The δ Expansion and the Principle of Minimal Sensitivity

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The δ-expansion is a nonperturbative approach for field theoretic models which combines the techniques of perturbation theory and the variational principle. Different ways of implementing the principle of minimal sensitivity to the δ-expansion produce in general different results for observables. For illustration we use the Nambu–Jona-Lasinio model for chiral symmetry restoration at finite density and compare results with those obtained with the Hartree-Fock approximation.

The standard application of the linear δ-expansion [1] to a theory with action S starts with an interpolation defined by $S(\delta) = (1-\delta)S_0(\mu) + \delta S$, where $S_0(\mu)$ is the action of a solvable theory. The action $S(\delta)$ interpolates between the solvable $S_0(\mu)$ (when $\delta = 0$) and the original S (when $\delta = 1$). Since S_0 is quadratic in the fields, arbitrary parameters (μ) with mass dimensions are required for dimensional balance. At the end one sets $\delta = 1$ fixing μ according to the principle of minimal sensitivity (PMS) [2] which requires a physical quantity $\Phi(\mu)$ to satisfy

$$\frac{\partial \Phi(\mu)}{\partial \mu}\Big|_{\bar{\mu}} = 0. \qquad (1)$$

Within this method, the general procedure is to apply the PMS directly to each different quantity of interest so as to adjust μ to the different energy scales of the theory [2]. A natural question which arises at this point is the uniqueness of the value of μ since different physical quantities might generate different values for the optimal μ. Of course this would not be catastrophic if the spread of the values of μ determined from different observables were not too large.

Alternatively, one could select only one among those observables to optimize the theory. This selection could be done by using some physical criterion or constraint (for example, in the case were only one of the calculated quantities satisfies the PMS equation). However, this strategy (referred as PMS1) does not completely specify a unique procedure and, as we shall see, can be misleading. One of our goals is to show that all these potential uncertanties could be avoided by demanding that fundamental quantities, such as the energy density, be used to fix μ whose optimal values are then used to calculate other observables. Using the energy momentum tensor of the original theory one can obtain the exact energy density written in terms of full vertices and propagators. Next, one uses the interpolated theory to evaluate self energies as well as vertex corrections perturbatively in powers of δ. These μ-dependent quantities are then plugged back into the energy density to which the PMS is applied. This approach (referred as PMS2) has been succesfully applied to the Walecka model for nuclear matter [3]. The fact that it is natural to demand stationarity of the energy with respect to unknow parameters uniquely selects this quantity as

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the generator of $\bar{\mu}$ so that all physical observables are determined from the same propagator.

In this paper we illustrate the problem with the PMS1 prescription by using the Nambu–Jona-Lasinio (NJL) model [4] for chiral symmetry restoration in a medium of finite density. Conventionally, the finite density chiral symmetry restoration problem within the NJL model has been tackled with the Hartree-Fock (HF) approximation. For the SU(2) case, this analytical approach shows that chiral symmetry is restored through a first-order phase transition at a critical density whose values depend on the choice of the parameters [5, 6]. We then follow the two alternatives, PMS1 and PMS2, and compare results with the traditional HF approach.

Some physical quantities of interest, whose values characterize the chiral symmetry restoration, are the quark condensate $\langle\bar{q}q\rangle$, the pion decay constant f_π and the constituent quark mass M_q. We calculate these quantities both with PMS1 and PMS2 and compare our results with the ones obtained in Ref. [5] with the HF approximation, where vertex corrections are neglected. Therefore, we shall also neglect vertex corrections. Of course, since the NJL model is essentially phenomenological, we shall pay more atention to the qualitative results (like the order of the phase transition) than to the quantitative ones (like the precise value of the critical density for which the phase transition takes place).

In the limit of zero current quark masses, the two-flavor Lagrangian density of the Nambu–Jona-Lasinio model is given by

$$\mathcal{L}_{\rm NJL} = \bar{q}(i\not\partial)q + G\left[(\bar{q}q)^2 - (\bar{q}\gamma_5\boldsymbol{\tau}q)^2\right] , \qquad (2)$$

where the quark field operators $q = q(x)$ represent the doublet of u and d quarks.

Let us start by deriving the energy density from the energy-momentum tensor of the original theory since this quantity will be necessary when using the PMS2. Using the Lagrangian density, Eq. (2), we have the energy-momentum tensor,

$$T^{\mu\nu}_{\rm NJL} = i\bar{q}\gamma^\mu\partial^\nu q - g^{\mu\nu}\mathcal{L}_{\rm NJL} = i\bar{q}\gamma^\mu\partial^\nu q - g^{\mu\nu}\left\{\bar{q}(i\not\partial)q + G\left[(\bar{q}q)^2 - (\bar{q}\gamma_5\boldsymbol{\tau}q)^2\right]\right\} . \qquad (3)$$

Note that we have not used the equation of motion for the quark field operator. Neglecting vertex corrections, the energy density is given by

$$\begin{aligned}
\mathcal{E}_{\rm NJL} &= \frac{1}{V}\int d^3x \ < T^{00} > \\
&= -i\int\frac{d^4q}{(2\pi)^4}q^0{\rm Tr}\left[\gamma^0 S(q)\right] + i\int\frac{d^4q}{(2\pi)^4}{\rm Tr}\left[\not q S(q)\right] - G\left\{-\left[\int\frac{d^4q}{(2\pi)^4}{\rm Tr}\left[S(q)\right]\right]^2\right. \\
&\quad + \int\frac{d^4q}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}{\rm Tr}\left[S(q)S(k)\right] + \left[\int\frac{d^4q}{(2\pi)^4}{\rm Tr}\left[\boldsymbol{\tau}\gamma_5 S(q)\right]\right]^2 \\
&\quad \left. - \int\frac{d^4q}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}{\rm Tr}\left[\gamma_5\tau^a S(q)\gamma_5\tau^a S(k)\right]\right\} , \qquad (4)
\end{aligned}$$

where $S(q)$ represents the dressed quark propagator.

The quark condensate, which is taken to be the parameter of order of the phase transition, is given by

$$\langle\bar{q}q\rangle = -i\int\frac{d^4p}{(2\pi)^4}{\rm tr}[S(p)] , \qquad (5)$$

where the trace is taken over spinor and color indices. As in Refs. [5, 6] we employ the Pagels-Stokar formula [7] to evaluate the pion decay constant (f_π),

$$iq^\mu f_\pi\delta^{ab} = \int\frac{d^4p}{(2\pi)^4}{\rm tr}\left[S(p+q)(g_{\pi q}\gamma^5\tau^a)S(p)(\frac{1}{2}\tau^b\gamma^\mu\gamma^5)\right] , \qquad (6)$$

where the trace is now over spinor, flavor and color. The quark-pion coupling can be obtained from the Golberger-Treiman relation. Of course, we could use other, perhaps more precise formulas for f_π, but for our purposes of comparing PMS1 and PMS2 results, Eq. (6) is sufficient.

To define the interpolated Lagrangian one needs to choose a solvable theory. Since we are looking for solutions which break chiral symmetry, the natural choice for $\mathcal{L}_0$ is

$$\mathcal{L}_0 = \bar{q}(i\partial\!\!\!/ - \mu)q \ , \tag{7}$$

where μ is an arbitrary mass parameter. Therefore, the interpolated NJL Lagrangian density can be written as

$$\begin{aligned} \mathcal{L}_{\rm NJL}(\delta) &= (1-\delta)\left[\bar{q}(i\partial\!\!\!/ - \mu)q\right] + \delta\left\{\bar{q}(i\partial\!\!\!/)q + G\left[(\bar{q}q)^2 - (\bar{q}\gamma_5\boldsymbol{\tau}q)^2\right]\right\} \\ &= \bar{q}(i\partial\!\!\!/ - \mu)q + \delta\left\{G\left[(\bar{q}q)^2 - (\bar{q}\gamma_5\boldsymbol{\tau}q)^2\right] + \mu\bar{q}q\right\} \ . \end{aligned} \tag{8}$$

Expressed in terms of self energy $\Sigma^\delta(p)$ the quark propagator reads $S^{-1}(p) = S_0^{-1}(p) - \Sigma^\delta(p)$ where $S_0^{-1}(p)$ is the inverse of the quark propagator corresponding to $\mathcal{L}_0$ ($S_0^{-1}(p) = \not{p} - \mu$), and the quark self-energy $\Sigma^\delta(p)$ is calculated as a power series in δ.

At zeroth order in δ, one is treating the free Lagrangian and hence $\Sigma^{(0)}(p) = 0$. The bare (zeroth order) in-medium quark propagator is then given by

$$S^{(0)}(p) = \frac{\not{p} + \mu}{p^2 - \mu^2 + i\epsilon} + \pi i \frac{\not{p} + \mu}{E_0(p)}\delta\left(p^0 - E_0(p)\right)\theta\left(P_F - |\mathbf{p}|\right) \ , \tag{9}$$

where $E_0(p) = \left(\mathbf{p}^2 + \mu^2\right)^{\frac{1}{2}}$, and P_F is the Fermi momentum which, for $N_f = 2$, relates to the quark density ρ via $P_F = (\pi^2\rho/2)^{1/3}$.

At this order in δ, no dynamical content from the model has been used. The dynamics of the model starts to show up at order δ. To O(δ) the self-energy ($\Sigma^{(1)}(p)$) is given by

$$\begin{aligned} \Sigma^{(1)}(p) = &- \delta\mu \\ &+ 2i\delta G\int\frac{d^4q}{(2\pi)^4}\left\{\mathrm{Tr}\left[S^{(0)}(q)\right] - S^{(0)}(q) - \gamma_5\tau^a\mathrm{Tr}\left[\tau^a S^{(0)}(q)\gamma_5\right] + \gamma_5\tau^a S^{(0)}(q)\tau^a\gamma_5\right\}, \end{aligned} \tag{10}$$

where a sum over the isospin index a is implied. Substituting Eq. (9) into this equation, we obtain for $\Sigma^{(1)}$ the expression

$$\Sigma^{(1)}(p) = -\delta\mu + M_1 - \gamma_0\Sigma_0 \ , \tag{11}$$

where

$$M_1 = \delta\frac{G}{\pi^2}\mu\left(N_cN_f + \frac{1}{2}\right)\left\{\Lambda\left(\Lambda^2+\mu^2\right)^{\frac{1}{2}} - P_F\left(P_F^2+\mu^2\right)^{\frac{1}{2}} - \mu^2\ln\left[\frac{\Lambda + \left(\Lambda^2+\mu^2\right)^{\frac{1}{2}}}{P_F + \left(P_F^2+\mu^2\right)^{\frac{1}{2}}}\right]\right\} \ , \tag{12}$$

and

$$\Sigma_0 = -4\delta G\int\frac{d^3q}{(2\pi)^3}\theta(P_F - |\mathbf{q}|). \tag{13}$$

One should note that, at this order, direct and exchange terms are treated at equal footing as implied by the factor $(N_cN_f + 1/2)$ in Eq. (12). Since the effect of Σ_0 is just to shift the chemical potential [6], one may write the constituent quark mass to $O(\delta)$ as

$$M_q = \mu - \delta\mu + M_1 \ . \tag{14}$$

Substituting Eq. (9) into Eqs. (5) and (6), one gets for the order parameter per flavor and for the pion decay constant the following lowest order expressions,

$$\langle\bar{q}q\rangle_0 = -\frac{N_c}{2\pi^2}\mu\left\{\Lambda(\Lambda^2+\mu^2)^{\frac{1}{2}} - p_F(p_F^2+\mu^2)^{\frac{1}{2}} - \mu^2\ln\left[\frac{\Lambda+(\Lambda^2+\mu^2)^{\frac{1}{2}}}{p_F+(p_F^2+\mu^2)^{\frac{1}{2}}}\right]\right\}, \quad (15)$$

and

$$f_\pi^2 = \frac{N_c N_f \mu^2}{4\pi^2}\left\{\ln\left[\frac{\Lambda+(\Lambda^2+\mu^2)^{\frac{1}{2}}}{p_F+(p_F^2+\mu^2)^{\frac{1}{2}}}\right] - \left(1+\frac{\mu^2}{\Lambda^2}\right)^{-\frac{1}{2}} + \left(1+\frac{\mu^2}{p_F^2}\right)^{-\frac{1}{2}}\right\}, \quad (16)$$

where the lowest order Goldberger-Treiman relation ($g_{\pi q} = \mu/f_\pi(0)$) has been used.

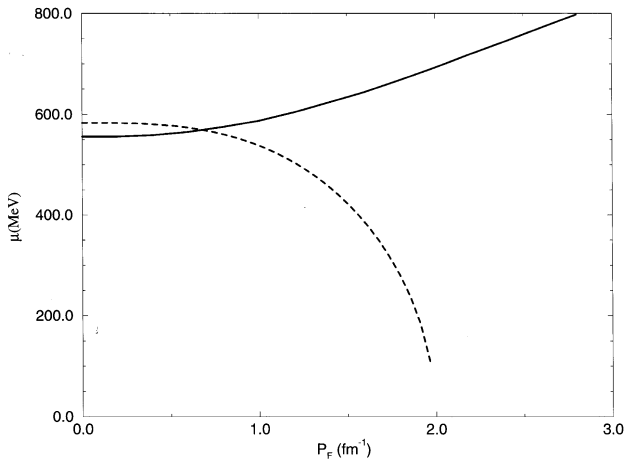

Figure 1. P_F dependence of $\bar{\mu}$ obtained with the PMS applied to f_π (solid line - PMS1) and to (dashed line - PMS2).

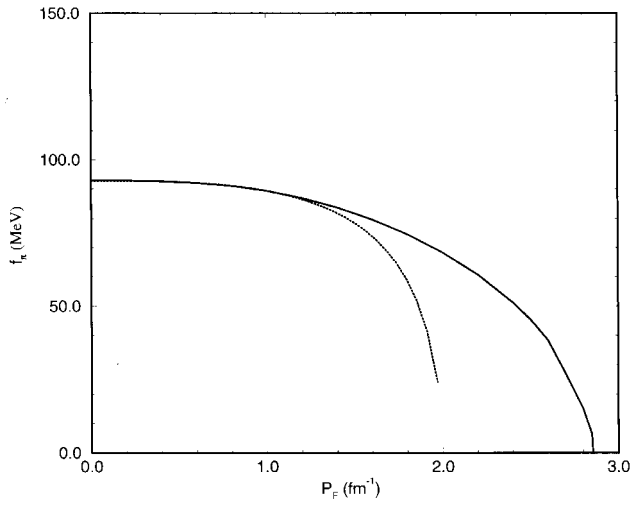

Figure 2. P_F dependence of f_π. The solid and dotted lines give respectivelly the PMS1 and the PMS2 solutions.

We now have the three quantites of interest (M_q, $\langle\bar{q}q\rangle_0$ and f_π) obtained at lowest order in δ and the next step is the optimization procedure. Let us start with the PMS1. Of the three calculated quantities the only one which satisfies the PMS condition (the one which has extremum points) is f_π. Moreover, at zero density, this quantity has a well established empirical value and can be chosen to fix μ. A direct application of the PMS condition to f_π gives $\bar{\mu} = 0.97\times\Lambda$. Using the zero density empirical value $f_\pi = 93$ MeV one gets the noncovariant cut-off $\Lambda = 571$ MeV. In principle, the fact that the cut-off can be fixed (with a value which agrees with the ones used in the literature) without any previous knowledge of the quark mass could be seen as an advantage of the method. However, one must be careful with the interpretation of this result since it has been obtained without any information about the model, because the coupling constant G does not appear at this lowest order evaluation of f_π. If one takes this value for Λ and proceeds blindly by applying the PMS to f_π for different values of P_F one obtains $\bar{\mu}$ as a function of the density as shown by the continuous line of Fig. 1. We note that $\bar{\mu}$ obtained with the PMS1 has a very peculiar behavior increasing with the density. This odd behavior is reflected in Fig. 2 where one sees that f_π goes smoothly to zero, indicating chiral symmetry restoration, through a second-order phase transition, contrary to the HF predictions. The same values of $\bar{\mu}$ can be used to evaluate the quark condensate and quark mass. The numerical zero density results for these quantities, $\langle\bar{q}q\rangle_0 = -(250\ \text{MeV})^3$ and $M_q = 574$ MeV (where the value $G = 8.86\times10^{-6}\ \text{MeV}^{-2}$ was used in Eq. (12) for M_q) are not far from the ones predicted in the literature when a noncovariant cut-off is used. However, the finite density behavior of these two quantities again points out towards a smooth second-order phase transition.

Let us now evaluate the same quantities using the PMS2 to generate the density dependent optimal values for $\bar{\mu}$. Substituting the lowest order quark propagator given by Eq. (9) into Eq.(4), we obtain

$$\mathcal{E}^{(0)}_{\rm NJL} = -2N_cN_f\int_{P_F}^{\Lambda}\frac{d^3q}{(2\pi)^3}\frac{\mathbf{q}^2}{E_0(q)} - 2GN_cN_f(2N_cN_f+1)\left[\int_{P_F}^{\Lambda}\frac{d^3q}{(2\pi)^3}\frac{\mu}{E_0(q)}\right]^2 . \tag{17}$$

The requirement that $\mathcal{E}$ be stationary with respect to variations in μ leads to

$$\bar{\mu} = 4G\left(N_cN_f+\frac{1}{2}\right)\int_{P_F}^{\Lambda}\frac{d^3q}{(2\pi)^3}\frac{\bar{\mu}}{E_0(q)} , \tag{18}$$

from where we immediately see that, even at zeroth order in δ, the value of μ depends on G, in contrast to the result obtained with PMS1. Note that this is the familiar Hartree-Fock gap equation of the model, where $\bar{\mu}$ has the interpretation of the dynamically generated mass as can also be seen from its behavior at finite densities displayed in Fig. 1 (dashed line). As expected, when these optimal values are injected in f_π, $\langle\bar{q}q\rangle_0$ and M_q, one predicts the restoration of chiral symmetry through a first-order phase transition in agreement with the HF results as can be seen by the dotted line in Fig. 2.

Next, one could try to improve these results by using the $O(\delta)$ quark propagator in the evaluation of the energy density. Inversion of Dyson's equation leads to

$$S^{(1)}(p) = \frac{\not{p}_1+M_1}{p_1^2-M_1^2+i\epsilon} + \pi i\frac{\not{p}_1+M_1}{E_1(p)}\delta\left(p_1^0-E_1(p)\right)\theta\left(P_F-|\mathbf{p}|\right) , \tag{19}$$

where

$$p_1^\mu = (p_1^0,\mathbf{p}) = (p^0+\Sigma_0,\mathbf{p}) \quad , \quad E_1(p) = \left[\mathbf{p}^2+(M_1)^2\right]^{\frac{1}{2}} , \tag{20}$$

with Σ_0 given by Eq. (13). The superscript (1) in $S^{(1)}$ indicates that the propagator has been obtained with a self-energy calculated up to first-order in δ (note that the term $\mu-\delta\mu$ appearing in Eq. (14) has already been discarded in Eq. (19)). Using the first-order quark propagator in the evaluation of the energy density one gets

$$\mathcal{E}^{(1)}_{\rm NJL} = -2N_cN_f\int_{P_F}^{\Lambda}\frac{d^3q}{(2\pi)^3}\frac{\mathbf{q}^2}{E_1(q)} - 2GN_cN_f(2N_cN_f+1)\left[\int_{P_F}^{\Lambda}\frac{d^3q}{(2\pi)^3}\frac{M_1}{E_1(q)}\right]^2 . \tag{21}$$

An application of the PMS to $\mathcal{E}^{(1)}_{\rm NJL}$,

$$\frac{d\mathcal{E}^{(1)}_{\rm NJL}}{d\mu} = \frac{d\mathcal{E}^{(1)}_{\rm NJL}}{dM_1}\frac{dM_1}{d\mu} = 0 , \tag{22}$$

leads to

$$M_1 = 4G\left(N_cN_f+\frac{1}{2}\right)\int_{P_F}^{\Lambda}\frac{d^3q}{(2\pi)^3}\frac{M_1}{E_1(q)} . \tag{23}$$

Again, we have obtained the familiar Hartree-Fock gap equation for the dynamically generated mass.

Higher-order corrections will in general introduce a momentum dependence for the dynamically generated mass. However, if one proceeds to higher orders in δ but neglect those graphs that correspond to vertex corrections, the higher-order quark propagator will always be of the form of Eq. (19), with M_1 replaced by another constant, say M, which is a function of μ. However, because of the PMS condition on $\mathcal{E}$, M at each order will always be given by the same value. This value is the one that satisfies the usual gap equation

$$M = 4G\left(N_cN_f+\frac{1}{2}\right)\int_{P_F}^{\Lambda}\frac{d^3q}{(2\pi)^3}\frac{M}{E(q)} , \tag{24}$$

where

$$E(q) = \left(\mathbf{q}^2+M^2\right)^{\frac{1}{2}} . \tag{25}$$

Therefore, the PMS condition on the energy density

(PMS2) is equivalent to the usual Hartree-Fock solution for the dynamically generated mass, when vertex corrections are neglected.

To conclude, in this paper we have used the NJL model to illustrate potential problems with the application of the PMS in the δ expansion. In order to specify a unique prescription to fix arbitrary parameters introduced by the δ expansion, we have studied two ways of introducing the PMS procedure. We have applied the PMS directly to f_π following the standard procedure (PMS1) [2]. We found that PMS1 leads to results for chiral symmetry restoration that disagree with the HF results. Having a close look in the way the PMS1 trades μ by the model parameters (the cut-off in this case) and its finite density behavior, we were able to identify the origin of this misleading result. We have also applied the PMS to the energy density (PMS2). We have shown that this prescription reproduces, already at lowest order, the HF results for chiral symmetry restoration at finite density within the NJL model. Moreover, this result can be reproduced at any order in δ provided that one ignores vertex contributions. This result should be compared with the one presented in Ref. [8] where, in the context of the effective potential, it was found that the δ expansion and the $1/N$ expansion are identical in the large N limit. Therefore, the PMS2 seems to be an adequate way of fixing the arbitrary parameters to generate nonperturbative results, and it is a promissing procedure since it allows the introduction of vertex corrections in a very direct way. Work in this direction is in progress [9].

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