# A Thermo-Hydrodynamic Theory Based on Informational Statistical Thermodynamics 

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#### Abstract

We consider the hydrodynamic description of a fluid of particles in the context of the classical approach to the Nonequilibrium Statistical Operator Method. It is based on the information entropy ensemble of Predictive Statistical Mechanics, and its accompanying Informational Statistical Thermodynamics. We start with a description of the macroscopic state of the system in terms of single- and two-particle reduced dynamic density functions in phase space, and the accompanying Lagrange multipliers (intensive nonequilibrium thermodynamic variables) that the method introduces. In terms of this basic set of dynamical variables we derive the equations of evolution for the mass, momentum, and energy densities, as well a the continuity equation for the informational entropy. It is shown how these equations are to be restricted in order to recover the results of classical hydrodynamics (based on linear irreversible thermodynamics), as well as a Gibbs relation defining local equilibrium. The differences between the generalized formalism and this classical limiting case are discussed.


## I. Introduction

Equilibrium Statistical Mechanics is well known to provide microscopic foundations to equilibrium Thermodynamics and so it is expected that Nonequilibrium Statistical Mechanics should play the same role in the case of the Thermodynamics of irreversible processes. This is the case not only in the linear regime (the so called Onsager's regime) but also in the nonlinear regime. The latter one is today of great interest because of the implicit possibility to describe complex behavior in dynamical systems. That is, dealing with
an unexpected, rich, and, in a certain sense, surprising behavior of the system, that is completely absent in the linear regime. This question is encompassed in the emerging theory of complexity [1, 2]. The role of irreversible dissipative evolution in open nonlinear systems away from equilibrium leading to complex behavior of a self-organized, coherent, and functional character was pointed out since the middle of this century by Ilya Prigogine and the Brussels' School, and the concept of (self-organized-synergetic) dissipative structures was introduced, a pioneering concept in the theory of complexity (see for example references [3-9]).

Linear nonequilibrium thermodynamics and statistical mechanics received a great deal of attention from the decade of the thirties onwards. It was mainly based on the transport theory originated in Boltzmann's work during last century, being largely used in solid state theory in conjunction with a quasi-particle approach (e.g. see reference [10]), having its culmination in the decade of the fifties with the fluctuation-dissipation theorem; Kubo's transport theory; and the method of the doubletime thermodynamic Green functions devised by Bogoliubov and Tyablikov (e.g. Ch. III in reference [11]).

Clearly, to a certain extent due to the increasing pressure put forward by recent noticeable developments of experimental techniques and technological advances in the area of physical-chemistry, as well as in biology, (see for example reference [12]) scientific research in the nonlinear regime expanded considerably. It is worth mentioning the statement of Ryogo Kubo in the opening address of the Oji Seminar [13] in 1978: "Statistical Mechanics of nonlinear nonequilibrium phenomena is just in its infancy [and] further progress can only be hoped by close cooperation with experiment." Since then some progress has been achieved.

We recall that the purpose of Statistical Mechanics of systems away from equilibrium is to determine their thermodynamic properties and the evolution in time of macroscopic observables of such systems in terms of the dynamical laws which govern the motion of their constitutive elements. The analysis of nonequilibrium systems presents far greater difficulties than those faced in the theory of equilibrium systems. This is mainly due to the fact that a non-equilibrium state is not uniquely defined and thus a more detailed discussion is necessary to determine the temporal dependence of measurable properties, and to calculate the time- and spacedependent transport coefficients associated with the irreversible processes that take place in these systems. It has been stated [14] that the basic goals of nonequilibrium statistical mechanics are: (i) to derive transport equations and to understand their structure; (ii) to understand how the approach to equilibrium occurs in closed and open systems; (iii) to study the properties of steady states; and (iv) to calculate the instantaneous values and the temporal evolution of the physical quantities which specify the macroscopic state of the system.

Nonequilibrium statistical mechanics has typically followed two directions: (1) The kinetic theory of dilute gases, where starting with a few, albeit drastic, hypotheses one obtains a description of how these sys-
tems evolve and approach equilibrium. The best example is Boltzmann's celebrated transport theory and $\mathcal{H}$ theorem. An extension of these ideas to dense systems follows several paths like, for example, the construction of a generalized theory of kinetic equations [15], and the equations of the BBGKY hierarchy [16]; (2) A generalization of the Brownian motion, where the complicated dynamic equations - the generalized Newton-Langevin equations - that follow from the laws of Mechanics are accompanied by statistical assumptions. Belonging to this approach are, for example the linear response theory in which the transport coefficients functions are given by the correlation functions due to Mori, Green, and Kubo [17] and the master equation method [18], first suggested by Pauli and later developed by van Kampen and others.

The approaches used to develop a theory encompassing the programme described by items (i) to (iv) stated above have been classified by Zwanzig [14] as: (a) Intuitive techniques; (b) Techniques based on the generalization of the kinetic theory of gases; (c) Techniques based on the theory of stochastic processes; (d) Expansions from an initial equilibrium ensemble; (e) Generalization of Gibbs' ensemble algorithm.

Within the last class of possibilities, namely item (e), the so called Nonequilibrium Statistical Operator Method (NSOM) is considered [14] to have an appealing structure and seems to be a very effective technique to deal with a large class of experimental situations. This formalism has been formulated by several authors, either using heuristic arguments [11,19-22] or projection operator techniques [23, 24]. These approaches can be brought together under a unifying variational method [25, 26], which appears to be encompassed in Jaynes' Predictive Statistical Physics [27] based on Bayesian inference techniques and the method of maximization of informational entropy (MaxEnt) [28]. We will call this formalism the MaxEnt-NSOM.

As mentioned above, the MaxEnt-NSOM seems to provide a formalism of very large scope to deal with dynamical systems arbitrarily away from equilibrium, for which it yields a generalized response function and scattering theories [25], as well as a nonlinear quantum transport theory of wide scope in applications [25, 29].

On the side of phenomenological nonequilibrium thermodynamics, it can be said that the problem has remained more or less stagnant as an open subject beyond the Onsager's regime, referred to as classical irreversible thermodynamics (CIT) or linear irreversible
thermodynamics LIT [30]. CIT is known to have drawbacks at the practical and conceptual levels [31]. New approaches are available to go beyond the domain of validity of CIT; two of them are Rational Thermodynamics, and Extended Irreversible Thermodynamics (EIT), the first one pioneered by Truesdell [32] and the second by Müller, Nettleton and others [31,33-39]. It has been shown that EIT admits kinetic foundations on the basis of Grad's moments approach to Boltzmann transport theory [37] and also in terms of the use of the maximum entropy formalism to obtain time-evolution equations for the steady-state conduction of heat in dense fluids [38], and a generalized Grad-type formalism with altered thermodynamic forces [37]. [See however GarciaColin and Veloso where the original Onsager formalism is shown to stem from Grad's method [37]]. These results can be encompassed and extended in the realm of MaxEnt-NSOM to provide what can be called an Informational Statistical Thermodynamics (IST) [40]. We notice that IST contains a generalization of Gibbs equation, Prigogine's minimum entropy production law (in the linear regime), and in the nonlinear regime Glansdorff-Prigogine universal criterion for evolution and (in)stability criterion [25], as well as generalized Maxwell-Cattaneo-Vernotte-like equations nonlocal in space and time and nolinear in the fluxes [41]. Further, a generalized Boltzmann formalism within the classical approach to the MaxEnt-NSOM has been described elsewhere [42]. It is therefore a natural extension of our work to consider the construction of a MaxEnt-NSOM hydrodynamic theory, what we attempt in this paper.

For a long time (classical) hydrodynamics has been based on CIT. As noted above CIT has shortcomings, that lead to the failure for an appropriate description of experiment at high frequencies and/or short wavelengths. This classical hydrodynamics yields equations of the so called Newton-Navier-Stokes-Fourier type, basically the conservation laws of mass and energy, complemented with auxiliary relations, namely the linear constitutive equations - that are simply proportionality relations between fluxes and forces. The latter are here defined as the gradients of concentration and temperature, that lead to Fick and Fourier diffusion laws respectively. Another difficulty inherent to classical hydrodynamics obtained through CIT arises from the fact that such linear laws lead to parabolic partial differential equations of evolution that contain the uncomfortable fact of propagation of the corresponding perturbation with infinite velocity.

Several attempts have been made in order to improve upon this situation (see for example references [43-47]). Recently, new approaches have been introduced to supersede CIT, like those of references [35] and [48]. These formalisms allow to explore the thermodynamic implications and consequences of the first deviations of the constitutive equations with respect to the classical ones at moderately high frequencies and long wavelengths. Microscopic approaches have also been used based upon the idea of retaining the form of the Navier-Stokes equations, namely the conservation laws and constitutive equations, but the transport coefficients depend on nonlocal in space and retardation effects. This is referred to as generalized hydrodynamics [49]. Microscopically this scheme stems out of Mori's formalism [50] in which the generalized NewtonLangevin equations obtained through a Fourier-Laplace transform of reduced versions of Liouville's equation leads to the formula for the dynamic structure factor, fundamental in the interpretation of scattering experiments [43, 49, 51]. Other kinetic approaches have been used to obtain a nonlocal generalized hydrodynamics with memory, one in the form of a dynamical generalization of the Ornstein-Zernike theory on the basis of the nonequilibrium statistical operator method [52, 53], and another based on Grad's moments approach[54].

In spite of the limitations associated to classical hydrodynamics, one can recognize that the Navier-Stokes equations constitute a set of equations which allow for a satisfactory description of a large variety of experimental situations concerning the space and temporal behavior of a fluid. However, besides the limitations referred to above, certain problems remain associated with the derivation of these equations from first principles, that is from Hamilton's equations describing the behavior of atoms and molecules in the classical case. This is of relevance in the sense that the absence of a satisfactory solution of this question impairs the gain of insight in the search for microscopic-dynamic descriptions beyond the domain of their validity [55]. Consequently, we address here this point, that is, starting at the classical level with IST we recover as a particular case the limit of classical hydrodynamics bringing into evidence the restrictions that need be imposed on the more general theory.

Resorting to the classical approach to the MaxEntNSOM, and its accompanying IST, briefly reviewed in section II, we introduce a truncated description for the macroscopic state of the system in terms of the
single- and two-particle distribution functions, and the accompanying Lagrange multipliers (intensive nonequilibrium thermodynamic variables) that the variational approach to the method defines. In terms of these distributions we write the equations of evolution for the mass, momentum, energy and entropy densities. This is done in section III, where it is shown how these equations are to be restricted in order to recover classical hydrodynamics and a Gibbs relation defining local equilibrium. The effect of the deviations from this limiting case are pointed out. In last section IV we summarize the results and comment upon them.

## II. The nonequilibrium statistical distribution

The first, and fundamental, step in NSOM is the choice of the basic set of variables deemed appropriate for the characterization of the macroscopic state of the system. This involves a contracted description in terms of, say, the dynamical quantities $P_{1}(\Gamma), P_{2}(\Gamma), \ldots$, $P_{n}(\Gamma)$, where $\Gamma$ is a point in phase space characterizing the state of the system at the microscopic mechanical level. The NSOM statistical distribution is a functional of this set of quantities, to be called $\varrho\left(\left\{P_{j}(\Gamma)\right\} \mid t\right)$, or, for short, $\varrho(\Gamma \mid t)$. The macroscopic state is characterized by a point in a thermodynamic state space, composed, at time $t$, of the set of macrovariables $Q_{1}(t)$, $Q_{2}(t), \ldots Q_{n}(t)$, which are the average values of the $P_{j}$, i.e. $Q_{j}(t)=\operatorname{Tr}\left\{P_{j} \varrho(\Gamma ; t)\right\}$. The choice of the basic variables is assisted by Bogoliubov's procedure of contraction of description based on a hierarchy of relaxation times [56], and - related to it - the ideas set forward, among others, by Mori [17], Zubarev [11], and Peletminskii [21, 22]. Here one introduces a separation of the total Hamiltonian into two parts, namely

$$
\begin{equation*}
H=H_{o}+H^{\prime} \tag{1}
\end{equation*}
$$

where $H_{o}$ is a "relevant" (or secular) part composed of the kinetic energies of the free subsystems and some of the interactions, namely those strong enough to have associated very short relaxation times (meaning those much smaller than the characteristic time scale of the experiment), and possessing certain symmetry properties. The other term, $H^{\prime}$, contains the interactions related to long-time relaxation mechanisms. The symmetry characteristics of the strong interactions depend on the problem under consideration: The required symmetry - to be called Zubarev-Peletminskii symmetry condition - is that

$$
\begin{equation*}
\left\{P_{j}, H_{o}\right\}=\sum_{k=1}^{n} \alpha_{j k} P_{k} \tag{2}
\end{equation*}
$$

where the left hand side stands for Poisson's bracket and $\alpha_{j k}$ are c-numbers determined by $H_{o}$. It should be noticed that quantities $P_{j}$ can be dependent on the space variable (densities), and in the classical description also on a momentum variable, namely, in general $P_{j}(\vec{r}, \vec{p} \mid \Gamma)$; in such case the quantities $\alpha$ also involve differential operators, as it is the case treated in next section.

Equation (2) provides a closure condition for the choice of the basic set of variables in a step by step procedure: First the secular part $H_{o}$ is adequately chosen in each particular problem under consideration. Second, one introduces a few dynamical variables $P$ deemed relevant for the description of the system under observation, and next Poisson's brackets with $H_{o}$ are calculated. The dynamical variables - different from those already introduced - that appear in the linear combination on the right of Eq. (2) are incorporated to the basic set; the procedure is repeated until a closure is attained. The first two references [40] relate this procedure to the question of choice of the basic variables in phenomenological irreversible thermodynamics, and an example is discussed. In practice one usually requires the introduction of an appropriate truncation along the chain in the procedure just described, hence resorting to approximations. It ought to be noticed that Eq. (2) also encompasses the case of quantities $P$ such that have associated null coefficients $\alpha$, namely, the case of constants of motion under the dynamics generated by $H_{0}$. Accordingly they are as expected, acceptable basic variables, and $H_{o}$ itself falls under this condition.

Assuming that the basic set $\left\{P_{j}(\Gamma)\right\}$ is given, the nonequilibrium statistical distribution (NSD) is constructed, within the context of Jaynes' Predictive Statistical Mechanics [27], using the principle of maximization of statistical-informational entropy, with memory and an ad hoc hypothesis [25, 42]. This is done first introducing Gibbs (fine-grained) statistical entropy

$$
\begin{equation*}
S_{G}(t)=-\int d \Gamma \varrho(\Gamma \mid t) \ln \varrho(\Gamma \mid t) \tag{3}
\end{equation*}
$$

with $\varrho$ defined in the interval $\left(t_{o}, t\right)$, where $t_{o}$ is the initial time of preparation of the sample and $t$ the time when a measurement is performed, and normalized at all times, i.e.

$$
\begin{equation*}
\int d \Gamma \varrho\left(\Gamma \mid t^{\prime}\right)=1 \tag{4}
\end{equation*}
$$

for $t_{0} \leq t^{\prime} \leq t$. Next, Gibbs entropy of Eq. (3) is maximized subjected to the constraints

$$
\begin{equation*}
Q_{j}\left(\vec{r}, \vec{p} ; t^{\prime}\right)=\int d \Gamma \varrho\left(\Gamma \mid t^{\prime}\right) P_{j}(\vec{r}, \vec{p} \mid \Gamma) ; j=1,2, \ldots, n \tag{5}
\end{equation*}
$$

again for $t_{0} \leq t^{\prime} \leq t$. Using Lagrange's method, and making a particular ad hoc hypothesis on the form of the Lagrange multipliers that the method introduces [25, 42], we obtain that

$$
\begin{equation*}
\varrho_{w}(\Gamma \mid t)=\exp \left\{\int_{t_{0}}^{t} d t^{\prime} w\left(t, t^{\prime}\right) \ln \bar{\varrho}\left(\Gamma \mid t^{\prime}, t^{\prime}-t\right)\right\} \tag{6}
\end{equation*}
$$

where the auxiliary NSD $\bar{\varrho}$ given by

$$
\begin{equation*}
\bar{\varrho}\left(\Gamma \mid t_{1}, t_{2}\right)=\exp \left\{-\phi\left(t_{1}\right)-\sum_{j=1}^{n} \int d^{3} r \int d^{3} p F_{j}\left(\vec{r}, \vec{p} ; t_{1}\right) P_{j}\left(\vec{r}, \vec{p} \mid \Gamma, t_{2}\right)\right\} \tag{7}
\end{equation*}
$$

has been introduced (see an alternative discussion of the process in [26]). In Eq. (7), $t_{1}$ refers to the time dependence of the intensive thermodynamic variables (arising from the Lagrange multipliers) $\phi$ and $F_{j}$ and $t_{2}$ to the time dependence of quantities governed by Hamilton equations of motion. In Eq. (6) $w\left(t, t^{\prime}\right)$ is a weight function with well defined properties [25, 42] that allows (1) to introduce the set of variables $F_{j}(t)$ that have the role of intensive variables thermodynamically conjugated to variables $Q_{j}$ in order to generate a complete connection with phenomenological irreversible thermodynamics, and (2) to fix an initial condition from which the irreversible evolution of the macrostate of the system proceeds. Finally, since $\varrho_{w}$ and $\bar{\varrho}$ define at each time $t$ the same macrostate, we have that

$$
\begin{equation*}
Q_{j}(\vec{r}, \vec{p} ; t)=\int d \Gamma \varrho_{w}(\Gamma \mid t) P_{j}(\vec{r}, \vec{p} \mid t)=\int d \Gamma \bar{\varrho}(\Gamma \mid t, 0) P_{j}(\vec{r}, \vec{p} \mid \Gamma), \tag{8}
\end{equation*}
$$

which ensures the normalization of $\bar{\varrho}$ once $\varrho_{w}$ is normalized, namely

$$
\begin{equation*}
\phi(t)=\ln \int d \Gamma \exp \left\{-\sum_{j=1}^{n} \int d^{3} r \int d^{3} p F_{j}\left(\vec{r}, \vec{p} ; t_{1}\right) P_{j}(\vec{r}, \vec{p} \mid \Gamma)\right\} \tag{9}
\end{equation*}
$$

and, together with itens (1) and (2) above, define the Lagrange multipliers $F_{j}$ as intensive nonequilibrium thermodynamic variables conjugated to the basic ones. Further, taking the logarithm in Eq. (6) and integrating by parts in time, we find that

$$
\begin{align*}
\ln \varrho_{\varepsilon}(\Gamma \mid t)= & \ln \bar{\varrho}(\Gamma \mid t, 0)- \\
& -\int_{-\infty}^{t} d t^{\prime} e^{\varepsilon\left(t^{\prime}-t\right)} \frac{d}{d t^{\prime}} \ln \bar{\varrho}\left(\Gamma \mid t^{\prime}, t^{\prime}-t\right) \tag{10}
\end{align*}
$$

where have explicitly used Zubarev's approach to NSOM (and we call $\varrho_{\varepsilon}$ the corresponding NSD), that is, taking $w\left(t, t^{\prime}\right)=\varepsilon e^{\varepsilon\left(t^{\prime}-t\right)}, t_{o} \rightarrow-\infty$, and $\varepsilon$ goes to zero after the trace operation in the calculation of averages has been performed [11]. Clearly, Eq. (10) tells us that the initial (at $t_{o} \rightarrow-\infty$ ) condition is

$$
\begin{equation*}
\varrho_{\varepsilon}\left(\Gamma \mid t_{o}\right)=\bar{\varrho}\left(\Gamma \mid t_{0}, 0\right), \tag{11}
\end{equation*}
$$

from which the irreversible evolution of the macrostate of the system [57] follows because of the properties of the weight function $w$. This fact implies a condition that mimics Prigogine's principle of dynamic condition for dissipativity [58].

Equation (10) allows to write the NSD as a sum of two terms, namely

$$
\begin{equation*}
\varrho_{\varepsilon}(\Gamma \mid t)=\bar{\varrho}(\Gamma \mid t, 0)+\varrho_{\varepsilon}^{\prime}(\Gamma \mid t) \tag{12}
\end{equation*}
$$

where $\bar{\varrho}$ is given in Eq. (7) and

$$
\begin{equation*}
\varrho_{\varepsilon}^{\prime}(\Gamma \mid t)=\sum_{k=1}^{\infty} \frac{1}{k!}\left[\int_{-\infty}^{t} d t^{\prime} e^{\varepsilon\left(t^{\prime}-t\right)} \widehat{\sigma}\left(\Gamma \mid t^{\prime}, t^{\prime}-t\right)\right]^{k} \bar{\varrho}(\Gamma \mid t, 0) \tag{13}
\end{equation*}
$$

with

$$
\begin{equation*}
\widehat{\sigma}\left(\Gamma \mid t^{\prime}, t^{\prime}-t\right)=-\frac{d}{d t^{\prime}} \ln \bar{\varrho}\left(\Gamma \mid t^{\prime}, t^{\prime}-t\right) \tag{14}
\end{equation*}
$$

Equation (12) indicates that the NSD is composed of an instantaneous ("frozen" or coarse-grained) generalized Gibbsian distribution $\bar{\varrho}$, which also defines the instantaneous values of the averages of the basic variables, [Cf. Eq. (8)], plus a deviation that accounts for the microscopic processes that produce dissipative effects in the system [57].

The MaxEnt-NSD thus obtained can be shown to satisfy a Liouville equation with a so called BoltzmannPrigogine symmetry, namely the presence of infinitesimal sources that breaks its otherwise time-reversal symmetry, i.e.

$$
\begin{equation*}
\left[\frac{\partial}{\partial t}+i \mathcal{L}\right] \ln \varrho_{\varepsilon}(\Gamma \mid t)=R_{\varepsilon}(t) \ln \varrho_{\varepsilon}(\Gamma \mid t) \tag{15}
\end{equation*}
$$

where

$$
\begin{equation*}
R_{\varepsilon}(t) \ln \varrho_{\varepsilon}(\Gamma \mid t)=-\varepsilon\left[\ln \varrho_{\varepsilon}(\Gamma \mid t)-\ln \bar{\varrho}(\Gamma \mid t, 0)\right] \tag{16}
\end{equation*}
$$

and $\mathcal{L}$ is the Liouville operator that governs the systems' dynamics. We stress that the weight function $w$ is taken as going to zero $(\varepsilon \rightarrow+0)$ after the operation of average is performed [25], that is, the NSOM implies in calculations of quasi-averages in Bogoliubov's sense [59]. In the present case Bogoliubov's procedure breaks the time-reversal symmetry of Liouville equation by disregarding the sub-group of advanced solutions.

Differentiation in time of Eqs. (8) leads to the evolution equations

$$
\begin{equation*}
\frac{\partial}{\partial t} Q_{j}(\vec{r}, \vec{p} ; t)=\int d \Gamma\left\{P_{j}(\vec{r}, \vec{p} \mid \Gamma), H(\Gamma)\right\} \varrho_{\varepsilon}(\Gamma \mid t) \tag{17}
\end{equation*}
$$

Using Eqs. (1), (2) ,(8) and (12) we can write that,

$$
\begin{equation*}
\frac{\partial}{\partial t} Q_{j}(\vec{r}, \vec{p} ; t)=J_{j}^{(0)}(\vec{r}, \vec{p} ; t)+J_{j}^{(1)}(\vec{r}, \vec{p} ; t)+\mathcal{J}_{j}(\vec{r}, \vec{p} ; t) \tag{18}
\end{equation*}
$$

where

$$
\begin{align*}
J_{j}^{(0)}(\vec{r}, \vec{p} ; t) & =\int d \Gamma\left\{P_{j}(\vec{r}, \vec{p} \mid \Gamma), H_{o}(\Gamma)\right\} \bar{\varrho}(\Gamma \mid t, 0),  \tag{19}\\
J_{j}^{(1)}(\vec{r}, \vec{p} ; t) & =\int d \Gamma\left\{P_{j}(\vec{r}, \vec{p} \mid \Gamma), H^{\prime}(\Gamma)\right\} \bar{\varrho}(\Gamma \mid t, 0),  \tag{20}\\
\mathcal{J}_{j}(\vec{r}, \vec{p} ; t) & =\int d \Gamma\left\{P_{j}(\vec{r}, \vec{p} \mid \Gamma), H^{\prime}(\Gamma)\right\} \varrho_{\varepsilon}^{\prime}(\Gamma \mid t) \tag{21}
\end{align*}
$$

The collision integral $\mathcal{J}_{j}$ is shown to be associated to dissipative processes, which can alternatively be written in the interesting form

$$
\begin{equation*}
\mathcal{J}_{j}(\vec{r}, \vec{p} ; t)=\int_{-\infty}^{t} d t^{\prime} e^{\varepsilon\left(t^{\prime}-t\right)}\left\{\left\{P_{j}(\vec{r}, \vec{p} \mid \Gamma), H^{\prime}(\Gamma)\right\} ; \widehat{\sigma}\left(\Gamma \mid t^{\prime}, t^{\prime}-t\right) \mid t\right\} \tag{22}
\end{equation*}
$$

where $\hat{\sigma}$ is given by Eq. (14), and we have introduced the supercorrelation function for, say, any pair of quantities $A$ and $B$ defined as

$$
\begin{equation*}
\{A(\Gamma) ; B(\Gamma) \mid t\}=\int d \Gamma A(\Gamma) Y_{\varepsilon}(\Gamma) B(\Gamma) \bar{\varrho}(\Gamma \mid t, 0) \tag{23}
\end{equation*}
$$

with

$$
\begin{equation*}
Y_{\varepsilon}(\Gamma)=1+\sum_{k=1}^{\infty} \frac{1}{k!}\left[\int_{-\infty}^{t} d t^{\prime} e^{\varepsilon\left(t^{\prime}-t\right)} \widehat{\sigma}\left(\Gamma \mid t^{\prime}, t^{\prime}-t\right)\right]^{k} \tag{24}
\end{equation*}
$$

In Eq. (22) the collision integral is then given by a supercorrelation function involving the time evolution of the basic quantity $P_{j}$ with the part of the Hamiltonian associated to the slow relaxing processes and the quantity $\hat{\sigma}$ whose average value is the MaxEnt-NSOM informational entropy production, also dependent on $H^{\prime}[25,57]$.

The collision integral of Eq. (21) becomes, because of its expression as given by Eq. (22) and the use of Eqs. (23) and (24), a series of contributions of higher and higher order in the dissipation processes that develop in the system (they consist of correlations over the auxiliary ensemble characterized by $\bar{\varrho}$ involving higher and higher powers of $\widehat{\sigma})$. Evidently, this collision integral is extremely difficult to calculate; however, it can be brought in a form that allows a more accessible and practical mathematical handling, namely [29]

$$
\begin{equation*}
\mathcal{J}_{j}(\vec{r}, \vec{p} ; t)=\sum_{n=2}^{\infty} J_{j}^{(n)}(\vec{r}, \vec{p} ; t) \tag{25}
\end{equation*}
$$

where the $J_{j}^{(n)}$ are partial contributions that are instantaneous in time and organized in increasing order $n$ in the strengths of the interactions contained in $H^{\prime}$. The lowest order, $n=2$ only, is a local in time approximation referred to as the quasi-linear theory of relaxation [60, 61]. These partial collision operators, which have an important role in the definition of the MaxEnt-NSOM entropy production function, are composed of several terms consisting of (1) the mechanical effects of collisions (in order $n$ ) averaged over the auxiliary coarse-grained ensemble, (2) terms that account for the evolution of the thermodynamic state of the system, and (3), for $n>2$, terms arising from memory effects [29]. They are of ever larger complexity with increasing $n$, but a truncation procedure may be introduced in the series of Eq. (25) up to a certain order $n$ in the interaction.

The connection of the MaxEnt-NSOM with phenomenological irreversible thermodynamics [40,57] is done introducing the NSOM entropy, or coarse-grained entropy, namely

$$
\begin{align*}
\bar{S}(t) & =-\int d \Gamma \varrho_{\varepsilon}(\Gamma \mid t) \ln \bar{\varrho}(\Gamma \mid t, 0)= \\
& =\phi(t)+\sum_{j=1}^{n} \int d^{3} r \int d^{3} p F_{j}(\vec{r}, \vec{p} ; t) Q_{j}(\vec{r}, \vec{p} ; t) \tag{26}
\end{align*}
$$

which satisfies the Pfaffian form (generalized Gibbs relation)

$$
\begin{equation*}
d \bar{S}(t)=\sum_{j=1}^{n} \int d^{3} r \int d^{3} p F_{j}(\vec{r}, \vec{p} ; t) d Q_{j}(\vec{r}, \vec{p} ; t) \tag{27}
\end{equation*}
$$

where

$$
\begin{equation*}
F_{j}(\vec{r}, \vec{p} ; t)=\delta \bar{S}(t) / \delta Q_{j}(\vec{r}, \vec{p} ; t) \tag{28}
\end{equation*}
$$

and we used that

$$
\begin{equation*}
Q_{j}(\vec{r}, \vec{p} ; t)=-\delta \phi(t) / \delta F_{j}(\vec{r}, \vec{p} ; t) \tag{29}
\end{equation*}
$$

Equations (28) and (29) define the previously mentioned conjugate property between the $Q$ and $F$ variables, being the generalization of the results in equilibrium thermodynamics, namely, they are nonequilibrium equations of state.
Here $\delta$ stands for functional derivative.

The MaxEnt-NSOM entropy production is given by

$$
\begin{equation*}
\sigma(t)=\frac{d \bar{S}(t)}{d t}=\sum_{j=1}^{n} \int d^{3} r \int d^{3} p F_{j}(\vec{r}, \vec{p} ; t) \frac{\partial}{\partial t} Q_{j}(\vec{r}, \vec{p} ; t), \tag{30}
\end{equation*}
$$

or, after introducing Eq. (18) we find that

$$
\begin{equation*}
\sigma(t)=\sum_{j=1}^{n} \int d^{3} r \int d^{3} p F_{j}(\vec{r}, \vec{p} ; t) \mathcal{J}_{j}(\vec{r}, \vec{p} ; t) \tag{31}
\end{equation*}
$$

where use was made of the fact that

$$
\begin{equation*}
\sum_{j=1}^{n} \int d^{3} r \int d^{3} p F_{j}(\vec{r}, \vec{p} ; t)\left[J_{j}^{(0)}(\vec{r}, \vec{p} ; t)+J_{j}^{(1)}(\vec{r}, \vec{p} ; t)\right]=0 \tag{32}
\end{equation*}
$$

which implies that dissipative effects are accounted for in the collision integral $\mathcal{J}$, and, therefore, by the contribution $\varrho_{\varepsilon}^{\prime}$ to the NSD $\varrho_{\varepsilon}$ of Eq. (12) [57].

Having thus briefly reviewed the fundamentals of the method we proceed next to its application to the derivation of a hydrodynamic formalism.

## III. A microscopic approach to classical hydrodynamics

Let us consider a fluid of $N$ interacting particles whose Hamiltonian is of the form of Eq. (1) with

$$
\begin{align*}
H_{o}(\Gamma)= & \int d^{3} r \int d^{3} p \frac{p^{2}}{2 m} n_{1}(\vec{r}, \vec{p} \mid \Gamma)  \tag{33}\\
H^{\prime}(\Gamma)= & \int d^{3} r \int d^{3} p w(\vec{r}) n_{1}(\vec{r}, \vec{p} \mid \Gamma)+ \\
& +\int d^{3} r \int d^{3} p \int d^{3} r^{\prime} \int d^{3} p^{\prime} V\left(\left|\vec{r}-\vec{r}^{\prime}\right|\right) n_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} \mid \Gamma\right) \tag{34}
\end{align*}
$$

with $H_{0}$ containing the kinetic energy and $H^{\prime}$ the interaction with external systems, $w$, (sources and reservoirs), as well as the interaction between particles through a central force potential $V$. Further, $n_{1}$ and $n_{2}$ are the one-particle and two-particles dynamical reduced density functions, namely

$$
\begin{align*}
n_{1}(\vec{r}, \vec{p} \mid \Gamma) & =\sum_{j=1}^{N} \delta\left(\vec{r}-\vec{r}_{j}\right) \delta\left(\vec{p}-\vec{p}_{j}\right),  \tag{35}\\
n_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p} \mid \Gamma\right) & =\sum_{k \neq j=1}^{N} \delta\left(\vec{r}-\vec{r}_{j}\right) \delta\left(\vec{p}-\vec{p}_{j}\right) \delta\left(\vec{r}-\vec{r}_{k}\right) \delta\left(\vec{p}-\vec{p}_{k}\right) \tag{36}
\end{align*}
$$

where $\vec{r}_{j}, \vec{p}_{j}$, etc. are the coordinate and momentum of the $j$-th particle.

## III. 1 The MaxEnt-NSOM Description

For the NSOM-statistical description of this system we take as basic dynamical variables the set

$$
\begin{equation*}
\left\{h_{o}(\vec{r}, \vec{p} \mid \Gamma), n_{1}(\vec{r}, \vec{p} \mid \Gamma), n_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} \mid \Gamma\right)\right\} \tag{37}
\end{equation*}
$$

that is, the kinetic energy

$$
\begin{equation*}
h_{\circ}(\vec{r}, \vec{p} \mid \Gamma)=\frac{p^{2}}{2 m} n_{1}(\vec{r}, \vec{p} \mid \Gamma) \tag{38}
\end{equation*}
$$

and $n_{1}$ and $n_{2}$ which are sufficient to write any relevant dynamical property of the system. The set of Eq. (37) satisfies Zubarev-Peletminskii closure condition of Eq. (2) since

$$
\begin{align*}
\left\{h_{o}(\vec{r}, \vec{p} \mid \Gamma), H_{o}(\Gamma)\right\} & =-\frac{1}{m} \frac{p^{2}}{2 m}(\vec{p} \cdot \nabla) n_{1}(\vec{r}, \vec{p} \mid \Gamma)  \tag{39}\\
\left\{n_{1}(\vec{r}, \vec{p} \mid \Gamma), H_{o}(\Gamma)\right\} & =-\frac{1}{m}(\vec{p} \cdot \nabla) n_{1}(\vec{r}, \vec{p} \mid \Gamma)  \tag{40}\\
\left\{n_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} \mid \Gamma\right), H_{o}(\Gamma)\right\} & =-\frac{1}{m}\left(\vec{p} \cdot \nabla+\vec{p}^{\prime} \cdot \nabla^{\prime}\right) n_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} \mid \Gamma\right) . \tag{41}
\end{align*}
$$

Therefore, in terms of this basic set of variables the auxiliary (coarse-grained) nonequilibrium statistical distribution for the system [cf. Eq.(7)] takes the form

$$
\begin{align*}
& \bar{\varrho}(\Gamma \mid t, 0)=\exp \left\{-\phi(t)-\int d^{3} r \int d^{3} p \beta(\vec{r}, t) \frac{p^{2}}{2 m} n_{1}(\vec{r}, \vec{p} \mid \Gamma)-\right. \\
& -\int d^{3} r \int d^{3} p \varphi_{1}(\vec{r}, \vec{p} ; t) n_{1}(\vec{r}, \vec{p} \mid \Gamma)- \\
& \left.-\int d^{3} r \int d^{3} p \int d^{3} r^{\prime} \int d^{3} p^{\prime} \varphi_{2}\left(\vec{r}, \vec{p}^{\prime} \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right) n_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} \mid \Gamma\right)\right\} \tag{42}
\end{align*}
$$

where, we recall, $\phi(t)$ ensures the normalization of this $\operatorname{NSD}$, and $\beta, \varphi_{1}$, and $\varphi_{2}$ are the Lagrange multipliers (intensive nonequilibrium thermodynamic variables) conjugated to the basic macrovariables, that the method introduces. These parameters are to be identified in the context of hydrodynamics later on. The basic set of macrovariables (extensive nonequilibrium thermodynamic variables) are

$$
\begin{align*}
\varepsilon_{o}(\vec{r}, \vec{p} ; t) & =\left\langle h_{o}(\vec{r}, \vec{p} \mid \Gamma) \mid t\right\rangle  \tag{43}\\
f_{1}(\vec{r}, \vec{p} ; t) & =\left\langle n_{1}(\vec{r}, \vec{p} \mid \Gamma) \mid t\right\rangle  \tag{44}\\
f_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right) & =\left\langle n_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} \mid \Gamma\right) \mid t\right\rangle \tag{45}
\end{align*}
$$

where for any basic variable $P_{j}$ of Eq. (37)

$$
\begin{equation*}
\left\langle P_{j}(\Gamma) \mid t\right\rangle=\int d \Gamma P_{j}(\Gamma) \bar{\varrho}(\Gamma \mid t, 0) \tag{46}
\end{equation*}
$$

and we recall that these average values coincide with the average values taken with the fine-grained distribution as imposed by Eq. (8) [25, 26, 42, 57].

Let us next consider the equations of evolution for the basic variables, that is Eqs. (18)-(21). First it should be noticed that $h_{o}$ is simply proportional to $f_{1}$ and so we need to derive the equations for $f_{1}$ and $f_{2}$. Using eqs. (17) and after some algebra we find that

$$
\begin{gather*}
{\left[\frac{\partial}{\partial t}+\frac{1}{m} \vec{p} \cdot \nabla+m \mathcal{F}(\vec{r}) \cdot \nabla_{\vec{p}}\right] f_{1}(\vec{r}, \vec{p} ; t)=} \\
=\int d^{3} r^{\prime} \int d^{3} p^{\prime} \nabla V\left(\left|\vec{r}-\vec{r}^{\prime}\right|\right) \cdot \nabla_{\vec{p}} f_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right)+ \\
+\mathcal{J}_{1}(\vec{r}, \vec{p} ; t),  \tag{47}\\
{\left[\frac{\partial}{\partial t}+\frac{1}{m}\left(\vec{p} \cdot \nabla+\vec{p}^{\prime} \cdot \nabla^{\prime}\right)+\left\{m \mathcal{F}(\vec{r})-\nabla V\left(\left|\vec{r}-\vec{r}^{\prime}\right|\right)\right\} \cdot \nabla_{\vec{p}}+\right.}
\end{gather*}
$$

$$
\begin{gather*}
\left.+\left\{m \mathcal{F}\left(\vec{r}^{\prime}\right)-\nabla^{\prime} V\left(\left|\vec{r}-\vec{r}^{\prime}\right|\right)\right\} \cdot \nabla_{\vec{p}}\right] f_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right) \\
=\widetilde{\mathcal{J}}_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right) \tag{48}
\end{gather*}
$$

where we have written

$$
\begin{gather*}
m \overrightarrow{\mathcal{F}}(\vec{r})=-\nabla w(\vec{r})  \tag{49}\\
\tilde{\mathcal{J}}_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right)=\int d^{3} r_{1} \int d^{3} p_{1}\left[\nabla V\left(\left|\vec{r}-\vec{r}_{1}\right|\right) \cdot \nabla_{\vec{p}}+\right. \\
\left.+\nabla^{\prime} V\left(\left|\vec{r}^{\prime}-\vec{r}_{1}\right|\right) \cdot \nabla_{\vec{p}^{\prime}}\right] f_{3}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime}, \vec{r}_{1}, \vec{p}_{1} ; t\right), \tag{50}
\end{gather*}
$$

and it is verified that

$$
\begin{equation*}
\overrightarrow{\mathcal{J}}_{1}(\vec{r}, \vec{p} ; t)=0 \tag{51}
\end{equation*}
$$

and the three-particle density distribution function has been introduced, namely

$$
\begin{equation*}
f_{3}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime}, \vec{r}_{1}, \vec{p}_{1} ; t\right)=\int d \Gamma \varrho_{w}(\Gamma \mid t) n_{3}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p} \prime, \vec{r}_{1}, \vec{p}_{1} \mid \Gamma\right) \tag{52}
\end{equation*}
$$

Before proceeding further in the direction of building a hydrodynamic approach three facts concerning the equations of evolution (47) and (48) must be emphasized. First they are not closed since they involve the three-particle distribution function on the right hand side of Eq. (48): If we incorporate $n_{3}$ as a basic variable it leads to an equation of evolution containing $n_{4}$, which in turn if incorporated as a basic variable implies another equation for $n_{5}$ and so on. Thus we would introduce a kind of $B B G K Y$ hierarchy in the realm of this statistical approach to irreversible thermodynamics. This question will be presented in a future article. Second, if in Eq. (48) we neglect the three-particle correlations, namely, we disregard the right hand side, and associated with this we take $f_{2}$ as a weakly timevarying function, solving the resulting equation for $f_{2}$ and next replacing the solution in Eq. (48), followed by a factorization of $f_{2}$ in terms of products of $f_{1}$ (Stosszahlanzatz), we recover the classical Boltzmann equation and Boltzmann's $\mathcal{H}$-theorem [42]. Third, once the basic set of variables has been chosen, in this case, $n_{1}$ and $n_{2}$, to close the system of coupled equations (47) and (48) the method requires to express $f_{3}$ and $\tilde{\mathcal{J}}_{2}$ in terms of them. This can be done resorting, for example, to a perturbation method for averages [22], as it was illustrated in previous articles [40, 42]. Let us next
consider how to construct a hydrodynamic formalism from Eqs. (47) and (48).

## III. 2 Classical Theory for a Microscopic Derivation of Balance Equations

Let us consider now the steps leading to the equations of hydrodynamics to compare them with the phenomenological classical ones. For this purpose we introduce: (i) the particle density

$$
\begin{equation*}
n(\vec{r}, t)=\int d^{3} p f_{1}(\vec{r}, \vec{p} ; t) \tag{53}
\end{equation*}
$$

(ii) the mass density

$$
\begin{equation*}
g(\vec{r}, t)=m n(\vec{r}, t) \tag{54}
\end{equation*}
$$

(iii) the mean momentum density (in terms of a drift velocity)

$$
\begin{equation*}
g(\vec{r}, t) \vec{v}(\vec{r}, t)=\int d^{3} p \vec{p} f_{1}(\vec{r}, \vec{p} ; t)=\vec{p}(\vec{r}, t) \tag{55}
\end{equation*}
$$

(iv) the mean kinetic energy density

$$
\begin{equation*}
g(\vec{r}, t) \varepsilon_{k}(\vec{r}, t)=\int d^{3} p[\vec{p}-m \vec{v}(\vec{r}, t)]^{2} f_{1}(\vec{r}, \vec{p}, t) \tag{56}
\end{equation*}
$$

(v) the mean interparticle-potential energy density

$$
\begin{gather*}
g(\vec{r}, t) \varepsilon_{v}(\vec{r}, t)= \\
=\frac{1}{2} \int d^{3} p \int d^{3} r^{\prime} \int d^{3} p^{\prime} V\left(\left|\vec{r}-\vec{r}^{\prime}\right|\right) f_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right), \tag{57}
\end{gather*}
$$

(vi) the internal energy density

$$
\begin{equation*}
u(\vec{r}, t)=\varepsilon_{k}(\vec{r}, t)+\varepsilon_{v}(\vec{r}, t)=h(\vec{r}, t) \tag{58}
\end{equation*}
$$

Resorting to the use of Eqs. (47) and (48) and following standard procedures [18] we can derive the balance equations for the quantities of Eqs. (54), (55) and (58), namely

$$
\begin{gather*}
\frac{\partial}{\partial t} g(\vec{r}, t)+\operatorname{div}[g(\vec{r}, t) \vec{v}(\vec{r}, t)]=0,  \tag{59}\\
\frac{\partial}{\partial t}[g(\vec{r}, t) \vec{v}(\vec{r}, t)]+\operatorname{div}[g(\vec{r}, t) \vec{v}(\vec{r}, t): \vec{v}(\vec{r}, t)+\widetilde{P}(\vec{r}, t)]= \\
=g(\vec{r}, t) \overrightarrow{\mathcal{F}}(\vec{r}, t)  \tag{60}\\
\frac{\partial}{\partial t}[g(\vec{r}, t) u(\vec{r}, t)]+\operatorname{div}\left[g(\vec{r}, t) u(\vec{r}, t) \vec{v}(\vec{r}, t)+\vec{I}_{q}(\vec{r}, t)\right]= \\
=\widetilde{P}(\vec{r}, t) \odot[\nabla: \vec{v}(\vec{r}, t)]+ \\
+\frac{1}{2} \int d^{3} p \int d^{3} r^{\prime} \int d^{3} p^{\prime} V\left(\left|\vec{r}-\vec{r}^{\prime}\right|\right) \tilde{\mathcal{J}}_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right) \tag{61}
\end{gather*}
$$

where $\widetilde{P}$ is the pressure tensor

$$
\begin{equation*}
\widetilde{P}(\vec{r}, t)=\widetilde{P}_{K}(\vec{r}, t)+\widetilde{P}_{V}(\vec{r}, t) \tag{62}
\end{equation*}
$$

with

$$
\begin{align*}
\widetilde{P}_{K}(\vec{r}, t)= & \frac{1}{m} \int d^{3} p[\vec{p}-m \vec{v}(\vec{r}, t)]:[\vec{p}-m \vec{v}(\vec{r}, t)] f_{1}(\vec{r}, \vec{p} ; t)  \tag{63}\\
\widetilde{P}_{V}(\vec{r}, t)= & -\frac{1}{2} \int_{0}^{1} d \lambda \int d^{3} p \int d^{3} \xi \int d^{3} p^{\prime} \frac{\vec{\xi} \cdot \vec{\xi}}{\xi} \frac{d V(\xi)}{d \xi} \times \\
& \times f_{2}\left[\vec{r}+(1-\lambda) \vec{\xi}, \vec{p}, \vec{r}-\lambda \vec{\xi}, \vec{p}^{\prime} ; t\right] \tag{64}
\end{align*}
$$

and $\vec{\xi}=\vec{r}-\vec{r}^{\prime}$; double dots stand for tensorial product of vectors and $\odot$ for fully contracted product of tensors. Moreover, we introduce the quantity

$$
\begin{equation*}
\vec{I}_{q}(\vec{r}, t)=\vec{I}_{K}(\vec{r}, t)+\vec{I}_{V 1}(\vec{r}, t)+\vec{I}_{V 2}(\vec{r}, t) \tag{65}
\end{equation*}
$$

interpreted as the heat flux, and where

$$
\begin{align*}
\vec{I}_{K}(\vec{r}, t)= & \frac{1}{2 m^{2}} \int d^{3} p[\vec{p}-m \vec{v}(\vec{r}, t)]^{2}[\vec{p}-m \vec{v}(\vec{r}, t)] f_{1}(\vec{r}, \vec{p} ; t)  \tag{66}\\
\vec{I}_{V 1}(\vec{r}, t)= & \frac{1}{2 m} \int d^{3} p \int d^{3} r^{\prime} \int d^{3} p^{\prime} V\left(\left|\vec{r}-\vec{r}^{\prime}\right|\right)[\vec{p}-m \vec{v}(\vec{r}, t)] \times \\
& \times f_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right) \tag{67}
\end{align*}
$$

$$
\begin{align*}
\vec{I}_{V 2}(\vec{r}, t)= & -\frac{1}{4 m} \int_{0}^{1} d \lambda \int d^{3} p \int d^{3} \xi \int d^{3} p^{\prime} \frac{\vec{\xi} \cdot \vec{\xi}}{\xi}[\vec{p}-m \vec{v}(\vec{r}, t)] \times \\
& \times \frac{d V(\xi)}{d \xi} f_{2}\left[\vec{r}+(1-\lambda) \vec{\xi}, \vec{p}, \vec{r}-\lambda \vec{\xi}, \vec{p}^{\prime} ; t\right] \tag{68}
\end{align*}
$$

We call the attention to the fact that the quantities above depend on $f_{1}$ and $f_{2}$, or, alternatively, on the thermodynamic variables (Lagrange multipliers) $\varphi_{1}$ and $\varphi_{2}$. The results embedded in eqs. (59)-(68) are not new. They were derived first by Kirkwood in 1946 [62] using a time smoothing approximation and later on, in 1951 by Irving and Zwanzig [63]. Here we have followed very closely the procedure of ref. [18] to obtain these equations. The main accomplishment of this treatment lies in the fact that our formalism not only allows for the derivation of the hydrodynamic, or better, the conservation equations, but also yields the thermodynamical aspects consistent with then. In fact we have an explicit form both for an entropy and an entropy production provided by the method itself so we can now proceed to study these quantities. For this purpose we introduce a NSOM entropy density of the form,

$$
\begin{equation*}
\bar{S}(t)=\int d^{3} r g(\vec{r}, t) s(\vec{r}, t) \tag{69}
\end{equation*}
$$

where $\bar{S}$ is given by Eq. (26). Further, for the present case the NSOM-entropy production of Eq. (30) is given by

$$
\begin{align*}
& \sigma(t)=\int d^{3} r \int d^{3} p\left\{\left[\beta(\vec{r}, t)\left(p^{2} / 2 m\right)+\varphi_{1}(\vec{r}, \vec{p} ; t)\right] \frac{\partial}{\partial t} f_{1}(\vec{r}, \vec{p} ; t)+\right. \\
& \left.\quad+\int d^{3} r \int d^{3} p \int d^{3} r^{\prime} \int d^{3} p^{\prime} \varphi_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right) \frac{\partial}{\partial t} f_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right)\right\} \tag{70}
\end{align*}
$$

and also from eq. (69) we may compute $\sigma$, namely,

$$
\begin{equation*}
\sigma(t)=\frac{d}{d t} \int d^{3} r g(\vec{r}, t) s(\vec{r}, t)=\int d^{3} r \frac{\partial}{\partial t}[g(\vec{r}, t) s(\vec{r}, t)] \tag{71}
\end{equation*}
$$

Hence,

$$
\begin{align*}
& \frac{\partial}{\partial t}[g(\vec{r}, t) s(\vec{r}, t)]=\int d^{3} p\left[\beta(\vec{r}, t) \frac{p^{2}}{2 m}+\varphi_{1}(\vec{r}, \vec{p} ; t)\right] \frac{\partial}{\partial t} f_{1}(\vec{r}, \vec{p} ; t)+ \\
& \quad+\int d^{3} p \int d^{3} r^{\prime} \int d^{3} p^{\prime} \varphi_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right) \frac{\partial}{\partial t} f_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right) \tag{72}
\end{align*}
$$

with $\beta, \varphi_{1}$ and $\varphi_{2}$ being the Lagrange parameters given by

$$
\begin{align*}
\delta \phi(t) / \delta \beta(\vec{r}, t) & =-\int d^{3} p \frac{p^{2}}{2 m} f_{1}(\vec{r}, \vec{p} ; t)  \tag{73}\\
\delta \phi(t) / \delta \varphi_{1}(\vec{r}, \vec{p} ; t) & =-f_{1}(\vec{r}, \vec{p} ; t)  \tag{74}\\
\delta \phi(t) / \delta \varphi_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right) & =-f_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right) \tag{75}
\end{align*}
$$

according to Eqs. (29). Let us also define

$$
\begin{equation*}
\phi(t)=-\int d^{3} r \beta(\vec{r}, t) \Omega(\vec{r}, t) \tag{76}
\end{equation*}
$$

the meaning of $\Omega(\vec{r}, t)$ to be discussed below.
In terms of these quantities we can write

$$
g(\vec{r}, t) s(\vec{r}, t)=-\beta(\vec{r}, t) \Omega(\vec{r}, t)+
$$

$$
\begin{align*}
& +\int d^{3} p\left\{\left[\beta(\vec{r}, t)\left(p^{2} / 2 m\right)+\varphi_{1}(\vec{r}, \vec{p} ; t)\right] f_{1}(\vec{r}, \vec{p} ; t)+\right. \\
& \left.+\int d^{3} r^{\prime} \int d^{3} p^{\prime} \varphi_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right) f_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right)\right\} \tag{77}
\end{align*}
$$

which is the general expression for the NSOM entropy density in the fluid. This result is important and interesting enough by itself. Indeed it shows how one can obtain an explicit expression for the entropy consistent with the conservation equations expressed in terms of the first two distribution functions.

We are now able to show that the NSOM conservation equations so far developed contain as a limiting case the usual ones arising from the local equilibrium assumption. For that purpose we compare the entropy density, given in the local equilibrium approximation of CIT, namely [18]

$$
\begin{equation*}
g_{o}(\vec{r}, t) s_{o}(\vec{r}, t)=\beta_{o}(\vec{r}, t) \frac{1}{3} \operatorname{Tr} \widetilde{P}_{o}(\vec{r}, t)+\beta_{o}(\vec{r}, t) g_{o}(\vec{r}, t)\left[u_{o}(\vec{r}, t)-\mu_{o}(\vec{r}, t)\right] \tag{78}
\end{equation*}
$$

with the entropy density given in Eq. (77). The lower index naught indicates the local equilibrium approximation and $\mu_{0}$ is the chemical potential. We can thus see that if we write

$$
\begin{gather*}
\varphi_{1}(\vec{r}, \vec{p} ; t)=\varphi_{1, o}(\vec{r}, \vec{p} ; t)+\Delta \varphi_{1}(\vec{r}, \vec{p} ; t)= \\
=\beta_{o}(\vec{r}, t)\left[\frac{1}{2} m v_{o}^{2}(\vec{r}, t)-\vec{p} \cdot \vec{v}_{o}(\vec{r}, t)-\mu_{o}(\vec{r}, t)+w(\vec{r}, t)\right]+\Delta \varphi_{1}(\vec{r}, \vec{p} ; t),  \tag{79}\\
\varphi_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right)=\varphi_{2 o}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right)+\Delta \varphi_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right)= \\
=\frac{1}{4}\left[\beta_{o}(\vec{r}, t) V\left(\left|\vec{r}-\vec{r}^{\prime}\right|\right)+\beta_{o}\left(\vec{r}^{\prime}, t\right) V\left(\left|\vec{r}^{\prime}-\vec{r}\right|\right)\right]+ \\
+\Delta \varphi_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right)  \tag{80}\\
\Omega(\vec{r}, t)=\Omega_{o}(\vec{r}, t)+\Delta \Omega(\vec{r}, t)=-\frac{1}{3} \operatorname{Tr} \widetilde{P}_{o}(\vec{r}, t)+\Delta \Omega(\vec{r}, t), \tag{81}
\end{gather*}
$$

and

$$
\begin{align*}
f_{1}(\vec{r}, \vec{p} ; t) & =f_{1, o}(\vec{r}, \vec{p} ; t)+\Delta f_{1}(\vec{r}, \vec{p} ; t)  \tag{82}\\
f_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right) & =f_{2, o}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right)+\Delta f_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right) \tag{83}
\end{align*}
$$

with $\widetilde{P}_{o}$ and $\mu_{o}$ defined in terms of $f_{1, o}$ and $f_{2, o}$, it follows that

$$
\begin{equation*}
g(\vec{r}, t) s(\vec{r}, t)=g_{0}(\vec{r}, t) s_{0}(\vec{r}, t)+\omega(\vec{r}, t) \tag{84}
\end{equation*}
$$

where $\omega(\vec{r}, t)$ includes all the terms involving $\Delta \varphi_{1}, \Delta \varphi_{2}, \Delta \Omega, \Delta f_{1}$, and $\Delta f_{2}$. Eq. (81) defines the quantity $\Omega$ of Eq. (76) as the usual contribution of local equilibrium thermodynamics plus the modifications arising out of the extensions introduced by the present theory.

Consequently, the conservation equations, from the local equilibrium approximation of LIT, result from a very particular choice of the Lagrange multipliers of MaxEnt-NSOM. Since classical hydrodynamics follows from these conservation equations when coupled to the well known constitutive equations, we may conclude that the correction terms involved in $\Delta f_{1}, \Delta f_{2}$ and similar quantities will lead to a more general version of hydrodynamics which will demand constitutive laws different from the linear ones. This subject is discussed in a forthcoming paper. Returning to the original expression for the auxiliary (coarse grained) NSD of Eq. (42), the classical conservation equations amounts then to the choice

$$
\bar{\varrho}_{C H D}(t, 0)=\exp \left\{\int d^{3} r \beta_{o}(\vec{r}, t) \frac{1}{3} \operatorname{Tr} \widetilde{P}_{o}(\vec{r}, t)-\right.
$$

$$
\begin{gather*}
-\int d^{3} r \int d^{3} p \beta_{o}(\vec{r}, t)\left[\frac{1}{2 m}\left[\vec{p}-\vec{v}_{o}(\vec{r}, t)\right]^{2}+\mu_{o}(\vec{r}, t)-\omega(\vec{r}, t)\right] n_{1}(\vec{r}, \vec{p} \mid \Gamma)- \\
-\frac{1}{4} \int d^{3} r \int d^{3} p \int d^{3} r^{\prime} \int d^{3} p^{\prime}\left[\beta_{o}(\vec{r}, t) V\left(\left|\vec{r}-\vec{r}^{\prime}\right|\right)+\right. \\
\left.\left.+\beta_{o}\left(\vec{r}^{\prime}, t\right) V\left(\left|\vec{r}^{\prime}-\vec{r}\right|\right)\right] n_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} \mid \Gamma\right)\right\} \tag{85}
\end{gather*}
$$

which tantamounts to choosing as the basic set of variables the truncated one that correponds to the set of macrovariables composed of

$$
\begin{equation*}
\{n(\vec{r}, t), \vec{p}(\vec{r}, t), h(\vec{r}, t)\} \tag{86}
\end{equation*}
$$

that is, the density, momentum density, and energy density, respectively. The thermodynamically conjugated variables have been indecated by $-\beta_{o}(\vec{r}, t) \mu_{o}(\vec{r}, t), \quad-\beta_{o}(\vec{r}, t) \vec{v}_{o}(\vec{r}, t), \quad \beta_{o}(\vec{r}, t)$, which can be associated to a reciprocal local temperature $\beta_{o}$, a local chemical potential $\mu_{o}$, and a drift velocity $\vec{v}_{o}$.

Using Eqs. (81) and (82), Eq. (72) can be rewritten as,

$$
\begin{gather*}
\frac{\partial}{\partial t}[g(\vec{r}, t) s(\vec{r}, t)]+\operatorname{div}\left[g(\vec{r}, t) s(\vec{r}, t) \vec{v}(\vec{r}, t)+\beta(\vec{r}, t) \vec{I}_{q}(\vec{r}, t)\right]= \\
\quad=\vec{I}_{q}(\vec{r}, t) \cdot \nabla \beta(\vec{r}, t)-\beta(\vec{r}, t) \widetilde{P}(\vec{r}, t) \odot[\nabla: \vec{v}(\vec{r}, t)]+ \\
+g(\vec{r}, t) u(\vec{r}, t) \vec{v}(\vec{r}, t) \cdot \nabla \beta(\vec{r}, t)+\operatorname{div}[\beta(\vec{r}, t) \mathfrak{p}(\vec{r}, t) \vec{v}(\vec{r}, t)]+ \\
\quad+\operatorname{div}[\beta(\vec{r}, t) \omega(\vec{r}, t) \vec{v}(\vec{r}, t)]+\omega^{\prime}(\vec{r}, t) \tag{87}
\end{gather*}
$$

where

$$
\begin{align*}
\omega^{\prime}(\vec{r}, t)= & \int d^{3} p \Delta \varphi_{1}(\vec{r}, \vec{p} ; t) f_{1}(\vec{r}, \vec{p} ; t)+ \\
& +\int d^{3} p \int d^{3} r^{\prime} \int d^{3} p^{\prime}\left\{( 1 / 4 ) \left[\beta(\vec{r}, t) V\left(\left|\vec{r}-\vec{r}^{\prime}\right|\right)+\right.\right. \\
& \left.\left.+\beta\left(\vec{r}^{\prime}, t\right) V\left(\left|\vec{r}^{\prime}-\vec{r}\right|\right)\right] \overline{\mathcal{J}}_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right)\right\} \times \\
& \times \frac{\partial}{\partial t} f_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right),  \tag{88}\\
\mathfrak{p}(\vec{r}, t)= & \frac{1}{3} \operatorname{Tr} \widetilde{P}(\vec{r}, t) \tag{89}
\end{align*}
$$

and $\tilde{\mathcal{J}}_{2}$ is given by Eq. (50). But

$$
\begin{align*}
& \operatorname{div}[\beta(\vec{r}, t) \mathfrak{p}(\vec{r}, t) \vec{v}(\vec{r}, t)]=\beta(\vec{r}, t) \mathfrak{p}(\vec{r}, t) \operatorname{div} \vec{v}(\vec{r}, t)- \\
& \quad-g(\vec{r}, t) u(\vec{r}, t) \vec{v}(\vec{r}, t) \cdot \nabla \beta(\vec{r}, t)-\vec{e}(\vec{r}, t) \cdot \vec{v}(\vec{r}, t) \tag{90}
\end{align*}
$$

with

$$
\begin{gather*}
\vec{e}(\vec{r}, t)=\int d^{3} p\left\{\nabla \Delta \varphi_{1}(\vec{r}, \vec{p} ; t) f_{1}(\vec{r}, \vec{p} ; t)+\right. \\
+\int d^{3} r^{\prime} \int d^{3} p^{\prime}\left\{(1 / 4)\left[\beta(\vec{r}, t) V\left(\left|\vec{r}-\vec{r}^{\prime}\right|\right)+\beta\left(\vec{r}^{\prime}, t\right) V\left(\left|\vec{r}^{\prime}-\vec{r}\right|\right)\right]+\right. \\
\left.\left.+\Delta \varphi_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right)\right\} f_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right)\right\} \tag{91}
\end{gather*}
$$

Using Eqs. (87), (88), (89), (90) and (91), we can write the following compact expression for the MaxEnt-NSOM entropy balance equation

$$
\begin{equation*}
\frac{\partial}{\partial t}[g(\vec{r}, t) s(\vec{r}, t)]+\operatorname{div} \vec{I}_{s}(\vec{r}, t)=\sigma_{s}(\vec{r}, t)+\Delta \sigma(\vec{r}, t) \tag{92}
\end{equation*}
$$

where

$$
\begin{equation*}
\vec{I}_{s}(\vec{r}, t)=g(\vec{r}, t) s(\vec{r}, t) \vec{v}(\vec{r}, t)+\beta(\vec{r}, t) \vec{I}_{q}(\vec{r}, t) \tag{93}
\end{equation*}
$$

is the NSOM entropy flux, and on the right hand side of Eq. (92) we have

$$
\begin{equation*}
\sigma_{s}(\vec{r}, t)=\vec{I}_{q}(\vec{r}, t) \cdot \nabla \beta(\vec{r}, t)-\beta(\vec{r}, t)[\widetilde{P}(\vec{r}, t)-\mathfrak{p}(\vec{r}, t) \tilde{1}] \odot[\nabla: \vec{v}(\vec{r}, t)], \tag{94}
\end{equation*}
$$

with $\tilde{1}$ being the unit tensor, and

$$
\begin{equation*}
\Delta \sigma(\vec{r}, t)=\operatorname{div}[g(\vec{r}, t) \omega(\vec{r}, t) \vec{v}(\vec{r}, t)]+\omega^{\prime}(\vec{r}, t)-\vec{e}(\vec{r}, t) \cdot \vec{v}(\vec{r}, t), \tag{95}
\end{equation*}
$$

which account for the MaxEnt-NSOM entropy production density.

Eq. (95) deserves a very special comment. In fact, it expresses the general entropy production that in our approach is consistent with the conservation equations, Eqs. (59), (60) and (61), which clearly contain much more information than those computed using only the local equilibrium values of $f_{1}$ and $f_{2}$. This is a new and interesting result. For if we want to compute hydrodynamic equations from Eqs. (59), (60) and (61) we have to resort to constitutive equations that go beyond the linear regime. In this sense this paper sets the microscopic basis for hydrodynamic equations that are valid beyond LIT. The explicit form for these equations and their generating constitutive laws is the subject of a forthcoming paper. On the basis of the above paragraph, if in all terms in Eq. (92) we introduce the quantities corresponding to the limit of CIT [Cf. Eqs. (78), (79), (80), (81), (82) and (83)], we obtain the particular local equilibrium entropy balance equation which is the one compatible with the equations of classical (CIT) hydrodynamics, namely

$$
\begin{equation*}
\frac{\partial}{\partial t}\left[g_{o}(\vec{r}, t) s_{o}(\vec{r}, t)\right]+\operatorname{div} \vec{I}_{s, o}(\vec{r}, t)=\sigma_{s, o}(\vec{r}, t) \tag{96}
\end{equation*}
$$

Here, we recall, index naught means averages performed with the use of the NSD of Eq. (85). In that limit $\Delta \sigma(\vec{r}, t)=0$.

## III. 3 The Thermodynamic Parameters and a Local Maxwellian Distribution

It should be noticed that the expressions derived so far depend on both the extensive and intensive macrovariables, $P_{j}$ and $F_{j}$ respectively, which are related by the nonequilibrium equations of state, Eqs. (73), (74) and (75). We proceed next to make explicit that dependence. First we start looking for expressions for the partial distributions $f_{1, o}$ and $f_{2, o}$ of Eqs. $(82,83)$. For that purpose we note that the exponent in the NSD of Eq. (42) can be separated in the part that appears in the exponent of the CHD-NSD of Eq. (85) (call it A) plus the rest (call it B). On the basis of this separation we can apply the Heims-Jaynes perturbation expansion method for averages [64], to find that

$$
\begin{align*}
f_{1}(\vec{r}, \vec{p} ; t) & =f_{1, o}(\vec{r}, \vec{p} ; t)+\Lambda_{1, o}(\vec{r}, \vec{p} ; t)  \tag{97}\\
f_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right) & =f_{2, o}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right)+\Lambda_{2, o}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right) \tag{98}
\end{align*}
$$

where $\Lambda_{1, o}$ and $\Lambda_{2 o}$ are the expressions for the series in powers of the contribution

$$
\begin{align*}
B= & \Delta \Omega(\vec{r}, t)-\Delta \beta(\vec{r}, t) \frac{p^{2}}{2 m}-\Delta \varphi_{1}(\vec{r}, \vec{p} ; t) n_{1}(\vec{r}, \vec{p} \mid \Gamma)- \\
& -\int d^{3} r^{\prime} \int d^{3} p^{\prime} \Delta \varphi_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right) n_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} \mid \Gamma\right) \tag{99}
\end{align*}
$$

to the NSD, which we omit to write down explicity for the sake of brevity.
Next, we introduce a Boltzmann-like approach, as the one described in reference [42], namely, we first put into evidence in the form of a series expansion the effect of two-particle correlations, namely the part in the exponent of the CHD-NSD that contains $n_{2}$. Again we resort to Heims-Jaynes' method [64] to obtain that

$$
\begin{align*}
f_{1, o}(\vec{r}, \vec{p} ; t) & =f_{1, o}^{o}(\vec{r}, \vec{p}, t)+\Lambda_{1}(\vec{r}, \vec{p} ; t)  \tag{100}\\
f_{2, o}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right) & =f_{2, o}^{o}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right)+\Lambda_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right), \tag{101}
\end{align*}
$$

where

$$
\begin{equation*}
f_{1, o}^{o}(\vec{r}, \vec{p} ; t)=\int d \Gamma \varrho_{o}(\Gamma \mid t) n_{1}(\vec{r}, \vec{p} \mid \Gamma) \tag{102}
\end{equation*}
$$

and

$$
\begin{equation*}
\varrho_{o}(\Gamma \mid t)=\exp \left\{-\phi_{o}(t)-\int d^{3} r \int d^{3} p A(\vec{r}, \vec{p} \mid \Gamma, t)\right\} \tag{103}
\end{equation*}
$$

where

$$
\begin{equation*}
A(\vec{r}, \vec{p} \mid \Gamma, t)=\beta_{0}(\vec{r}, t)\left[(1 / 2 m)[\vec{p}-m \vec{v}(\vec{r}, t)]^{2}-\mu_{0}(\vec{r}, t)+\bar{\omega}(\vec{r}, t)\right] n_{1}(\vec{r}, \vec{p} \mid \Gamma) \tag{104}
\end{equation*}
$$

with $\phi_{0}$ ensuring its normalization. In Eqs. (100-101) $\Lambda_{1}$ and $\Lambda_{2}$ are given in Eqs. (57a) and (55a) of reference [42], and

$$
\begin{equation*}
\bar{\omega}(\vec{r}, t)=\omega(\vec{r}, t)+\int d^{3} r^{\prime} \int d^{3} p^{\prime} V\left(\left|\vec{r}-\vec{r}^{\prime}\right|\right) f_{1, o}^{o}\left(\vec{r}^{\prime}, \vec{p}^{\prime} ; t\right) \tag{105}
\end{equation*}
$$

is an effective potential composed of the external potential and the (self-consistent) average molecular field potential in this quasiparticle approximation. Performing the calculation in Eq. (102) we find that

$$
\begin{equation*}
f_{1, o}^{o}(\vec{r}, \vec{p} ; t)=\frac{N}{V} \exp \left\{-\varphi_{o}(t)-\beta_{o}(\vec{r}, t) \xi(\vec{r}, \vec{p}, t)\right\} \tag{106}
\end{equation*}
$$

with

$$
\begin{equation*}
\xi(\vec{r}, \vec{p}, t)=(1 / 2 m)[\vec{p}-m \vec{v}(\vec{r}, t)]^{2}-\mu_{0}(\vec{r}, t)+\bar{\omega}(\vec{r}, t) \tag{107}
\end{equation*}
$$

and where

$$
\varphi_{o}(t)=\ln \int d^{3} r \int d^{3} p \exp \left\{-\beta_{o}(\vec{r}, t) \xi(\vec{r}, \vec{p}, t)\right\}
$$

is the normalization (to $\mathrm{N} / \mathrm{V}$ ) condition. Moreover, resorting to the obvious relationship,

$$
\begin{equation*}
n_{o}(\vec{r}, t)=\int d^{3} p f_{1, o}^{o}(\vec{r}, \vec{p} ; t) \tag{108}
\end{equation*}
$$

we can write that

$$
\begin{equation*}
f_{1, o}^{o}(\vec{r}, \vec{p} ; t)=n_{o}(\vec{r}, t)\left[\frac{\beta_{o}(\vec{r}, t)}{2 \pi m}\right]^{3 / 2} \exp \left\{-\frac{\beta_{o}(\vec{r}, t)}{2 m}[\vec{p}-m \vec{v}(\vec{r}, t)]^{2}\right\}, \tag{109}
\end{equation*}
$$

which is a local-equilibrium Maxwell-Boltzmann distribution, where

$$
\begin{equation*}
n_{o}(\vec{r}, t)=\frac{N}{V}\left[\frac{2 \pi m}{\beta_{o}(\vec{r}, t)}\right]^{3 / 2} \exp \left\{\beta_{o}(\vec{r}, t)\left[\mu_{o}(\vec{r}, t)-\bar{\omega}(\vec{r}, t)\right]\right\} \tag{110}
\end{equation*}
$$

and we interpret $\beta_{o}^{-1}(\vec{r}, t)=k_{B} T_{o}(\vec{r}, t)$ as defining a local-nonequilibrium temperature, which according to the theory [Cf. Eq. (78)] is given by

$$
\begin{equation*}
\beta_{o}(\vec{r}, t)=\delta s_{o}(\vec{r}, t) / \delta \mu_{o}(\vec{r}, t) \tag{111}
\end{equation*}
$$

We may also verify that a local equipartition of energy holds true, namely

$$
\begin{equation*}
\frac{1}{m} g_{o}(\vec{r}, t) \varepsilon_{k, o}(\vec{r}, t)=\frac{3}{2} n_{o}(\vec{r}, t) k_{B} T_{o}(\vec{r}, t) \tag{112}
\end{equation*}
$$

As a final note in this section we stress that, according to Eqs. (97), (98), (100) and (101), we have that

$$
\begin{align*}
& f_{1, o}(\vec{r}, \vec{p} ; t)=f_{1, o}^{o}(\vec{r}, \vec{p} ; t)+\Lambda_{1, o}(\vec{r}, \vec{p} ; t)+\Lambda_{1}(\vec{r}, \vec{p} ; t),  \tag{113}\\
& f_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right)=f_{1, o}^{o}(\vec{r}, \vec{p} ; t) f_{1, o}^{o}\left(\vec{r}^{\prime}, \vec{p}^{\prime} ; t\right)+ \\
& +\Lambda_{2, o}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right)++\Lambda_{2}\left(\vec{r}, \vec{p}, \vec{r}^{\prime}, \vec{p}^{\prime} ; t\right) \tag{114}
\end{align*}
$$

where $\Lambda_{1}$ and $\Lambda_{2}$ account for the effects of two-particle correlations, while $\Lambda_{1, o}$ and $\Lambda_{2, o}$ are the terms that are responsible for the contributions that broaden the local equilibrium approximation.

## IV. Concluding remarks

In the present paper we have derived a hydrodynamic description of a fluid of particles interacting through a central force while under, in principle, general nonequilibrium conditions. This was done in the spirit of the IST based on the nonequilibrium statistical operator method, a formalism which appears as belonging to the realm of Jaynes' Predictive Statistical Mechanics involving Bayesian and maximum informational-entropy methods. The description of the macroscopic state of the system is done in terms of the single- and twoparticle reduced dynamic distribution functions, which, in principle, allow to express any dynamic observable of relevance in terms of them.

The equations of evolution for the nonequilibrium averages of those quantities, namely, the density distribution functions, were obtained within the nonlinear generalized transport theory that the MaxEnt-NSOM provides. These equations were next used to obtain the equations of evolution for the usual hydrodynamic variables, namely the particle density, momentum density, and internal energy density. In that way there followed their balance equations including dissipative sources. We call the attention to the fact that these equations
involve the three-particle distribution function, which if it is incorporated as a basic variable would lead us to a generalization to arbitrary nonequilibrium condition of the BBGKY hierarchy. To introduce only $n_{1}$ and $n_{2}$ implies the need of a truncation procedure necessary for the practical use of the method (as discussed elsewhere [40]) in a way reminiscent of a truncation of Grad's moments method. The balance equation for the MaxEnt-NSOM entropy is obtained in its general form for the chosen set of macrovariables, following from it the expression for the NSOM-entropy flux and NSOMentropy production.

At this point, once in possession of the general expressions we have derived, within the framework of IST, we have made contact with classical conservation equations based on CIT. We showed that in fact a particular choice of the MaxEnt-NSOM Lagrange multipliers, what is equivalent to a NSOM approach that includes as basic variables only the particle density, momentum density, and energy density, leads to the well known results of classical hydrodynamics, mainly to Gibbs relation defining local equilibrium [Cf. Eqs. (78) and (85)].

Hence, on the one hand, this proves that essentially classical hydrodynamics is contained in IST as a limiting case, and on the other that the method provides a way to go beyond the limitations of CIT. The procedure developed in section III evidences the contributions that are beyond the local equilibrium and other restrictions imposed in CIT [Cf. Eqs. (97-98)]. Of course in Eqs. $(97-98)$ it is necessary to determine what $\Delta \varphi_{1}$ and $\Delta \varphi_{2}$ are. One way to do that is to introduce the fluxes, that is, all higher order fluxes are to be incorporated to the basic set of variables to be used to describe the macroscopic state of the system and its evolution in arbitrarily far away from equilibrium conditions. This includes nonlocality and retro-effects, thus allowing to deal with short wavelength and high frequency phenomena in a generalized extended hydrodynamics in, eventually, far-from-equilibrium conditions. Also, the constitutive equations of classical hydrodynamics (that lead to the uncomfortable question of propagation of thermal perturbations at infinite speed) become generalized equations which in restricted conditions (first
order extension of CIT) are of the hyperbolic type in the form of the so called telegraphist equation [40, 41]. Next step is of course a full development of these subject, presently under way and to be reported in future articles.

Returning to the results of section III, we considered in the third subsection a connection between the density distribution functions and the MaxEnt-NSOM Lagrange multipliers. To obtain a general relation [except the implicit ones given by Eqs. (73), (74) and (75)] is a formidable task, and so one needs to resort to approximations. We used a quite simple one consisting in, as already noted, a truncated description that would produce classical conservation equation [Cf. Eqs. (85) and (86)], and so acceptable under the proviso that the local-equilibrium approximation holds. Next, neglecting two-particle correlation [Cf. Eqs. (85), (97) and (98)], viz, considering a dilute fluid, we finally arrived at the result that there follows a local Maxwell-Boltzmann distribution function, with the presence of a shift in momentum distribution, and the presence of a spacedependent potential composed of the external potential and an average field potential accounting for an approximation (quasiparticles in the dilute weakly correlated fluid) to the two-particle central-force interaction. Local (and time-dependent) temperature, chemical potential, drift velocity, and pressure, can be defined and the average kinetic energy satisfies, locally, the equipartition theorem. As noted before, the extension of these concepts, mainly that of nonequilibrium temperature, entropy and entropy production under quite general dissipative conditions is an open question.

As final words, we comment that the MaxEntNSOM, which possesses a remarkable compactness, having by far a most appealing structure, and based on reasonable sound principles [14,21-23,25-27] offers a very effective method for dealing with nonlinear and nonlocal in space and time irreversible processes in far-from-equilibrium conditions in many-body systems. As noticed, it provides microscopic foundations to IST, and in this paper we have stressed the fact that it provides a seemingly far-reaching-generalized hydrodynamics. We have shown how classical conservation equations are recovered when imposing very limiting restrictive con-
ditions, bringing into evidence - at least in a general way - the contributions to be expected to allow for going outside the domain of validity of CIT. In a sense this anticipates that besides the quasi-conserved quantities (basically density and energy density) one needs to introduce, to an appropriate degree depending on the problem in hands, relaxing fluxes of ever increasing order.

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