Critical Behavior of an Ising Model with Aperiodic Interactions*

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We write exact renormalization-group recursion relations for a ferromagnetic Ising model on the diamond hierarchical lattice with an aperiodic distribution of exchange interactions according to a class of generalized two-letter Fibonacci sequences. For small geometric fluctuations, the critical behavior is unchanged with respect to the uniform case. For large fluctuations, the uniform fixed point in the parameter space becomes fully unstable. We analyze some limiting cases, and propose a heuristic criterion to check the relevance of the fluctuations.

I. Introduction

The experimental discovery of quasi-crystals motivated many investigations of the effects of geometric fluctuations produced by different types of aperiodic structures. There are several specific results for phonon and electronic spectra of linear problems[1]. There are also some results for more difficult non-linear problems, as the analysis of the critical properties of a ferromagnetic Ising model on a layered lattice with aperiodic exchange interactions along an axial direction[2].

In a recent publication[3], Luck presented a detailed study of the critical behavior in the ground state of the quantum Ising chain in a transverse field (which is known to be related to the transition at finite temperatures of the two-dimensional Ising model). The nearest-neighbor ferromagnetic exchange interactions are chosen according to some (generalized) Fibonacci sequences. The geometric fluctuations are gauged by a wandering exponent ω associated with the eigenvalues of the substitution matrix of each sequence. The critical behavior remains unchanged (that is, of Onsager type) for bounded fluctuations (small values of ω). Large fluctuations induce much weaker singularities, similar to the case of a disordered Ising ferromagnet.

In this paper, we take advantage of the simplifications introduced by a hierarchical lattice to give another example of the effects of geometrical fluctuations on the critical behavior of a ferromagnetic model. It should be pointed out that hierarchical lattices have been widely used as a toy model to test critical properties. In some cases the (approximate) Migdal-Kadanoff renormalization-group transformations on a Bravais lattice are identical to the (exact) transformations on a suitable hierarchical structure[4]. There are rather detailed studies of these exact transformations for the Ising model on a variety of hierarchical structures[5]. There are also some recent investigations of an Ising spin-glass on the diamond hierarchical lattice[6].

This paper is organized as follows. In Section II, we make some comments on Fibonacci sequences and introduce a nearest-neighbor ferromagnetic Ising model on a diamond hierarchical lattice with q branches. To simulate a layered system, the exchange interactions ($J_A > 0$ and $J_B > 0$) are distributed along the branches according to the same aperiodic rule. For a large class of two-letter Fibonacci sequences, we write exact renormalization-group recursion relations in terms of two parameters, $x_A = \tanh K_A$, and $x_B = \tanh K_B$, where $K_A = \beta J_A$, $K_B = \beta J_B$, and β is the inverse of the temperature. In Sections III and IV, we analyze two typical and distinct examples. We obtain the fixed points and the flows of the recursion relations for

*This paper is dedicated to Prof. Roberto Luzzi in his 60th anniversary.

these specific cases. In the first example, for small ω, the critical behavior is unchanged with respect to the uniform case ($J_A = J_B$); the physical fixed point in parameter space is a saddle point. In the example of Section IV, however, the fluctuations turn the physical fixed point fully unstable. We also analyze some limiting cases. For $q = 1$ (Ising chain), there is no transition at finite temperatures. The fixed point at zero temperature, however, may change its character depending on the value of the wandering exponent. In a particular infinite-branching limit ($q \to \infty$), we obtain rather simple results. In Section V, we present some conclusions, as well as a heuristic adaptation for the diamond hierarchical lattice of Luck's criterion[7] for the relevance of geometric fluctuations.

II. Definition of the model

Consider a particular two-letter generalized Fibonacci sequence given by the substitutions

$$\begin{array}{c} A \to AB, \\ B \to AA. \end{array} \tag{1}$$

If we start with letter A, the successive application of this inflation rule produces the sequences

$$A \to AB \to ABAA \to ABAAABAB \to \cdots. \tag{2}$$

At each stage of this construction, the numbers N_A and N_B, of letters A and B, can be obtained from the recursion relations

$$\left(\begin{array}{c} N'_A \\ N'_B \end{array} \right) = \mathbf{M} \left(\begin{array}{c} N_A \\ N_B \end{array} \right), \tag{3}$$

with the substitution matrix

$$\mathbf{M} = \left(\begin{array}{cc} 1 & 2 \\ 1 & 0 \end{array} \right). \tag{4}$$

The eigenvalues of this matrix, $\lambda_1 = 2$ and $\lambda_2 = -1$, govern most of the geometrical properties. In a more general case (that is, for a more general rule), at a large order n of the construction, the total number of letters is given by the asymptotic expression

$$N^n = N_A^n + N_B^n \sim \lambda_1^n, \tag{5}$$

where $\lambda_1 > |\lambda_2|$. The smaller eigenvalue governs the fluctuations with respect to these asymptotic values,

$$\Delta N^n \sim \Delta N_A^n \sim \Delta N_B^n \sim |\lambda_2|^n. \tag{6}$$

From these equations, we can write the asymptotic expression

$$\Delta N \sim N^{\omega}, \tag{7}$$

with the wandering exponent

$$\omega = \frac{\ln|\lambda_2|}{\ln \lambda_1}. \tag{8}$$

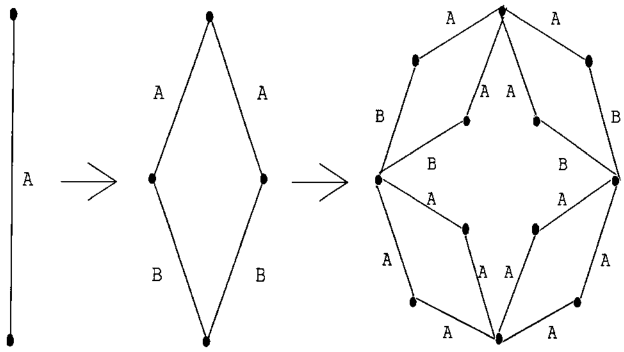

Figure 1. Some stages of the construction of a diamond hierarchical lattice with $q = 2$ branches. Letters A and B indicate the exchange interactions (J_A and J_B).

Now we consider a nearest-neighbor Ising model, given by the Hamiltonian

$$\mathcal{H} = -\sum_{(i,j)} J_{i,j}\sigma_i\sigma_j, \tag{9}$$

with the spin variables $\sigma_i = \pm 1$ on the sites of a diamond hierarchical structure. In Fig. 1, which is suitable for the period-doubling Fibonacci rule of Eq.(1), we draw the first stages of the construction of a diamond lattice with a basic polygon of four bonds (ramification $q = 2$). As indicated in this figure, we simulate a layered system by the introduction of the interactions $J_A > 0$ and $J_B > 0$ along the branches of the structure. Considering the elementary transformations of Fig.2, and using the rules of Eq. (1), it is straightforward to establish the recursion relations

$$\tanh K'_A = \tanh\left[2\tanh^{-1}(\tanh K_A \tanh K_B)\right], \tag{10}$$

and

$$\tanh K'_B = \tanh\left[2\tanh^{-1}\left(\tanh^2 K_A\right)\right], \tag{11}$$

where $K_{A,B} = \beta J_{A,B}$. In this particular case, these equations can also be written as

$$x'_A = \frac{2x_A x_B}{1 + x_A^2 x_B^2}, \tag{12}$$

and

$$x'_B = \frac{2x_A^2}{1+x_A^4}, \tag{13}$$

where

$$x_{A,B} = \tanh K_{A,B}. \tag{14}$$

In Section 3, we analyze the fixed points associated with these simple recursion relations.

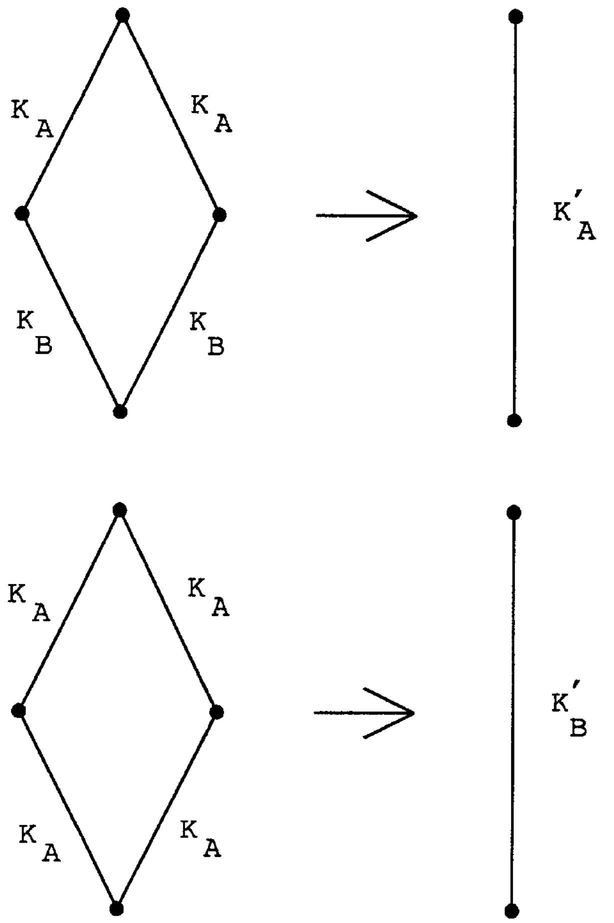

Figure 2. Basic graphs to obtain the recursion relations for an Ising model on a diamond hierarchical lattice with $q = 2$ and exchange interactions given by the generalized Fibonacci rules, $A \to AB$, and $B \to AA$.

We can use similar procedures to consider much more general Fibonacci rules. To avoid any changes in the geometry of the hierarchical lattice, in this paper we restrict the analysis to period-multiplying Fibonacci inflation rules of two letters (the largest eigenvalue of the substitution matrix, λ_1, gives the multiplication factor of the period). For example, let us consider the substitutions

$$\begin{array}{l} A \to A^k B^l, \\ B \to A^{k+l}, \end{array} \tag{15}$$

where $k, l \geq 1$ are two integers. From the substitution matrix,

$$\mathbf{M} = \begin{pmatrix} k & k+l \\ l & 0 \end{pmatrix}, \tag{16}$$

we have the eigenvalues, $\lambda_1 = k + l$ and $\lambda_2 = -l$, and the wandering exponent, $\omega = \ln l / \ln(k+l)$. Now we consider a diamond lattice with q branches and a basic polygon of $q\,(k+l)$ bonds. The exchange interactions are chosen according to the general Fibonacci rule of Eq. (15). We then write the recursion relations

$$\tanh K'_A = \tanh\left[q \tanh^{-1}\left(\tanh^k K_A \tanh^l K_B\right)\right], \tag{17}$$

and

$$\tanh K'_B = \tanh\left[q \tanh^{-1}\left(\tanh^{k+l} K_A\right)\right]. \tag{18}$$

III. Irrelevant fluctuations

Consider again the Fibonacci inflation rules given by Eq. (1). From the eigenvalues of the substitution matrix, $\lambda_1 = 2$ and $\lambda_2 = -1$, we have $\omega = 0$. For the branching number $q = 2$, the recursion relations are given by Eqs. (12) and (13). The fixed points and some orbits of the second iterates of this map are shown in Fig. 3. Besides the trivial fixed points, there is also a non-trivial fixed point, given by

$$x_A^* = x_B^* = 0.543689..., \tag{19}$$

which comes from the solution of the polynomial equation

$$x_A^8 + 2x_A^4 - 4x_A^2 + 1 = 0. \tag{20}$$

The linearization about this uniform fixed point yields the asymptotic expression

$$\begin{pmatrix} \Delta x'_A \\ \Delta x'_B \end{pmatrix} = C\mathbf{M}^T \begin{pmatrix} \Delta x_A \\ \Delta x_B \end{pmatrix}, \tag{21}$$

where

$$\mathbf{M}^T = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \tag{22}$$

is the transpose of the substitution matrix, and

$$C = \frac{1-(x_A^*)^4}{2x_A^*} = 0.839286.... \tag{23}$$

The diagonalization of this linear form gives the eigenvalues

$$\Lambda_1 = C\lambda_1 = 2C = 1.678573..., \tag{24}$$

and

$$\Lambda_2 = C\lambda_2 = -C = -0.839286.... \quad (25)$$

As $\Lambda_1 > 1$ and $-1 < \Lambda_2 < 0$, the fixed point is a saddle point with a flipping approximation (in Fig. 3, we draw the trajectories of some second iterates of this map). Moreover, if we make $J_A = J_B$, it is easy to see that the same eigenvalue Λ_1 characterizes the (unstable) fixed point of the uniform model (see the diagonal flow in Fig. 3). Therefore, in this particular example the geometric fluctuations are unable to change the critical behavior of the uniform system.

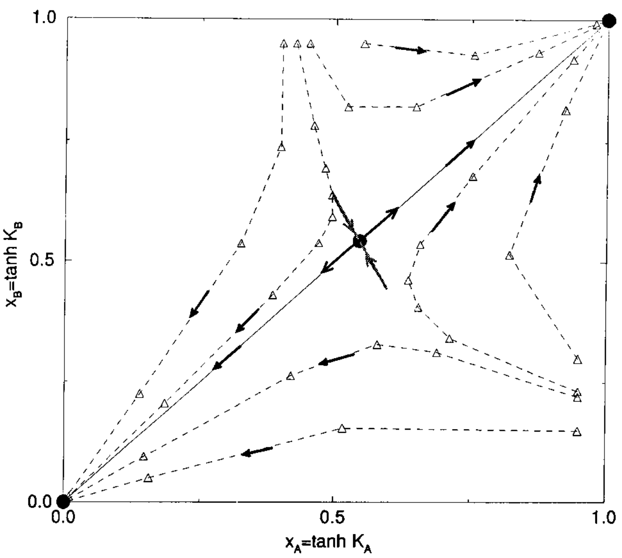

Figure 3. Second iterates and fixed points (black circles) of the recursion relations in the parameter space, $x_A = \tanh K_A$ versus $x_B = \tanh K_B$, for an Ising model on a diamond hierarchical lattice with $q = 2$ and exchange interactions given by the generalized Fibonacci rules, $A \rightarrow AB$, and $B \rightarrow AA$ (irrelevant fluctuations). We draw the stable trivial fixed points (at zero and infinite temperatures) and the physical saddle point. The dashed lines and the black arrows indicate the flows of the second iterates of the map. The uniform model is recovered along the diagonal, $x_A = x_B$. The light arrows indicate the stable direction in the neighborhood of the physical fixed point.

It is not difficult to check that the same sort of behavior (saddle point; largest eigenvalue associated with the uniform system) still holds for all finite values of the branching number q of the diamond structure. In the limit of infinite branching ($q \rightarrow \infty$, $K_A, K_B \rightarrow 0$, with $q^2 K_A$ and $q^2 K_B$ fixed), the recursion relations are particularly simple,

$$y_A' = y_A y_B, \quad (26)$$

and

$$y_B' = y_A^2, \quad (27)$$

where $y_A = q^2 K_A$ and $y_B = q^2 K_B$. The linearization about the uniform fixed point, $y_A^* = y_B^* = 1$, yields the relations

$$\left(\begin{array}{c} \Delta y_A' \\ \Delta y_B' \end{array} \right) = \mathbf{M}^T \left(\begin{array}{c} \Delta y_A \\ \Delta y_B \end{array} \right), \quad (28)$$

which correspond to a limiting case of Eq. (21), with $C \rightarrow 1$. As the limiting eigenvalues are given by $\Lambda_1 = \lambda_1 = 2$, and $\Lambda_2 = \lambda_2 = -1$, a linear analysis is not enough to check the (flipping saddle-point) character of this marginal case.

Another particular case of interest is the simple Ising chain ($q = 1$). The recursion relations are given by

$$x_A' = x_A x_B, \quad (29)$$

and

$$x_B' = x_A^2, \quad (30)$$

with the same form of Eqs. (26) and (27), but with the parameters x_A and x_B given by Eq. (14). As shown in Fig. 4(a), the zero-temperature fixed point displays the character of a saddle point. As there is no phase transition at finite temperatures, it cannot be reached from physically acceptable initial conditions.

IV. Relevant fluctuations

To give an example of relevant fluctuations, consider the generalized Fibonacci substitutions,

$$\begin{array}{c} A \rightarrow ABB, \\ B \rightarrow AAA. \end{array} \quad (31)$$

From the substitution matrix,

$$\mathbf{M} = \left(\begin{array}{cc} 1 & 3 \\ 2 & 0 \end{array} \right), \quad (32)$$

we have the eigenvalues, $\lambda_1 = 3$ and $\lambda_2 = -2$, and the wandering exponent, $\omega = \ln 2/\ln 3 = 0.630929....$ For a general branching number q, we can use Eqs. (17) and (18), to write the recursion relations

$$\tanh K_A' = \tanh\left[q \tanh^{-1}\left(\tanh K_A \tanh^2 K_B\right)\right], \quad (33)$$

and

$$\tanh K_B' = \tanh\left[q \tanh^{-1}\left(\tanh^3 K_A\right)\right]. \quad (34)$$

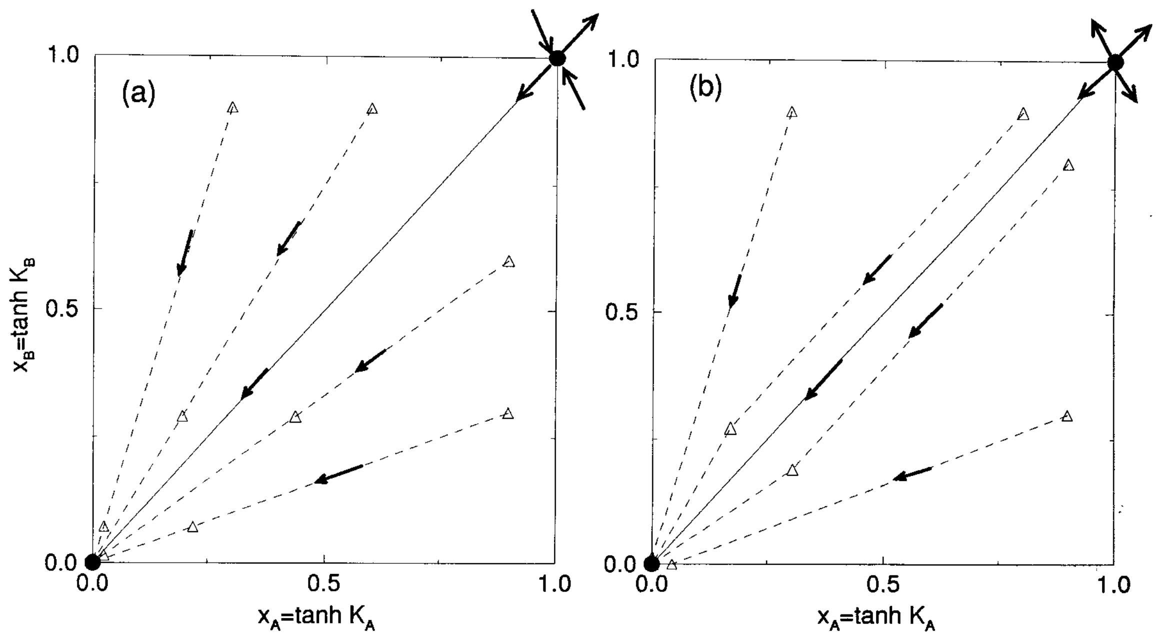

Figure 4. Second iterates and fixed points for the Ising chain: (a) exchange interactions according to the rules $A \to AB$, and $B \to AA$ (irrelevant fluctuations); (b) exchange interactions according to the rules $A \to ABB$, and $B \to AAA$ (relevant fluctuations).

For the particular case $q = 2$, Eqs. (33) and (34) reduce to the simple relations

$$x'_A = \frac{2x_A x_B^2}{1 + x_A^2 x_B^4}, \qquad (35)$$

and

$$x'_B = \frac{2x_A^3}{1 + x_A^6}, \qquad (36)$$

where the parameters x_A and x_B are given by Eq. (14). In Fig. 5, we show the fixed points and some second iterates associated with these recursion relations. The nontrivial fixed point, given by

$$x_A^* = x_B^* = 0.786151..., \qquad (37)$$

comes from the physical solution of a polynomial equation. The linearization about this uniform fixed point yields the relations

$$\begin{pmatrix} \Delta x'_A \\ \Delta x'_B \end{pmatrix} = C\mathbf{M}^T \begin{pmatrix} \Delta x_A \\ \Delta x_B \end{pmatrix}, \qquad (38)$$

where the substitution matrix is given by Eq. (32), and

$$C = \frac{2\left(x_A^*\right)^2 \left[1 - \left(x_A^*\right)^6\right]}{\left[1 + \left(x_A^*\right)^6\right]^2} = 0.618033.... \qquad (39)$$

From the diagonalization of this linear form we have the eigenvalues

$$\Lambda_1 = C\lambda_1 = 1.854101..., \qquad (40)$$

and

$$\Lambda_2 = C\lambda_2 = -1.236067.... \qquad (41)$$

The absolute values of Λ_1 and Λ_2 larger than unit indicate the relevance of the geometric fluctuations. The uniform fixed point is fully unstable (and there might be no transition for $J_A \neq J_B$).

Again, it is not difficult to check that the uniform fixed point remains unstable for all values of the branching number q. In particular, for $q \to \infty$, and $K_A, K_B \to 0$, with $q^{1/2}K_A$ and $q^{1/2}K_B$ fixed, we have the limiting recursion relations,

$$y'_A = y_A y_B^2, \qquad (42)$$

and

$$y'_B = y_A^3, \qquad (43)$$

where $y_A = q^{1/2}K_A$ and $y_B = q^{1/2}K_B$. The linearization about the uniform fixed point, $y_A^* = y_B^* = 1$, yields the relations

$$\begin{pmatrix} \Delta y'_A \\ \Delta y'_B \end{pmatrix} = \mathbf{M}^T \begin{pmatrix} \Delta y_A \\ \Delta y_B \end{pmatrix}, \qquad (44)$$

with the substitution matrix given by Eq. (32). We then have the limiting eigenvalues, $\Lambda_1 = \lambda_1 = 3$, and

$\Lambda_2 = \lambda_2 = -2$, which confirm the unstable character of this fixed point.

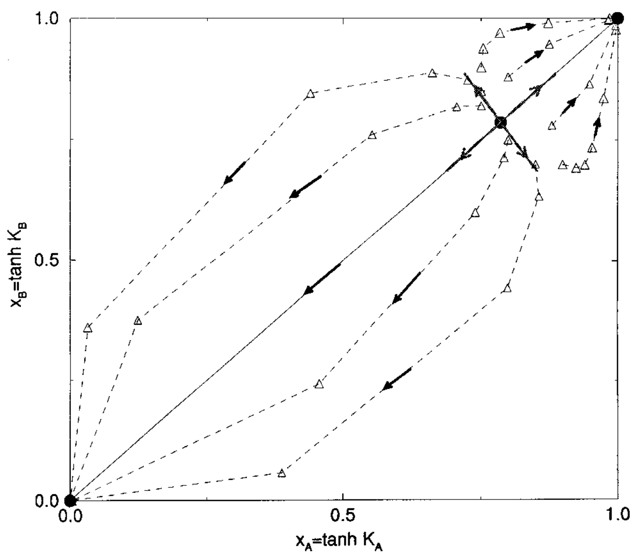

Figure 5. Second iterates and fixed points (black circles) of the recursion relations in the parameter space, $x_A = \tanh K_A$ versus $x_B = \tanh K_B$, for an Ising model on a diamond hierarchical lattice with $q = 2$ and exchange interactions given by the generalized Fibonacci rules, $A \to ABB$, and $B \to AAA$ (relevant fluctuations). We draw the stable trivial fixed points (at zero and infinite temperatures) and the unstable node. The uniform model is recovered along the diagonal, $x_A = x_B$. The dashed lines are indications of the flow associated with the second iterates of the map. The light arrows indicate the directions of the eigenvectors in the neighborhood of the physical fixed point.

For the particular case of the linear chain ($q = 1$), we have

$$x'_A = x_A x_B^2, \tag{45}$$

and

$$x'_B = x_A^3. \tag{46}$$

In agreement with the lack of a phase transition, the fixed point at zero temperature is unstable [see Fig. 4(b)].

V. Conclusions

From the analysis of the exact renormalization-group recursion relations associated with an Ising model on a variety of hierarchical diamond structures, we show that aperiodic fluctuations of the ferromagnetic exchange interactions may change the character of a (uniform) fixed point. In a particular example, with a small wandering exponent, the fluctuations are irrelevant. In this case, the critical behavior is still characterized by the same exponents of the corresponding uniform system. In another example, however, stronger geometric fluctuations turn the physical fixed point unstable in the parameter space. Even in one dimension, although there is no phase transition at finite temperatures, we show that the geometric fluctuations change the stability of the uniform fixed point at zero temperature.

As in the work of Luck[7], it should be interesting to devise a general criterion to gauge the influence of the geometric fluctuations on the critical behavior of the Ising model on the diamond hierarchical lattice. For a large lattice, with q branches, the total fluctuation ΔJ in the exchange interactions should be proportional to ΔN. Thus we can write the asymptotic relation

$$\Delta J \sim \Delta N \sim N^{\omega} \sim L^{q\omega}, \tag{47}$$

where L is a measure of the total length. The critical temperature, T_c, should be proportional to the total value of the exchange (that is, to L^q). We can then define a reduced temperature, $t = (T - T_c)/T_c$, whose fluctuations are given by the asymptotic form

$$\delta t \sim \frac{L^{q\omega}}{L^q} = L^{q(\omega-1)}. \tag{48}$$

In the neighborhood of the critical point, we have $L \sim \xi \sim t^{-\nu}$, where ξ is a correlation length, and ν is a critical exponent. Thus, we can write

$$\frac{\delta t}{t} \sim \frac{t^{-\nu q(\omega-1)}}{t} = t^{\phi}, \tag{49}$$

with the exponent

$$\phi = -\nu q(\omega - 1) - 1. \tag{50}$$

If $\phi > 0$, the fluctuations are irrelevant. If $\phi < 0$, however, the critical behavior is changed drastically. To calculate a suitable value of ν, we consider the largest eigenvalue of the diagonal form about the physical fixed point, and write the usual renormalization-group expression

$$\Lambda_1 = C\lambda_1 = b^{y_1} = b^{1/\nu}. \tag{51}$$

As the largest eigenvalue λ_1 of the substitution matrix gives the multiplication factor of the Fibonacci rule, we write

$$b = (\lambda_1)^{1/q}. \tag{52}$$

From Eqs. (51) and (52), we have

$$\nu = \frac{\ln \lambda_1}{q \ln (C\lambda_1)}. \tag{53}$$

Inserting into Eq. (50), we finally have

$$\phi = -\frac{\ln \lambda_1}{\ln (C\lambda_1)} (\omega - 1) - 1. \tag{54}$$

The fluctuations are relevant for

$$\omega > -\frac{\ln C}{\ln \lambda_1}, \tag{55}$$

where the prefactor C can be obtained from the linear analysis of the unstable fixed point of the uniform ($J_A = J_B$) system. Although the existence of some counterexample should not be ruled out, the validity of this criterion has been confirmed by the application to a fair number of cases (including the examples of Sections 3 and 4). In the limit of infinite branching, the prefactor C tends to unit, and the criterion of relevance is reduced to the Pisot condition[3], $\omega > 0$.

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