Field of a Charged Particle in a Nonsymmetric Gravitational Theory

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The field equations of Moffat's nonsymmetric gravitational field are derived when electromagnetic fields are present, by adopting a non-minimal coupling which ensures the validity of the equivalence principle. The static, spherically symmetric solution of the field of a charged point particle is obtained.

I. Introduction

Soon after establishing his nonsymmetrical gravitational theory (NGT), Moffat[1] studied the gravitational field produced by an electric charge[2] adopting a minimal coupling of the electromagnetic field to gravitation. Later, Will[3] showed that such a coupling violates the weak equivalence principle (WEP) because it predicts that the test body gravitational acceleration depends upon its internal electrostatic structure. Then, by writing a more general and suitable coupling, Mann et al.[4] were able to maintain the validity of the WEP. Here we wish to study first the consequences of this more general coupling on the electromagnetic field equations and corresponding energy-momentum tensor. This we do in Sect. II. In Sect. III we go back to the original problem of the field produced by an electric charge in the new scheme.

II. The electromagnetic action

Mann et al. give the electromagnetic action as $I_{em} = -\frac{1}{16\pi}\int d^4x\sqrt{-g}fg^{\mu\alpha}g^{\nu\beta}[ZF_{\mu\nu}F_{\alpha\beta} + (1 - Z)F_{\alpha\nu}F_{\mu\beta} + YF_{\mu\alpha}F_{\nu\beta}]$, where, as usual $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ is the electromagnetic field tensor. The matrix $g^{\mu\alpha}$ is the inverse of the nonsymmetric gravitational field $g_{\mu\alpha}$ defined by $g^{\mu\alpha}g_{\nu\alpha} = g^{\alpha\mu}g_{\alpha\nu} = \delta^\mu_\nu$. Z and Y are constants while f is a scalar function whose value is unity when the antisymmetric part of $g_{\mu\nu}$ vanishes, $g_{[\mu\nu]} \rightarrow o$, implying that f is a function of $\sqrt{-g}/\sqrt{-\gamma}$ where $g = detg_{\mu\nu}$ and γ is the determinant of the symmetric part, $\gamma = detg_{(\mu\nu)}$. When $f = 1$, $Z = 1$ and $Y = 0$ one recovers the minimally coupled action used by Moffat [2], which according to Will [3] leads to a violation of the WEP. Mann et. al. showed that this could be avoided with the choice $Z - 1 = Y$ and

$$f = \frac{\sqrt{-g}}{\sqrt{-\gamma}} \tag{2.1}$$

In this way, the action for a spherically symmetric field becomes the action of the $TH\epsilon\mu$ formalism[5], except for a term, quadractic in the scalar product of the magnetic field and the off diagonal component $g_{[oi]}$, which makes no contribution to the acceleration of the body to electrostatic order[3].
The resulting action is then

$$I_{em} = \int d^4x\sqrt{-g}L_{em} \tag{2.2}$$

where

$$L_{em} = \frac{-1}{16\pi} f g^{\mu\alpha} g^{\nu\beta} \left[ZF_{\mu\nu}F_{\alpha\beta} + (1-Z)\left(F_{\alpha\nu}F_{\mu\beta} + F_{\mu\alpha}F_{\nu\beta}\right)\right], \tag{2.3a}$$

which can also be written as

$$L_{em} = -\frac{1}{16\pi} f F_{\mu\nu}F_{\alpha\beta}\left[g^{(\mu\alpha)}g^{(\nu\beta)} + (2Z-1)gFro^{[\mu\alpha]}g^{[\nu\beta]} + (1-Z)g^{[\mu\nu]}g^{[\alpha\beta]}\right), \tag{2.3b}$$

where f is given by (2.1). We have then one parameter left, Z, but as we shall see the field of a spherically symmetric charge turns out to be independent of Z. Varying the Lagrangian density with respect to $g^{\mu\alpha}$ one obtains

$$\delta\left(\sqrt{-g}L_{em}\right) = \frac{\sqrt{-g}}{2}\delta g^{\mu\alpha}E_{\mu\alpha} \tag{2.4}$$

where

$$E_{\mu\alpha} = \frac{1}{4\pi}\left[\frac{1}{8}\left(fg_{\mu\alpha} - 2\frac{\partial f}{\partial g^{\mu\alpha}}\right)g^{\lambda\sigma}g^{\nu\beta}\Big((ZF_{\lambda\nu}F_{\sigma\beta} + (1-Z)(F_{\sigma\nu}F_{\lambda\beta} + F_{\lambda\sigma}F_{\nu\beta})\Big)\right.$$
$$\left. -fg^{\nu\beta}\Big((ZF_{\mu\nu}F_{\alpha\beta} + (1-Z)(F_{\mu\beta}F_{\alpha\nu} + F_{\mu\alpha}F_{\nu\beta})\Big)\right] \tag{2.5}$$

is the energy-momentum tensor of the electromagnetic field. This is a traceless tensor, $g^{\mu\alpha}E_{\mu\alpha} = 0$, since we have the relation

$$g^{\mu\alpha}\frac{\partial f}{\partial g^{\mu\alpha}} = 0. \tag{2.6}$$

This can be proved by direct calculation from the relations $g^{-1} = \varepsilon_{\alpha\beta\gamma\delta}g^{\alpha o}g^{\beta 1}g^{\gamma 2}g^{\delta 3}$ and $\gamma^{-1} = \varepsilon_{\alpha\beta\gamma\delta}g^{(\alpha o)}g^{(\beta 1)}g^{(\gamma 2)}g^{(\delta 3)}$.

Next we consider the variation with respect to A_μ of the field plus interaction Lagrangian, $\pounds_{em} = -\sqrt{-g}J^\mu A_\mu$. We get

$$\partial_\nu[\sqrt{-g}f\Big(g^{(\mu\alpha)}g^{(\nu\beta)} + (2Z-1)g^{[\mu\alpha]}g^{[\nu\beta]} + (1-Z)g^{[\alpha\beta]}g^{[\mu\nu]}\Big)F_{\alpha\beta}] = -4\pi\sqrt{-g}J^\mu, \tag{2.7}$$

which is the inhomogeneous Maxwell equation in the presence of the nonsymmetric field. The homogeneous equation is, from $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$,

$$F_{\mu\nu,\alpha} + F_{\alpha\mu,\nu} + F_{\nu\alpha,\mu} = 0, \tag{2.8}$$

which can be indicated by $F_{[\mu\nu,\alpha]} = 0$.

III. The field of a charged particle

The gravitational vacuum field equations of the NGT are[2]

$$g_{\mu\nu,\sigma} - g_{\mu\rho}\Gamma^\rho_{\sigma\nu} - g_{\rho\nu}\Gamma^\rho_{\nu\sigma} = 0, \tag{3.1}$$

$$(\sqrt{-g}g^{[\mu\nu]})_{,\nu} = 0, \tag{3.2}$$

$$R_{(\mu\nu)}(\Gamma) = 8\pi E_{(\mu\nu)}, \tag{3.3}$$

$$R_{[\mu\nu,\sigma]}(\Gamma) = 8\pi E_{[\mu\nu,\sigma]}, \tag{3.4}$$

where

$$R_{\mu\nu}(\Gamma) = \Gamma^\beta_{\mu\nu,\beta} - \frac{1}{2}(\Gamma^\beta_{(\mu\beta),\nu} + \Gamma^\beta_{(\nu\beta),\mu}) - \Gamma^\beta_{\alpha\nu}\Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{(\alpha\beta)}, \tag{3.5}$$

For a static, spherically symmetric field $g_{\mu\nu}$, in spherical polar coordinates, has the form

$$g_{00}=\gamma(r)\quad,\quad g_{11}=-\alpha(r),$$
$$g_{22}=-r^2\quad,\quad g_{33}=-r^2\sin^2\Theta,$$
$$g_{01}=-w(r)=-g_{10},\tag{3.6}$$

and all other components equal to zero.
From (3.6) the inverse matrix non-zero elements are

$$g^{00}=\frac{\alpha}{\alpha\gamma-w^2}\quad,\quad g^{11}=-\frac{\gamma}{\alpha\gamma-w^2}\quad,$$
$$g^{22}=-\frac{1}{r^2}\quad,\quad g^{33}=-\frac{1}{r^2\sin^2\Theta}\quad,$$
$$g^{01}=\frac{w}{\alpha\gamma-w^2}=-g^{10}.\tag{3.7}$$

The solution of (3.2) is $wr^2(\alpha\gamma-w^2)^{-\frac{1}{2}}=\ell^2$, where ℓ^2, which is called $i\ell^2$ in Ref.[2] but ℓ^2 later in Ref.[6], is a constant, the conserved fermionic charge. Then,

$$w^2=\alpha\gamma\frac{\ell^4}{\ell^4+r^4}.\tag{3.8}$$

As $g=(w^2-\alpha\gamma)r^4\sin^2\Theta$ and $\gamma=-\alpha\gamma r^4\sin^2\Theta$, we get from (2.1),

$$f=\sqrt{1-\frac{w^2}{\alpha\gamma}}=\sqrt{1-\frac{g^{01}g^{10}}{g^{00}g^{11}}}.\tag{3.9}$$

The electric field is $E(r)=F_{01}$. Then, outside the source, Eq.(2.7) yield, for $\mu=0$,

$$\partial_r\left(\frac{r^2E}{\sqrt{\alpha\gamma}}\right)=0,\tag{3.10}$$

independently of Z. Upon integration

$$E=\frac{Q}{r^2}\sqrt{\alpha\gamma},\tag{3.11}$$

where the constant of integration has been put equal to the charge of the particle to reproduce the usual Reissner-Nordström [7] result when $\ell^2=0$ which implies, from the equation (3.18) below, $\alpha\gamma=1$. From (2.5) and (3.11) we obtain the following non-zero components of $E_{\mu\nu}$:

$$4\pi E_{00}=\frac{1}{2}\frac{Q^2}{r^4}\frac{\gamma}{\sqrt{1-\frac{w^2}{\alpha\gamma}}}\left(1+\frac{w^2}{\alpha\gamma}\right)\tag{3.12}$$

$$E_{11}=-\frac{\alpha}{\gamma}E_{00}\tag{3.13}$$

$$4\pi E_{22}=\frac{Q^2}{2r^2}\frac{1}{\sqrt{1-\frac{w^2}{\alpha\gamma}}}\tag{3.14}$$

$$E_{33}=\sin^2\Theta\ E_{22}\tag{3.15}$$

$$4\pi E_{[01]}=\frac{Q^2}{r^4\frac{w}{\sqrt{1-\frac{w^2}{\alpha\gamma}}}}.\tag{3.16}$$

The energy-momentum tensor is independent of Z and, therefore, the same will occur for the gravitational field equations. From now on the calculation proceeds as in Ref.[2]. A calculation of $R_{\mu\nu}$ has been presented before[6]. Since the only non-zero component of $R_{[\mu\nu]}$ is $R_{[10]}$, Eq.(3.4) is identically satisfied. As in [2] we obtain, from $\alpha^{-1}R_{00}+\gamma^{-1}R_{11}=0$,

$$\frac{\alpha'}{\alpha}+\frac{\gamma'}{\gamma}+\frac{4}{r}\frac{\ell^4}{\ell^4+r^4}=0,\tag{3.17}$$

which integrates to

$$\alpha\gamma=\frac{l^4+r^4}{r^4},\tag{3.18}$$

where the constant of integration has been chosen in such a way that $\alpha\gamma$ becomes equal to the Reissner-Nordström value, one, when $\ell^2=0$.
From (3.8) and (3.18) we obtain

$$w=\frac{\ell^2}{r^2}.\tag{3.19}$$

The R_{22} equation gives

$$\left(\frac{r}{\alpha}\right)'-1+\frac{r}{2\alpha}\left(\frac{\alpha'}{\alpha}+\frac{\gamma'}{\gamma}+\frac{4\ell^4}{r(\ell^4+r^4)}\right)=-\frac{Q^2}{r^2}\frac{1}{\left(1-\frac{w^2}{\alpha\gamma}\right)^{\frac{1}{2}}}.\tag{3.20}$$

This differs from the corresponding equation of Ref.[2] by the presence of the inverse square-root factor in the right-hand side. Using (3.8) and (3.17), Eq.(3.20) yields the following equation for α,

$$\left(\frac{r}{\alpha}\right)' - 1 = -Q^2 \frac{(r^4+\ell^4)^{\frac{1}{2}}}{r^4}. \qquad (3.21)$$

Choosing the constant of integration in such a way that the Reissner-Nordström result is obtained when $\ell^2 = 0$, we get

$$\frac{1}{\alpha} = 1 - \frac{2m}{r} - \frac{Q^2}{r} f(r), \qquad (3.22)$$

where

$$f(r) = \int \frac{(\ell^4+r^4)^{\frac{1}{2}}}{r^4} dr. \qquad (3.23)$$

Then, from (3.18) we get

$$\gamma = (1+\frac{\ell^4}{r^4})(1-\frac{2m}{r}-\frac{Q^2}{r}f(r)). \qquad (3.24)$$

For large values of r, $r \gg \ell$, we have $f(r) \to -r^{-1}$ and the solutions (3.22) and (3.24) go into the one obtained in Ref.[2]. For small values of r, $f(r) \to -\ell^2 r^{-3}$ and, therefore,

$$\frac{1}{\alpha} = 1 - \frac{2m}{r} + \frac{Q^2\ell^2}{3r^4}, \qquad (3.25a)$$

$$\gamma = (1+\frac{\ell^4}{r^4})(1-\frac{2m}{r}+\frac{Q^2\ell^2}{3r^4}). \quad (r \ll \ell) \qquad (3.25b)$$

The electric field is, from (3.11) and (3.18),

$$E = \frac{Q}{r^2}\left(1+\frac{\ell^4}{r^4}\right)^{\frac{1}{2}} \qquad (3.26)$$

For large values of r the electric field becomes the usual Coulomb field but for small values of r it increases as r^{-4}.

IV. Conclusions

By adopting a non-minimal coupling that ensures the validity of the equivalence principle [4] we have derived the field equations of Moffat's [1,6] nonsymmetric gravitational field when electromagnetic fields are present. The above mentioned non-minimal coupling contains one free parameter, Z. However, we have shown that for a static spherically symmetric field the electromagnetic equation is idependent of Z and, as the electromagnetic energy-momentum tensor also does not depend on Z the same will occur for the gravitational field equations. The solution of the field equations for the case of charged point particle is obtained. Apart from de strong deviation of the metric tensor from the usual Reissner-Nordström (R -N) solution, the electric field departs strongly from the Coulomb field value obtained in the R-N case, to reach it approaches only at large distances. At small distances it behaves as r^{-4}.

References

[1] Moffat, J.W. (1979). Phys.Rev.D **19**, 3554.

[2] Moffat, J.W. (1979). Phys.Rev.D **19**, 3562.

[3] Will, C.M. (1989). Phys.Rev.Lett. **62**, 369.

[4] Mann, R.B., Palmer, J.H. and Moffat, J.W. (1989). Phys.Rev.Lett. **62**, 2765.

[5] Lightman, A.P. and Lee, D.L. (1973). Phys.Rev.D **8**, 364

[6] Moffat, J.W. (1987). Phys.Rev.D **35**, 3733.

[7] Reissner, H. (1916). Ann.Phys. (Leipzig) **50**, 106. Nordström, G. (1918). K.Ned.Akad.Wet.Versl. Gewone Vergad . Afd.Natuurkd **20**, 1238.