int_0^{t_1} \langle\langle \xi(s_1)\xi(s_2) \rangle\rangle \, ds_1 ds_2 \tag{8}$$

A. Short-range noise correlation

Let us exemplify the use of the short-range correlation (1). In this case (8) gives

$$\frac{2}{,\,^2}\sigma_{short}(t) \equiv t - \theta\left(1 - e^{-t/\theta}\right) \tag{9}$$

From this expression it is simple to see that due to the non-Markovian effect $(\theta \neq 0)$ the $1-$time characteristic function $G_X(k, t) \equiv \langle \exp ik X(t) \rangle$ does not have the familiar Wiener's scaling:

$$G_X(\frac{k}{\sqrt{\Lambda}}, \Lambda t) = G_X(k, t) \tag{10}$$

Which would imply in the **s.p.** $\mathbf{X}(t)$ the scaling

$$\frac{1}{\sqrt{\Lambda}}\mathbf{X}(\Lambda t) = \mathbf{X}(t) \tag{11}$$

Only asymptotically for long-times, from (9) the following relation is held

$$\sigma_{short}(\Lambda t) \to \Lambda \; \sigma_{short}(t) \quad , \quad t \gg \theta \tag{12}$$

Then we can conclude that a *short-range* correlated noise does not affect—in the long time regime—the universal Wiener scaling (11) [i.e.: only induces *weak* non-Markovian effects on $\mathbf{X}(t)$].

B. Long-range noise correlation

From now on we will only be interested in *strong* non-Markovian effects. Thus, let us now use the long-range correlation (2). In this case (8) gives for $0 \le \mu \neq 1, 2$

$$\frac{2}{,\,^2}\sigma_{long}(t) = \frac{2t}{(\mu - 1)} + \frac{2\tau^{\mu-1}(t + \tau)^{2-\mu}}{(\mu - 1)(\mu - 2)} - \frac{2\tau}{(\mu - 1)(\mu - 2)} \tag{13}$$

For $\mu = 1$ we get

$$\frac{2}{,\,^2} \sigma_{long}(t)\big|_{\mu=1} = 2\tau + 2(t + \tau)\left(\log\left(1 + \frac{t}{\tau}\right) - 1\right) , \tag{14}$$

and for $\mu = 2$ reads

$$\frac{2}{,\,^2} \sigma_{long}(t)\big|_{\mu=2} = 2t - 2\tau \log\left(1 + \frac{t}{\tau}\right) \tag{15}$$

If $0 \le \mu \le 1$ from these expressions it is sim-

ple to see that due to the long-range effect of the noise, the $1-$time characteristic function $G_X(k_1, t_1) \equiv \langle \exp ik X(t_1) \rangle$ does not have, even at long times, Wiener's scaling. If the noise power-law parameter μ

belongs to the values $\mu \in [0,1)$, we see from (13) that a different long-time asymptotic scaling is obtained: $\sigma_{long}(\Lambda t) \rightarrow \Lambda^{2-\mu} \sigma_{long}(t)$. Then from (4) the 1−time characteristic function fulfills the asymptotic long-time scaling (with $X_0 = 0$)

$$G_X(\frac{k}{\sqrt{\Lambda^{2-\mu}}}, \Lambda t) \longrightarrow G_X(k,t) \quad , \quad t \gg \tau \qquad (16)$$

which implies in the **s.p.** $\mathbf{X}(t)$ the (anomalous) super-diffusion asymptotic scaling

$$\frac{1}{\sqrt{\Lambda^{2-\mu}}}\mathbf{X}(\Lambda t) \rightarrow \mathbf{X}(t) \quad , \quad \mu \in [0,1) \qquad (17)$$

Then we can conclude that a long-range correlated noise induces a *strong* non-Markovian effect which changes the long-time asymptotic scaling of **s.p.** $\mathbf{X}(t)$. If $\mu = 1$ there are logarithmic corrections, only if $\mu > 1$

Wiener's scaling is reobtained in the asymptotic long-time regime.

C. Spectral of the random fluctuations of the s.p. $\mathbf{X}(t)$

From (17) let us define the parameter $H = 1 - \frac{\mu}{2}$. Then, after the transient, in the asymptotic long-time regime the **s.p.** $\mathbf{X}(t)$ fulfills the scaling $\mathbf{X}(\Lambda t) \rightarrow \Lambda^H \mathbf{X}(t)$. Note that from our non-Markovian generalized Wiener process, the parameter H is bounded to the values $H \in (\frac{1}{2}, 1]$. From this relation it is possible to estimate the power spectrum of the **s.p.** $\mathbf{X}(t)$ in a simple way[8]. In order to calculate the spectral representation of the random fluctuations let us fix some Λ and define a **s.p.** $\mathbf{Y}(t)$ in the following way:

$$\mathbf{Y}(t,T) = \begin{cases} \mathbf{Y}(t) = \Lambda^{-H} \mathbf{X}(\Lambda t) & if \quad 0 < t < T \\ 0 & otherwise \end{cases} \qquad (18)$$

Adopting the Fourier notations

$$
\begin{array}{ll}
F_X(f,T), \ F_Y(f,T) & \text{Fourier transform of } \mathbf{X}(t,T) \text{ and } \mathbf{Y}(t,T) \\
S_X(f,T), \ S_Y(f,T) & \text{Spectral densities of } \mathbf{X}(t,T) \text{ and } \mathbf{Y}(t,T) \\
S_X(f), \ S_Y(f) & \text{Spectral densities of } \mathbf{X}(t) \text{ and } \mathbf{Y}(t)
\end{array} \qquad (19)
$$

for example $F_Y(f,T) = \int_{-\infty}^{\infty} \mathbf{Y}(t,T) \exp(-2\pi i f t) dt = \int_0^T \mathbf{Y}(t) \exp(-2\pi i f t) dt$ and $S_Y(f,T) = \frac{1}{T} \mid F_Y(f,T) \mid^2$, etc. Let us now compute $F_Y(f,T)$ using its scaling relation (18)

$$F_Y(f,T) = \int_0^T \mathbf{Y}(t) \exp(-2\pi i f t) dt = \frac{1}{\Lambda} \int_0^T \Lambda^{-H} \mathbf{X}(\Lambda t) \ \exp(\frac{-2\pi i f \Lambda t}{\Lambda}) \Lambda dt$$

$$= \int_0^{T\Lambda} \Lambda^{-H-1} \mathbf{X}(t') \ \exp(\frac{-2\pi i f t'}{\Lambda}) \ dt' \qquad (20)$$

Then it follows

$$F_Y(f,T) = \frac{1}{\Lambda^{H+1}} F_X(\frac{f}{\Lambda}, T\Lambda) \qquad , \qquad (21)$$

so the spectral density is given by

$$S_Y(f,T) = \frac{1}{T} \mid F_Y(f,T) \mid^2 = \frac{1}{T} \left(\frac{1}{\Lambda^{H+1}}\right)^2 \mid F_X(\frac{f}{\Lambda}, T\Lambda) \mid^2 = \frac{1}{\Lambda^{2H+1}} S_X(\frac{f}{\Lambda}, T\Lambda) \qquad (22)$$

and in the limit $T \rightarrow \infty$ we get

$$S_Y(f) = \frac{1}{\Lambda^{2H+1}} S_X(\frac{f}{\Lambda}) \qquad (23)$$

Now because $\mathbf{Y}(t)$ is nothing more than a properly rescaled version of $\mathbf{X}(t)$, their spectral densities must coincide. Take for example $\Lambda \equiv 1$, so $S_Y(f) = S_X(f)$.

Hence, it follows the relation

$$S_X(f) = \frac{1}{\Lambda^{2H+1}} S_X(\frac{f}{\Lambda}) \qquad (24)$$

Now, formally putting $f = 1$ and replacing $1/\Lambda$ by f in

(24) the desire result is obtained

$$S_X(f) \propto \frac{1}{f^{2H+1}} \qquad (25)$$

This expression is the spectral representation of the random fluctuations of a **s.p.** having the scaling property: $\Lambda^{-H} \mathbf{X}(\Lambda t) = \mathbf{X}(t)$.

Our non-Markovian **s.p.** fulfills in the long-time regime the scaling (17), then it follows that its corresponding spectral behaves asymptotically like

$$S_X(f) \propto \frac{1}{f^{3-\mu}} \qquad , \qquad \mu \in [0, 1) \qquad (26)$$

Thus, only approaching the limit $\mu \to 1$ the fluctuations of our **s.p.** $\mathbf{X}(t)$ approaches Brownian's noise spectral $\to 1/f^2$. On the other hand, after the transient, and in the asymptotic long-time regime the non-Markovian **s.p.** $\mathbf{X}(t)$ gets Wiener's spectral only if $\mu > 1$, see (13)-(15).

D. Fractal dimension of the record $\mathbf{X}(t)$

The $1-$time probability distribution $P\left(X[t]\right)$ of our non-Markovian generalized Wiener process can be obtained from (4) and (7), leading to the exact result

$$P\left(x, t\right) = \left(4\pi\sigma(t)\right)^{-1/2} \exp\left[-\left(x - X_0\right)^2 / 4\sigma(t)\right] \quad (27)$$

where $\sigma(t)$ is given in (9) for a short-range model, and by (13)-(15) for a long-range correlated model [*strong* non-Markovian case]. In Eq. (27) X_0 is the initial

condition of the **s.p.** $\mathbf{X}(t)$, so to be precise $P\left(x, t\right)$ is in fact a conditional probability distribution. In general due to the non-Markovian effects this distribution does not satisfy—at short times—a scaling relation, only in its asymptotic long-time regime a scaling law is reached, which in fact is analogous to the scaling relation of self-similar objects. However there is an important difference, the distribution (27) with $\sigma(t)$ given by (13) [for the *weak* non-Markovian case use (12)] has an asymptotic scaling with different ratios in time and in position. Because of this fact, it is very important to take into account the concept of self-affinity[9]. In fact the probability distribution of our **s.p.** $\mathbf{X}(t)$ is related (asymptotically) to the concept of *statistically self-affine* objects[9].

The fractal dimension of self-affine fractal objects is not uniquely defined, but fortunately the *box dimension* can be evaluated "mechanically" for a set of points such as the record of our strong non-Markovian generalized Wiener **s.p.** $\mathbf{X}(t)$. If $\mathcal{N}(\Lambda, a, t)$ is the number of "boxes" of width (in time) Λt and length (in space) Λa needed to cover the record, and if this number scale with Λ in the form $\mathcal{N}(\Lambda, a, t) \sim \Lambda^{-D_B}$ we say that D_B is the box dimension of the record. To calculate this fractal dimension we closely follow Feder's arguments[9]. Let the time-span of the record be $\mathcal{T}$, then we need $\mathcal{T}/\Lambda t$ segments of length Λt to cover the time axis. Now, let the scaling be $\mathbf{X}(\Lambda t) = \Lambda^H \mathbf{X}(t)$, then it follows that

$$\Delta\mathbf{X}(\Lambda t) \equiv \mathbf{X}(\Lambda t) - \mathbf{X}(0) = \Lambda^H \left(\mathbf{X}(t) - \mathbf{X}(0)\right) \equiv \Lambda^H \, \Delta\mathbf{X}(t)$$

thus in each segment the range of the record is of the order $\Delta\mathbf{X}(\Lambda t) = \Lambda^H \, \Delta\mathbf{X}(t)$ and we need a stack of $\Lambda^H \, \Delta\mathbf{X}(t)/\Lambda a$ boxes of height Λa to cover that range. Therefore the number of boxes to cover the set is of the order

$$\mathcal{N}(\Lambda, a, t) = \frac{\Lambda^H \, \Delta\mathbf{X}(t)}{\Lambda a} \times \frac{\mathcal{T}}{\Lambda t} \sim \Lambda^{H-2}$$

which leads to the (local) box counter fractal dimension[9] $D_B = 2 - H$. Now, from the asymptotic

scaling (17) and the definition $H = 1 - \frac{\mu}{2}$ we get, for our strong non-Markovian **s.p.** the box dimension

$$D_B = 1 + \frac{\mu}{2} \qquad , \qquad \mu \in [0, 1) \qquad (28)$$

Note that in this argument we have used boxes that were small with respect to both the length of the record $\mathcal{T}$ and the range of the record, thus the relation $D_B = 1 + \frac{\mu}{2}$ holds in high-resolution, so this is a *local* fractal dimension[1]. In the case $\mu = 0$ we obtain a ballis-

---

[1] It should be pointed out that in Feder's argument Λ is supposed to be a small parameter. So in order to apply this argument to

tic behavior (highly persistent record $D_B = 1$). On the contrary, in the limit $\mu \to 1$ the **s.p.** $\mathbf{X}(t)$ approaches Wiener's box counter fractal dimension $D_B \to \frac{3}{2}$

Owing to this analysis we can conclude that in the asymptotic long-time regime, $t \gg \tau$, our non-Markovian **s.p.** can be mapped to the fractional Brownian motion[10] (fBm) with $H \in (\frac{1}{2}, 1]$, in according with the *persistent* behavior of the fBm. Here persistence means that if the motion has been in an increasing

direction during a period t, then it is expected to continue to increase for a similar period of time. This fact is in total agreement with the idea of a strong correlated stochastic force $\xi(t)$ appearing in the SDE $\dot{X}(t) = \xi(t)$, see also the appendix for the exact calculation of the correlation $\langle\langle X(t_1) X(t_2)\rangle\rangle$.

To be more precise let us here calculate a normalized correlation function of future increments $[\mathbf{X}(t) - \mathbf{X}(0)]$ with past increments $[\mathbf{X}(0) - \mathbf{X}(-t)]$

$$
\begin{aligned}
C(t) &\equiv \frac{1}{\langle\mathbf{X}(t)^2\rangle} \langle[\mathbf{X}(0) - \mathbf{X}(-t)] [\mathbf{X}(t) - \mathbf{X}(0)]\rangle \\
&= \frac{1}{\langle\mathbf{X}(t)^2\rangle} \left(\langle\mathbf{X}(0)\mathbf{X}(t)\rangle - \langle\mathbf{X}(0)^2\rangle - \langle\mathbf{X}(-t)\mathbf{X}(t)\rangle + \langle\mathbf{X}(-t)\mathbf{X}(0)\rangle\right)
\end{aligned}
\tag{29}
$$

From (46) it is possible to show that (for $0 < \mu \neq \{1, 2\}$)

$$
C(t) = \frac{\tau + \tau^{\mu-1} (\tau + 2t)^{2-\mu}}{2(\mu - 2)t - 2\tau + 2\tau^{\mu-1} (\tau + t)^{2-\mu}}
\tag{30}
$$

Therefore if $0 \leq \mu < 1$, past increments are correlated with future increments, i.e.: a long-range correlated noise with $\mu \in [0, 1)$ induces infinitely long-run correlations in the **s.p.** $\mathbf{X}(t)$ like in the persistent fBm (anomalous super-diffusion). On the contrary if $\mu > 1$

the normalized correlation function $C(t)$ goes to zero in the limit $t \to \infty$, in agreement with a Wiener-like behavior.

It is also possible to see, using that the noise $\xi(t)$ is symmetric and adopting the initial condition $\mathbf{X}(0) = X_0 = 0$ in (42), that the variance of an arbitrary increment of **s.p.** $\mathbf{X}(t)$ is given by [for $0 < \mu \neq \{1, 2\}$ and assuming $t_1 \leq t_2$]

$$
\left\langle [\mathbf{X}(t_2) - \mathbf{X}(t_1)]^2 \right\rangle = \frac{2, 2}{(\mu - 1)(\mu - 2)} \left[(t_2 - t_1)(\mu - 2) - \tau + \tau^{\mu-1}(\tau + t_2 - t_1)^{2-\mu}\right]
\tag{31}
$$

Thus, for large $t_2 - t_1$ such that $t_2 - t_1 \gg \tau$ and if the noise parameter $\mu \in [0, 1)$, we see that $\left\langle [\mathbf{X}(t_2) - \mathbf{X}(t_1)]^2 \right\rangle$ increases with time as $\sim (t_2 - t_1)^{2-\mu}$ in agreement with the picture of a persistent fBm, i.e.: a long-range correlated noise $\xi(t)$ induces an anomalous power-law behavior in the variance of the increments of our **s.p.** $\mathbf{X}(t)$. If the noise parameter is $\mu > 1$ we reobtain—in the asymptotic long-time regime—the Wiener result $\left\langle [\mathbf{X}(t_2) - \mathbf{X}(t_1)]^2 \right\rangle \sim |t_2 - t_1|$.

It is important to remark that our analytical approach can also be extended to the case when the noise $\xi(t)$ is also non-Gaussian and non-white. The interplay between both effects is an interesting phenomena to be studied (non-Gaussian vs. strong non-Markovian effects), which is currently under investigation. Note that the same type of *long-run* fluctuations are also expected for the generalized Orstein-Uhlenbeck process—in presence of a strong correlated noise—this case can also be worked out in a similar way[6, 11].

---

our non-Markovian **s.p.** we must assume that we have allowed the **s.p.** to run for a long time so that any transient has disappeared, $t \gg \tau$. Therefore the portion of the record $\mathbf{X}(t)$ (time-span of the record $\mathcal{T}$) that we want to measure is in fact in the long-time regime, so it fulfills the asymptotic scaling (17).

E. Fractal divider dimension of $\mathbf{X}(t)$

Another fractal dimension that can be evaluated "mechanically" is the *divider dimension* along a curve to measure its length. For self-similar fractal curves such as coastlines this fractal dimension can be esti- mated from the behavior of its total length $L \sim \delta^{1-D}$, where δ is the "length" of the rule[9]. The measured length, of a random walk path in the $x, t-$plane, with a rule of length δ, placed such that it covers a time step Λt gives a contribution to the length

$$\delta = \sqrt{(\Lambda t)^2 + \left(\frac{\Delta \mathbf{X}(\Lambda t)}{a}\right)^2} = \sqrt{(\Lambda t)^2 + \Lambda^{2H}\left(\frac{\Delta \mathbf{X}(t)}{a}\right)^2} \qquad , \tag{32}$$

where the last equality was written by virtue of the sta- tistical self-similar scaling $\Delta \mathbf{X}(\Lambda t) = \Lambda^H \Delta \mathbf{X}(t)$. Here, as before, a measures the scale in the $x-$axis. Then, depending on the magnification in the $x-$axis the be- havior of δ as a function of Λ will be different. Using a small a, the dominant behavior is $\delta \sim \Lambda^H$. Then the number of segments along the time axis is

$$\frac{\mathcal{T}}{\Lambda t} \sim \Lambda^{-1} \sim \delta^{-1/H} \qquad , \tag{33}$$

where the last equality was written by virtue of the small a dominant behavior $\delta \sim \Lambda^H$. Therefore the to- tal length will have the behavior

$$L = \left(\frac{\mathcal{T}}{\Lambda t}\right) \times \delta \sim \delta^{1-1/H} \qquad , \tag{34}$$

from which the *divider fractal dimension* reads[9, 8] $D = 1/H$. Because in the asymptotic long-time regime our **s.p.** $\mathbf{X}(t)$ fulfills the scaling (17), we obtain for $\mathbf{X}(t)$ the fractal *local* divider dimension[2]

$$D = \frac{2}{2-\mu} \qquad , \qquad \mu \in [0, 1) \tag{35}$$

A result which, in the limit $\mu \to 1$ is in agreement with Wiener's *divider* dimension $D \to 2$. In the opposite limit $\mu = 0$ the expected ballistic behavior is obtained.

III   Perspectives in the frame- work of $Tsallis-$statistics

A. Anomalous diffusion

In recent years there have been some interest in re- lating non-extensive entropies with the phenomena of anomalous diffusion. The starting point is the defi- nition of a $q-$entropy[12] (in a continuous representa- tion):

$$S_q[\rho(\mathbf{x})] \equiv -k \frac{1 - \int \rho^q(\mathbf{x})\, d\mathbf{x}}{1-q} \qquad , \qquad q \in \mathcal{R}e \quad (36)$$

where k is a positive constant (such that in the $\lim_{q\to 1} k \to k_B$ is the Boltzmann constant) and $\mathbf{x}$ is an arbitrary $d-$dimensional vector. Thus under the normalization condition of probability $1 = \int \rho(\mathbf{x})\, d\mathbf{x}$, and the definition of a generalized mean value energy $\langle \epsilon(\mathbf{x}) \rangle_q \equiv \int \epsilon(\mathbf{x})\, \rho^q(\mathbf{x})\, d\mathbf{x} = \mathbf{E}_q$ ($q-$expectation value), the maximization of $S_q[\rho(\mathbf{x})]$ allows to obtain a general- ized thermodynamics which conserves Legendre's struc- ture [13]. The definition of the $q-$entropy (36) can also been used in the context of stochastic processes. We should remark that (36) is the definition of a general- ized (equilibrium) $q-$entropy, therefore in (36) it should be understood that the distribution $\rho(\mathbf{x})$ is related with a stationary measure. In fact this is the concept which was used when studying irreversible thermodynamics in the neighborhood of its equilibrium distribution[14].

In order to study systems far away from equilibrium the approach is quite different. For instead it is possi- ble to use (36) as the starting point in the definition of the probability transition $\rho(\mathbf{x})$ (hopping of length $\mid \mathbf{x}_1 - \mathbf{x}_2 \mid = \mathbf{x}$) in a random walk scheme. These hopping transitions can be of the short or long-range class. A short-range hopping transition can be obtained by using

---

[2]As we pointed out in the previous section, in order to apply Feder's argument to our non-Markovian **s.p.** we ought to assume that any transient has disappeared, so the portion of the record $\mathbf{X}(t)$ (time-span $\mathcal{T}$) that we want to analyze fulfills the scaling (17).

the well-posed optimization procedure [15]. Nevertheless, this is not the case if we want to find a long-range hopping transition, as the one proposed by Lévy [16]. It is here, where the concept of $q-$entropy comes into the approach, to be able to give an unified picture concerning short or long-range hopping transitions, from a *single optimization procedure*. The question may be put in the following way: can Lévy's stable[3] distributions ($\rho(\mathbf{x}) \sim \mathbf{x}^{-1-\gamma}$, $0 < \gamma < 2$) be obtained form a *simple* constraint in the context of the optimization technique? The answer is yes, but only if a generalization of the entropy is used.

Define the generalized constraint $\sigma_q^2 \, d =$ $\int x^2 \, \rho^q(\mathbf{x}) \, dx$, then applying the variational principle to $S_q[\rho(\mathbf{x})]$ upon the normalization condition on $\rho(\mathbf{x})$ and the constraint $\sigma_q^2 \, d = finite$; the resulting hopping probability distribution is (in $d-$dimension)

$$\rho(\mathbf{x}) \;=\; \left[\frac{kq}{\alpha(q-1)} + \frac{\beta q}{\alpha}x^2\right]^{1/(1-q)} \qquad (37)$$

where α and β are the variational Lagrange parameters[17]. For large x this hopping transition has a Lévy stable form, then we can identify (in any dimension) $q = (3 + \gamma)/(1 + \gamma)$.

In the context of the random walk theory, the Lévy parameter γ can be mapped with the Hausdorff-Besicovitch fractal dimension of the set of points visited by the random walk (thus $0 < \gamma < d$). These requirements determine that the value of q must satisfy: $(3 + d)/(1 + d) < q < (2 + d)/d$, which in $1-$dimension give $1 < q < 3$ (a result which was also obtained in the context of the analysis of the fluctuations in irreversible thermodynamics[14]).

Going one step further it is possible to study the temporal evolution of a Markovian random walk in the context of $q-$entropies[18]. To be more precise, let

$P(s, n)$ be the $1D$ random walk probability to be at site s in its $n - th$ step. This is nothing more than a Markovian chain in a regular infinite domain. Then $P(s, n)$ fulfills the recurrence relation

$$P(s, n) = \sum_{s'} \rho(|\, s - s'\,|)P(s', n - 1) \qquad (38)$$

which can immediately be solved[4] by using the Fourier inversion of $\hat{P}(k, n)$, i.e.:

$$P(s, n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{\rho}(k)^n \hat{P}(k, n = 0)e^{-iks}dk \qquad (39)$$

where $\hat{P}(k, n = 0)$ is the Fourier transform of the random walk initial condition $P(s, n = 0)$. From (39) it is simple to see that all the information concerning the evolution of the random walk is contained in the function $\hat{\rho}(k)^n$, which is just given in terms of the Fourier transform of the hopping transition. Then it is possible to see that if the hopping transition is of the long-range type: $\rho(\mathbf{x}) \sim \mathbf{x}^{-1-\gamma}$ (i.e.: its Fourier transform is the Lévy stable definition $\hat{\rho}(k) = e^{-b|k|^\gamma}$), none of the random walk moments $\langle s(n)^m \rangle \equiv \sum_s s^m P(s, n)$ are defined. This is so because this class of hopping transition has not an intrinsic length scale, this fact can easily be seen through the non-analyticity, around $k = 0$, of the Fourier transform $\hat{\rho}(k)$.

In the context of the $q-$statistics we would like to characterize the evolution of these moments avoiding these intrinsic divergencies. In particular we can be interested in characterize the $q-$expectation random walk variance $\langle s(n)^2 \rangle_q$, and for example we would like study under which conditions (values of q), for large n, the central limit theorem applies. Invoking a generalized Levy-Gnedenko Markov convolution the result in $1D$ gives[18]

---

[3] They are called *stable* distributions because the addition of N of such variables preserve the same probability distribution.

[4] Equation (38) can be rewritten by defining the auxiliary generating function: $R(s, z) = \sum_{n=0}^{\infty} z^n P(s, n)$, for $|\, z\,| < 1$. Multiplying (38) by z^n and summing from $n = 1$ to ∞ we get

$$R(s, z) - P(s, n = 0) = z \sum_{s'} \rho(|\, s - s'\,|)R(s', z)$$

Now, taking the Fourier transform $\hat{P}(k, n) \equiv \mathcal{F}[P(s, n)]$, etc. ; with the help of the convolution theorem and the geometric series representation, we get

$$R(s, z) = [1 - z\hat{\rho}(k)]^{-1} \hat{P}(k, n = 0)$$

Then $\hat{P}(k, n)$ can be obtained from the expansion coefficients of $\hat{R}(k, z)$ in power of z.

$$\left\langle s(n)^2 \right\rangle_q = \begin{cases} D_q^G kT\, n & if & -\infty \le q \le 5/3 \\[2mm] D_q^L kT\, n^{(q-1)} & if & 5/3 < q < 3 \end{cases} \qquad (40)$$

where D_q^G and D_q^L are well behaved functions, showing a divergency for $q = 5/3$, and kT is related to a thermal energy. Thus measuring the $q-$expectation variance (40), can provide a possible experimental verification of the $q-$statistics.

Naturally the existence of non-Markovian correlations (in the random walk) could modify the predicted critical exponents (40). Therefore it is very important to know *exact* non-Markovian solutions [as the ones presented in previous sections] in order to analyze possible generalizations of (40), by considering non-Markovian effects in the random walk.

Strong non-Markovian processes, like the one characterized in section II.B, but also taking into account large-excursion hopping transitions—as the one characterized by a Lévy distribution—can be worked out in our approach and they are under investigation.

B. Irreversible Thermodynamics

In the context of $q-$entropies, Onsager's reciprocity relations were proved by using linear Markov processes (regression theorem) and the concept of $q-$expectation values[14], therefore it could be of interest to study its non-Markovian generalization.

If a friction term $-\eta X$ is added to (3), and we allow the Gaussian random force $\xi(t)$ to be a long-range correlated noise, this type of Gaussian Langevin-like equation can also be worked out, in a similar way, as we have presented here[11]. Thus, an open problem can be to study the $N-$dimensional linear stochastic dynamics

$$\frac{d}{dt} X_j = -\sum_{l=1}^{N} \eta_{jl}\, X_l + \xi_j(t) \qquad (41)$$

where $\langle\langle \xi_j(t_1)\xi_l(t_2)\rangle\rangle$ have long-range correlations as in (2). This type of model could help in understanding irreversible thermodynamics which takes into account long-range fluctuations. Thus providing another possible scenario where the application of the $q-$statistics can be of value.

IV Discussions

A generalized non-Markovian Wiener stochastic process $\mathbf{X}(t)$—with natural boundary condition—was completely characterized in terms of its functional $G_X([Z(t)])$. Two models of correlated noises have been worked out, particular stress has been put in long-range correlations $\langle\langle \xi(t_1)\xi(t_2)\rangle\rangle$, like in (2), giving rise to anomalous super-diffusion. This fairly general method is based upon knowing the characteristic functional of the noise $G_\xi([k(t)])$, which in the present paper has been assumed to be Gaussian (generalizations to non-Gaussian statistics can also be worked out in our approach). Any $m-$time moment of the **s.p.** $\mathbf{X}(t)$ follows straightforwardly by taking functional derivative of $G_X([Z(t)])$, see (6).

One of the questions addressed in this paper has been the characterization of the spectrum and the fractal dimensions of our strong non-Markovian particle in a generalized medium, i.e.: in presence of a long-range correlated noise. If the noise parameter $\mu \in [0, 1)$, past increments of the **s.p.** $\mathbf{X}(t)$ are correlated with future increments, so a long-range correlated noise induces infinitely long-run correlations in the **s.p.** $\mathbf{X}(t)$ like in the *persistent* fBm (super-diffusion with $H \in (\frac{1}{2}, 1]$).

Using our approach, the medium can be represented by an arbitrary correlated random force $\xi(t)$, [additive noise appearing in the SDE (3)]. The present analytical formulation provides a systematic starting point to obtain higher-order moments and also to compute the whole Kolmogorov hierarchy.

We remark that there is not limitation in the exact calculation of any higher-order moment of our non-Markovian process. In the present paper we have used a functional technique to solve a problem with natural boundary conditions, the application of this method to problems with non-natural boundary condition is under investigation[19]. Also in order to study anoma-

lous *super*-diffusion, a possible connection with the $q-$statistics has been pointed out.

Acknowledgments

References

[1] K. Ishii and K. Kitahara, Prog. Theor. Phys. **68**, 665 (1982).

[2] A. Morita, Phys. Rev. **A 41**, 754 (1990).

[3] M. Gitterman, Phys. Rev. **E 52**, 303 (1995), and reference therein.

[4] A. Compte and M.O. Cáceres, "*Fractional Dynamics in Random Velocity Fields*", Phys. Rev. Lett. **81**, 3140 (1998).

[5] N.G. van Kampen, in: *Stochastic Processes in Physics and Chemistry,* Second Edition, (North-Holland, Amsterdam,1992).

[6] M.O. Cáceres, A.A. Budini, J. Phys. A Mat. & Gen. **30**, 8427 (1997).

[7] J.M. Sancho and M. San Miguel, in : *Noise in Nonlinear Dynamical Systems,* Vol.1, Chap.3, Cambridge University Press, Eds. F. Moss, and P.V.E. McClintock, (1989).

[8] D. Saupe, in: *The Science of Fractal Images,* Chap. 2, Eds. Heinz-Otto Peitgen and Dietmar Saupe, (Springer-Verlag, Berlin, 1988).

[9] J. Feder, in: *Fractals*, (Plenum Press, London, 1989).

[10] B.B. Mandelbrot and J.W. van Ness, SIAM Review 10, 422 (1968).

[11] M.O. Cáceres, "*Stochastic Harmonic Potential Driven by Long-range Correlated Noises*", submitted (1998).

[12] C. Tsallis, J. Stat. Phys. **52**, 479 (1988).

[13] C. Tsallis, "*Some Comments on Boltzmann-Gibbs Statistical Mechanics* ", Chaos, Solitons and Fractals **6**, 539 (1995), and references therein.

[14] M.O. Cáceres, Physica **A 218**, 471 (1995).

[15] E.W. Montroll and M. Shlensinger; J. Stat. Phys. **32**, 209 (1983).

[16] P. Lévy; in: *Théorie de láddition des variables aléatoires,* (Guthier-Villars, Paris, 1937).

[17] (a) P.A. Alemany and D. Zanete, Phys. Rev. E **49**, R956 (1994); (b) P.A. Alemany, PhD. Thesis, Instituto Balseiro, (1995).

[18] C. Tsallis, S.V.F. Levy, A.M.C. Souza, and R. Maynard, Phys. Rev. Lett. **75**, 3589 (1995); erratum Phys. Rev. Lett. **77**, 5442 (1996); M.O. Cáceres and C.E. Budde, Phys. Rev. Lett. **77**, 2589 (1996).

[19] A.A Budini and M.O. Cáceres, "*The Generalized Wiener Process II: Finite Systems*", J. Phys. A Mat. & Gen. (submitted 1998).

APPENDIX A. The 2−time second cumulant of the non-Markovian s.p. $\mathbf{X}(t)$

Let us now study the correlation function $\langle\langle X(t_1)X(t_2)\rangle\rangle$, which can be calculated in an exact way by using functional derivatives as in (6), from (4) we get

$$\langle\langle X(t_1)X(t_2)\rangle\rangle = i^{-2}\left.\frac{\delta^2 \log G_X([Z(t)])}{\delta Z(t_1)\delta Z(t_2)}\right|_{Z=0} = i^{-2}\frac{\delta^2}{\delta Z(t_1)\delta Z(t_2)}\left[iX_o\int_0^\infty Z(s)ds\right.$$

$$\left.-\frac{\Gamma_2}{2}\int_0^\infty\int_0^\infty\left(\int_{s_1}^\infty Z(s)ds\right)\left(\int_{s_2}^\infty Z(s')ds'\right)\langle\langle\xi(s_1)\xi(s_2)\rangle\rangle\,ds_1ds_2\right]_{Z=0} \tag{42}$$

Without loss of generality we now assume that $t_1 \leq t_2$, then we get

$$\langle\langle X(t_1)X(t_2)\rangle\rangle = {}_{,2}\int_0^{t_1}\int_0^{t_2}\langle\langle\xi(s_1)\xi(s_2)\rangle\rangle\,ds_1ds_2 \tag{43}$$

so using the definition (8) we can write

$$\langle\langle X(t_1)X(t_2)\rangle\rangle = 2\sigma(t_1) + {}_{,2}\int_0^{t_1}ds_1\int_{t_1}^{t_2}ds_2\,\langle\langle\xi(s_1)\xi(s_2)\rangle\rangle \tag{44}$$

1. Short-range

For the particular case of a Gaussian short-range correlated noise the result gives

$$\frac{1}{,2}\langle\langle X(t_1)X(t_2)\rangle\rangle = t_1 - \theta\left(1 - e^{-t_1/\theta}\right) + \frac{\theta}{2}\left(e^{t_1/\theta} - 1\right)\left(e^{-t_1/\theta} - e^{-t_2/\theta}\right) \tag{45}$$

which of course in the limit $\theta \to 0$ goes to Wiener's result: $\langle\langle X(t_1)X(t_2)\rangle\rangle = {}_{,2}\min(t_1, t_2)$. Note that due to the non-Markovian character (if $\theta \neq 0$) of the **s.p.** $\mathbf{X}(t)$, both times are always present in this non-stationary second cumulant. If $t_1 \gg \theta$ and $t_2 \gg \theta$ such that $t_2 - t_1 > 0$, the non-Markovian signature of the **s.p.** can still be seen in the transient appearing in (45), which gives rise to a small θ–correction of the form $\frac{1}{\Gamma_2}\langle\langle X(t_1)X(t_2)\rangle\rangle \sim t_1 - \frac{\theta}{2}\left(1 + e^{-(t_2 - t_1)/\theta}\right)$. Nevertheless in the asymptotic limit $t_1, t_2 \to \infty$ the dominant term is linear. This means that, in the asymptotic long-time regime, the non-Markovian character of the **s.p.** $\mathbf{X}(t)$ will be erased when the correlation of the noise is of the short-range class.

2. Long-range

For the particular case of a Gaussian noise $\xi(t)$ with a long-range correlation like the one in (2), the behavior is quite different, and gives rise to anomalous super-diffusion. From (44) the result, for $0 < \mu \neq \{1, 2\}$, assuming as before that $t_1 \leq t_2$, gives

$$\frac{1}{\Gamma_2}\langle\langle X(t_1)X(t_2)\rangle\rangle = \frac{2t_1}{(\mu-1)} - \frac{\tau}{(\mu-1)(\mu-2)}$$
$$+ \frac{\tau^{\mu-1}}{(\mu-1)(\mu-2)}\left[(t_1+\tau)^{2-\mu} + (t_2+\tau)^{2-\mu} - (t_2-t_1+\tau)^{2-\mu}\right]$$

(46)

In the asymptotic long-time regime $t_1 \gg \tau$ and $t_2 \gg \tau$, such that $t_2 - t_1 \simeq \mathcal{O}(\tau)$, and if the noise parameter is $\mu \in [0, 1)$ we see that the two-point second cumulant (46), increases in time with a power-law $\sim \left(t_1^{2-\mu} + t_2^{2-\mu}\right)$. Thus, a long-range correlated noise induces a super-diffusion behavior in the cumulant $\langle\langle X(t_1)X(t_2)\rangle\rangle$. Note that both times are always present in this anomalous non-stationary correlation function. Only if the noise parameter is $\mu > 1$ we reobtain, in the asymptotic long-time regime, the linear (Wiener) result $\langle\langle X(t_1)X(t_2)\rangle\rangle \propto \min(t_1, t_2)$.

In general the *increments* of our **s.p.** $\mathbf{X}(t)$ are *not* statistical independent, as it is—of course—required by a fundamental theorem that says: if the increments of any **s.p.** are independent therefore the **s.p.** is a Markov one (the reverse is not true!).