# Stochastic Mechanics of Nonequilibrium Systems* 

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#### Abstract

A nonequilibrium system of classical particles interacting through nonconservative forces is studied by a stochastic Markovian process defined over the space of the particle positions. The velocities of the particles are treated as independent stochocastic variables with a given probability distribution. We deduce expressions for the rate in which energy is exchanged with the surrounding as well as the rate in which energy is dissipated. These results are applied to the special case in which the irrotational and solenoidal parts of the forces acting on the particles are orthogonal. We also solve exactly a model in which these two forces are linear functions of positions.


## I. Introduction

We consider here the stochastic Markovian approach in the study of nonequilibrium system of classical particles interacting through forces that may be nonconservative. According to this approach, the deterministic trajectory of the system in phase space is replaced by a random trajectory governed by a stochastic or Langevin equation [1] [2] [3]. Our starting point is a Langevin equation of motion defined over the space of the particle positions only. The probability distribution defined on this space will evolve in time according to a Smoluchowski equation which is a Fokker-Planck equation in position space [1] [2] [3]. Such a stochastic approach can be understood as a coarse grained description in which the velocities become stochastic variables independent of positions, with a certain given probability distribution. The positions are also stochastic variables but depend on the velocities.

The coarse grained description used here is obtained by assuming that within a characteristic time interval $\tau$ the velocities of the particles become independent of the positions and acquire a distribution whose average speed of the particles is $u$. The description is then characterized by the velocity relaxation time $\tau$ and by the average speed $u$ of the particles after relaxation. We assume also that the forces among particles vary so slowly over a distance $u \tau$ that may be considered constant
within the time interval $\tau$.
Within the above assumptions we obtain the Langevin equation and the corresponding FokkerPlanck equation in position space. We consider systems described by nonconservative forces and deduce from these equations expressions for the rate in which energy is exchanged with the surrounding as well as the rate in which energy is dissipated. These results are applied to the special case in which the irrotational and solenoidal parts of the forces acting on the particles are orthogonal. We solve exactly a model in which these two orthogonal forces are linear functions of positions.

There are other ways to justify the position space description used here. For instance, one could start from a Kramers equation, which is a Fokker-Planck equation in position and velocity space, and perform the elimination of velocities treated as being fast variables [1] [2] [3]. But in this case we would have to introduce first the stochastic equation that describes the trajectory of the system in the position and velocity space and then proceed to the elimination of the fast variables. Moreover this procedure involves the use of a friction term that we avoided in our approach.

The point of view we use here in the study of nonequilibrium system described by continuous variables is the same one adopted by Tomé [4] in the study of nonequilibrium stochastic lattice gas models. In this case the analysis of lattice gas models by a stochas-

[^0]tic dynamics [5] [6] [7] has been successful in describing the phase transitions in system far from equilibrium [7-15]. Other approaches, macroscopic and microscopic, in which fluctuations play an important role, have been used in explaining self-organization and dissipative structures. These include the pioneering work of Nicolis and Prigogine[16] and the one based on the nonequilibrium statistical operator method introduced by Zubarev [17]. This last formalism has been explored extensively and successfully by Luzzi and Vasconcellos [18] in the study of dissipative structures in condensed matter.

## II. Stochastic equation of motion

Consider a classical system composed of a certain number of identical particles of mass $m$ each one interacting with each other and possibly with the external environment. Let $x_{i}(t)$ and $v_{i}(t)$ be the position and the velocity of the $i$-th particle at time $t$ and let $f_{i}(x)$ be the total force on particle $i$ due to the other particles which is supposed to depend only on the positions. The forces acting on particles may be conservative or nonconservative. The notation $x$ stands for the collection $\left\{x_{i}\right\}$ of positions of the particles and $v$ will denote the set $\left\{v_{i}\right\}$ of their velocities. Assuming that the forces $f_{i}(x)$ can be considered constant during a time interval $\tau$, the position of the $i$-th particle at time $t+\tau$ will be given by

$$
\begin{equation*}
x_{i}(t+\tau)=x_{i}(t)+\tau v_{i}(t)+\frac{\tau^{2}}{2} \frac{f_{i}(x(t))}{m} \tag{1}
\end{equation*}
$$

where $f_{i}(x(t)) / m$ is the acceleration and $v_{i}(t)$ is the velocity of the particle at the beginning of the time interval, and $m$ is the mass of the particle.

According to our assumptions, the velocity $v(t+\tau)$ of the particles at time $t+\tau$ will not depend on the positions $x(t)$ of the particles at time $t$ but could depend on the velocity $v(t)$ at time $t$. However, we assume that it will not depend on the velocity $v(t)$ of the particles on the previous time so that $\left\langle v_{i}(t+\tau) v_{j}(t)\right\rangle=0$ for any pair $i, j$. The velocity probability distribution of distinct particles are identical and are characterized by having zero mean and dispersion $u^{2}$, that is,

$$
\begin{equation*}
<v_{i}(t)>=0 \quad \text { and } \quad<v_{i}(t) v_{j}(t)>=u^{2} \delta_{i j} \tag{2}
\end{equation*}
$$

Let us define $\alpha=2 m / \tau$ and $\zeta_{i}(t)=\alpha v_{i}(t)$ so that equation (1) is written in the form of a stochastic difference equation

$$
\begin{equation*}
\alpha \frac{x_{i}(t+\tau)-x_{i}(t)}{\tau}=\zeta_{i}(t)+f_{i}(x(t)) \tag{3}
\end{equation*}
$$

where the noise $\zeta_{i}(t)$ has the properties

$$
\begin{equation*}
<\zeta_{i}(t)>=0 \quad, \quad<\zeta_{i}(t) \zeta_{j}\left(t^{\prime}\right)>=\frac{2 \alpha \Theta}{\tau} \delta_{i j} \delta_{t t^{\prime}} \tag{4}
\end{equation*}
$$

where $\Theta=m u^{2}$ is proportional to the average kinetic energy of the particles

Notice that there are only three independent parameters: the relaxation time $\tau$, the average speed $u$ and the mass $m$ of the particles. The others, $\Theta$ and $\alpha$, are defined in terms of these three. In what follows however we will adopt as independent parameters $\tau, \Theta$ and $\alpha$ so that $m=\tau \alpha / 2$ and $u=\sqrt{2 \Theta / \tau \alpha}$. Formally we may take the limit $\tau \rightarrow 0$, with $\Theta$ and $\alpha$ constants to obtain the following stochastic differential equation or Langevin equation,

$$
\begin{equation*}
\alpha \frac{d x_{i}}{d t}=\zeta_{i}(t)+f_{i}(x) \tag{5}
\end{equation*}
$$

where $\zeta_{i}(t)$ is a white noise with the properties

$$
\begin{equation*}
<\zeta_{i}(t)>=0 \quad, \quad<\zeta_{i}(t) \zeta_{j}\left(t^{\prime}\right)>=2 \alpha \Theta \delta_{i j} \delta\left(t-t^{\prime}\right) \tag{6}
\end{equation*}
$$

In this case we are left with only two parameters which are $\Theta$ and $\alpha$. In any case, we will always consider equations (5) and (6) as meaning equation (3) and (4) with small $\tau$. Eventually we may take the limit $\tau \rightarrow 0$ in case it makes sense.

The Fokker-Planck equation corresponding to the Langevin equation (5) with properties (6), that is, the equation that governs the time evolution of the probability $P(x, t)$ of the system being at position $x$ at time $t$, is given by

$$
\begin{align*}
\alpha \frac{\partial}{\partial t} P(x, t) & =-\sum_{i} \frac{\partial}{\partial x_{i}}\left[f_{i}(x) P(x, t)\right] \\
& +\Theta \sum_{i} \frac{\partial^{2}}{\partial x_{i}^{2}} P(x, t) \tag{7}
\end{align*}
$$

which we call Smoluchowski equation.
By solving this equation we may obtain the average $<F>$ of any state function $F(x)$ by

$$
\begin{equation*}
<F>=\int F(x) P(x, t) d x \tag{8}
\end{equation*}
$$

Quantities that are not functions of positions cannot, in principle, be calculated from this formula. We shall see however that there are some quantities of this type that can be written as an average of a certain function of positions. One purpose of this paper is to show how this
can be done for dissipation of energy in nonequilibrium systems.

In the case of a free particle (Brownian motion) where the forces vanish, the equation

$$
\begin{equation*}
\alpha \frac{d x_{i}}{d t}=\zeta_{i}(t), \tag{9}
\end{equation*}
$$

with the properties (6) can be exactly solved and we get the following result

$$
\begin{equation*}
<x_{i}^{2}>=2 D t \quad, \quad D=\frac{\Theta}{\alpha} \tag{10}
\end{equation*}
$$

The Smoluchowski equation becomes the pure diffusion equation

$$
\begin{equation*}
\frac{\partial}{\partial t} P(x, t)=D \sum_{i} \frac{\partial^{2}}{\partial x_{i}^{2}} P(x, t) \tag{11}
\end{equation*}
$$

The diffusion constant can also be written as $D=$ $\tau u^{2} / 2=\ell^{2} / 2 \tau$ where $\ell=u \tau$.

## III. Equilibrium state

According to classical statistical mechanics the probability distribution $P(x)$ of a system interacting through a potential energy $E(x)$ and in equilibrium with a heat reservoir at an absolute temperature $T$ is given by

$$
\begin{equation*}
P(x)=\frac{1}{Q} \exp \{-\beta E(x)\} \tag{12}
\end{equation*}
$$

where $\beta=1 / k_{B} T$ and $Q$ is a normalization factor. The potential energy $E(x)$ is defined over the position space $x=\left\{x_{i}\right\}$ composed of the positions of the particles. Here we show that if the force $f_{i}(x)$ is conservative, that is, if $f_{i}(x)$ is the gradient of a potential energy $E(x)$, then (12) is the stationary solution of the Smoluchowski equation (7).

Consider the Langevin equation (5) with the noise having the properties (6). The corresponding FokkerPlanck equation (7) can be written in the form

$$
\begin{equation*}
\alpha \frac{\partial}{\partial t} P(x, t)=-\sum_{i} \frac{\partial}{\partial x_{i}} J_{i}(x, t) \tag{13}
\end{equation*}
$$

where $J_{i}(x, t)$ is the current defined by

$$
\begin{equation*}
J_{i}(x, t)=f_{i}(x) P(x, t)-\Theta \frac{\partial}{\partial x_{i}} P(x, t) \tag{14}
\end{equation*}
$$

The stationary solution $P(x)$ satisfies the equation

$$
\begin{equation*}
\sum_{i} \frac{\partial}{\partial x_{i}} J_{i}(x)=0 \tag{15}
\end{equation*}
$$

where the stationary $J_{i}(x)$ current is given by

$$
\begin{equation*}
J_{i}(x)=f_{i}(x) P(x)-\Theta \frac{\partial}{\partial x_{i}} P(x) . \tag{16}
\end{equation*}
$$

If the force $f_{i}(x)$ is the gradient of a potential energy $E(x)$, that is, if

$$
\begin{equation*}
f_{i}(x)=-\frac{\partial E(x)}{\partial x_{i}} \tag{17}
\end{equation*}
$$

then one can write

$$
\begin{equation*}
J_{i}(x)=-P(x) \frac{\partial}{\partial x_{i}}\{E(x)+\Theta \ln P(x)\} \tag{18}
\end{equation*}
$$

from which one easily sees that the stationary solution is

$$
\begin{equation*}
P(x)=A \exp \left\{-\frac{1}{\Theta} E(x)\right\} \tag{19}
\end{equation*}
$$

where $A$ is a normalization constant. This solution not only satisfies equation (15) but also satisfies $J_{i}(x)=0$. That is, the current vanishes at each point of position space.

We conclude that the probability distribution (19) becomes identical to the probability distribution given by equation (12) of classical equilibrium system at temperature $T$ such that

$$
\begin{equation*}
\Theta=k_{B} T . \tag{20}
\end{equation*}
$$

Hence, may say that the stationary state of a system interacting through conservative forces is an equilibrium state. To known a priori whether the stationary state of a given system is an equilibrium state we just have to check whether the condition

$$
\begin{equation*}
\frac{\partial f_{i}}{\partial x_{j}}=\frac{\partial f_{j}}{\partial x_{i}} \tag{21}
\end{equation*}
$$

is satisfied for any pair $(i, j)$. When this condition is satisfied the system has microscopic reversibility.

The results of this section serve as the basis of the numerical simulation method used in equilibrium statistical mechanics known as the Langevin method. Suppose we wish to estimate numerically averages of functions that depend only on the positions of a system described by the potencial energy $E(x)$. If we were able to generate a set of $M$ points $x^{(1)}, x^{(2)}, x^{(3)}, \ldots x^{(M)}$ in position space, each one sampled according to the probability distribution given by equation (12) then an estimative of the average

$$
\begin{equation*}
<F>=\int F(x) P(x) d x \tag{22}
\end{equation*}
$$

will be simply given by

$$
\begin{equation*}
\bar{F}=\frac{1}{M} \sum_{m=1}^{M} F\left(x^{(m)}\right) \tag{23}
\end{equation*}
$$

The numerical problem is then reduced to devise methods of sampling points in the position space according to a probability distribution (12) known a priori. These methods are englobed in basically two types that are known as the Monte Carlo methods and the method of the Langevin equation. Both approaches use a stochastic Markov process defined over the position space whose stationary probability distribution is $P(x)$. The stochastic process can be viewed as a trajectory in position space such that, for long times, the points of the trajectory will be emerging with the desired probability $P(x)$. If we simulate equation (5), with $f_{i}(x)=-\partial E(x) / \partial x_{i}$, we will generate, for long times, a set of points that are distributed according to equation (12).

## IV. Power of a conservative force

The average rate in which any given force $F_{i}(x)$ does work, that is, its average power, is determined by

$$
\begin{equation*}
\Omega=<\sum_{i} \frac{d x_{i}}{d t} F_{i}(x)> \tag{24}
\end{equation*}
$$

We want to find a state function $W(x)$ such that

$$
\begin{equation*}
\Omega=<W(x)> \tag{25}
\end{equation*}
$$

To this end we consider first the case in which the force is conservative, that is, $F_{i}(x)=-\partial V(x) / \partial x_{i}$. In this case

$$
\begin{align*}
\Omega & =-<\sum_{i} \frac{d x_{i}}{d t} \frac{\partial V(x)}{\partial x_{i}}>=-<\frac{d V(x)}{d t}> \\
& =-\frac{d}{d t}<V(x)> \tag{26}
\end{align*}
$$

so that the average power of a conservative force is the rate of the average potential energy $\langle V(x)\rangle$. It is clear that in the stationary state $\langle V(x)\rangle$ does not depend on time and the power $\Omega$ vanishes.

Now, expression (26) gives

$$
\begin{equation*}
\Omega=-\int V(x) \frac{\partial P(x, t)}{\partial t} d x \tag{27}
\end{equation*}
$$

Using the equation (13) we obtain that

$$
\begin{equation*}
\Omega=\frac{1}{\alpha} \sum_{i} \int V(x) \frac{\partial}{\partial x_{i}} J_{i}(x, t) d x . \tag{28}
\end{equation*}
$$

After an integration by parts and using the definition (16) of current and equation (17) we get

$$
\begin{equation*}
\Omega=\frac{1}{\alpha} \sum_{i} \int F_{i}(x)\left\{f_{i}(x) P(x, t)-\Theta \frac{\partial}{\partial x_{i}} P(x, t)\right\} d x \tag{29}
\end{equation*}
$$

Performing once more an integration by parts in the last integral we further obtain that

$$
\begin{equation*}
\Omega=\frac{1}{\alpha} \sum_{i} \int\left\{F_{i}(x) f_{i}(x)+\Theta \frac{\partial F_{i}(x)}{\partial x_{i}}\right\} P(x, t) d x \tag{30}
\end{equation*}
$$

and then

$$
\begin{equation*}
W(x)=\frac{1}{\alpha} \sum_{i}\left\{F_{i}(x) f_{i}(x)+\Theta \frac{\partial F_{i}(x)}{\partial x_{i}}\right\} \tag{31}
\end{equation*}
$$

as desired.

## V. Power of a generic force

Let us now consider the power developed by a generic force $F_{i}(x)$ that might not be conservative. In this case we cannot use the procedure of the preceding section since the force cannot in general be written as the gradient of a certain function. We proceed then as follows. We start by using the discretize form

$$
\begin{equation*}
\theta \frac{x_{i}(t+\tau)-x_{i}(t)}{\tau}+(1-\theta) \frac{x_{i}(t)-x_{i}(t-\tau)}{\tau} \tag{32}
\end{equation*}
$$

of the velocity $d x_{i} / d t$, where $0 \leq \theta \leq 1$. The expression for the power of the force $F_{i}$ is then

$$
\begin{equation*}
\Omega=\sum_{i}<\left(\theta \frac{x_{i}(t+\tau)-x_{i}(t)}{\tau}+(1-\theta) \frac{x_{i}(t)-x_{i}(t-\tau)}{\tau}\right) F_{i}(x(t))> \tag{33}
\end{equation*}
$$

Using the discretize form (3) of the Langevin equation this expression can be written as

$$
\begin{equation*}
\Omega=\frac{1}{\alpha} \sum_{i}<\left(\theta\left[\zeta_{i}(t)+f_{i}(x(t))\right]+(1-\theta)\left[\zeta_{i}(t-\tau)+f_{i}(x(t-\tau))\right]\right) F_{i}(x(t))> \tag{34}
\end{equation*}
$$

Now, in the limit $\tau \rightarrow 0$,

$$
\begin{equation*}
<\left\{\theta f_{i}(x(t))+(1-\theta) f_{i}(x(t-\tau))\right\} F_{i}(x(t))>\quad \rightarrow \quad<f_{i}(x) F_{i}(x)> \tag{35}
\end{equation*}
$$

Also

$$
\begin{equation*}
<\zeta_{i}(t) F_{i}(x(t))>=0 \tag{36}
\end{equation*}
$$

since $\zeta_{i}(t)$ is independent of $x_{i}(t)$. Next we perform the following expansion

$$
\begin{equation*}
F_{i}(x(t+\tau))=F_{i}(x(t))+\sum_{j} F_{i j}(x(t))\left[x_{j}(t+\tau)-x_{j}(t)\right], \tag{37}
\end{equation*}
$$

where $F_{i j}(x)=\partial F_{i} / \partial x_{j}$, or, using again the discretize form (3) of the Langevin equation, we find that

$$
\begin{equation*}
F_{i}(x(t+\tau))=F_{i}(x(t))+\frac{\tau}{\alpha} \sum_{j} F_{i j}(x(t))\left[\zeta_{j}(t)+f_{j}(x(t))\right] \tag{38}
\end{equation*}
$$

From this expression we obtain that

$$
\begin{equation*}
<\zeta_{i}(t) F_{i}(x(t+t))>=2 \Theta<F_{i i}(x(t))> \tag{39}
\end{equation*}
$$

when $\tau \rightarrow 0$. Therefore in the limit $\tau \rightarrow 0$ it follows that

$$
\begin{equation*}
\Omega=\frac{1}{\alpha} \sum_{i}\left\{<f_{i}(x) F_{i}(x)>+(1-\theta) 2 \Theta<F_{i i}(x)>\right\} \tag{40}
\end{equation*}
$$

Since this expression must be valid also for conservative forces we should choose $\theta=1 / 2$ so that it becomes identical with expression (30).

For any force $F_{i}(x)$, conservative or nonconservative, we have then

$$
\begin{equation*}
\Omega=<W(x)> \tag{41}
\end{equation*}
$$

where

$$
\begin{equation*}
W(x)=\frac{1}{\alpha} \sum_{i}\left\{F_{i}(x) f_{i}(x)+\Theta F_{i i}(x)\right\} \tag{42}
\end{equation*}
$$

The quantity vanishes in the case of a conservative force $F_{i}$ no matter whether the force $f_{i}(x)$ that describes the system is conservative or nonconservative.

## VI. Exchanged and dissipated power

Let us consider a system described by a force that might not be conservative, that is, one such that equation (21) might not be satisfied. When the force $f_{i}(x)$ is nonconservative the stationary state will not be an
equilibrium state and we expect that energy will be dissipated continuously. From the result of the previous section, the power of the force $f_{i}(x)$ will be

$$
\begin{equation*}
\Omega=<\frac{1}{\alpha} \sum_{i}\left\{\left[f_{i}(x)\right]^{2}+\Theta f_{i i}(x)\right\}> \tag{43}
\end{equation*}
$$

In the stationary state this expression will give the dissipated power since the exchanged power must vanish. However, if the system is in a transient state this expression gives the total power, that is, the sum of the exchanged and dissipated power. Let us try next to calculate each of these two contributions.

We start by splitting the force $f_{i}(x)$ into two parts

$$
\begin{equation*}
f_{i}(x)=f_{i}^{C}(x)+f_{i}^{D}(x) \tag{44}
\end{equation*}
$$

such that $f_{i}^{C}(x)$ is conservative, that is, $f_{i}^{C}(x)$ is the gradient of a potential $E(x)$, or

$$
\begin{equation*}
f_{i}^{C}=-\frac{\partial E}{\partial x_{i}}, \tag{45}
\end{equation*}
$$

and $f_{i}^{D}(x)$ is a nonconservative force with a vanishing divergence (solenoidal force), that is,

$$
\begin{equation*}
\sum_{i} \frac{\partial f_{i}^{D}}{\partial x_{i}}=0 \tag{46}
\end{equation*}
$$

That such a splitting can always be done can be seen as follows. This last equation implies that

$$
\begin{equation*}
\sum_{i} \frac{\partial f_{i}}{\partial x_{i}}=\sum_{i} \frac{\partial f_{i}^{C}}{\partial x_{i}} \tag{47}
\end{equation*}
$$

or yet

$$
\begin{equation*}
\sum_{i} \frac{\partial f_{i}}{\partial x_{i}}=-\sum_{i} \frac{\partial^{2} E}{\partial x_{i}^{2}} \tag{48}
\end{equation*}
$$

Therefore, if we are given a generic force $f_{i}(x)$, this last equation can be solved to obtain $E(x)$. From $E(x)$ we get $f_{i}^{C}$ by means of equation (45) and $f_{i}^{D}$ by using equation (44).

The total power is then splitted into two parts

$$
\begin{equation*}
\Omega=\Omega_{C}+\Omega_{D} \tag{49}
\end{equation*}
$$

where

$$
\begin{equation*}
\Omega_{C}=<\frac{1}{\alpha} \sum_{i}\left\{f_{i}^{C}(x)\left[f_{i}^{C}(x)+f_{i}^{D}(x)\right]+\Theta f_{i i}^{C}(x)\right\}> \tag{50}
\end{equation*}
$$

is the power of force $f_{i}^{C}(x)$ and

$$
\begin{equation*}
\Omega_{D}=<\frac{1}{\alpha} \sum_{i} f_{i}^{D}(x)\left[f_{i}^{C}(x)+f_{i}^{D}(x)\right]> \tag{51}
\end{equation*}
$$

is the power of force $f_{i}^{D}(x)$. These expression were obtained from the expression by setting $f_{i}=f_{i}^{C}+f_{i}^{D}$ and $F_{i}$ equal to $f_{i}^{C}$ and $f_{i}^{D}$ respectively. We have also used relation (46). Now $\Omega_{C}$ vanishes in the stationary state since it is the power of a conservative force and it is identified with the energy exchanged per unit time. The quantity $\Omega_{D}$ vanishes identically when the system is conservative $\left(f_{i}^{D}=0\right)$ and it is identified with the energy dissipated per unit time. It also must be positive. For the special case in which the two components of the force $f_{i}^{C}(x)$ and $f_{i}^{D}(x)$ are orthogonal this is indeed the case. A system described by a force whose components are of this type is considered in the next section.

Equation (49) may be interpreted by assuming, for instance, that the system is an open system in contact with two reservoir. One of them is the source of energy and the other is the recipient of the dissipated energy. In a given time interval $\Delta t$ the first reservoir releases an amount of energy $\Delta E$ to the system. A part of it $\Delta E_{C}$, is retained by the system whereas the remaining part $\Delta E_{D}$ is dissipated and released to the second reservoir. In the stationary state $\Delta E_{C}=0\left(\Omega_{C}=0\right)$ and all the energy released to the system is dissipated. If, moreover, the stationary state is an equilibrium state, then also $\Delta E_{D}=0\left(\Omega_{D}=0\right)$ and there is no exchange of energy between the source reservoir and the system.

## VII. Orthogonal forces

Consider the special case where the irrotational forces $f_{i}^{C}$ and the solenoidal forces $f_{i}^{D}$ are orthogonal that is

$$
\begin{equation*}
\sum_{i} f_{i}^{C}(x) f_{i}^{D}(x)=0 \tag{52}
\end{equation*}
$$

In this case

$$
\begin{equation*}
\Omega_{D}=<\frac{1}{\alpha} \sum_{i}\left[f_{i}^{D}(x)\right]^{2}> \tag{53}
\end{equation*}
$$

so that the dissipated power is positive and

$$
\begin{equation*}
\Omega_{C}=<\frac{1}{\alpha} \sum_{i}\left\{\left[f_{i}^{C}(x)\right]^{2}+\Theta f_{i i}^{C}(x)\right\}> \tag{54}
\end{equation*}
$$

Although the system is described by force $f_{i}=$ $f_{i}^{C}+f_{i}^{D}$ which is not conservative since $f_{i}^{D}$ is not conservative we show that in this case it is possible to find the nonequilibrium stationary state. To this end, consider the Fokker-Planck equation (7) in the stationary regime, that is,

$$
\begin{equation*}
-\sum_{i} \frac{\partial}{\partial x_{i}}\left[f_{i}^{C}(x) P(x)\right]-\sum_{i} \frac{\partial}{\partial x_{i}}\left[f_{i}^{D}(x) P(x)\right]+\Theta \sum_{i} \frac{\partial^{2}}{\partial x_{i}^{2}} P(x)=0 \tag{55}
\end{equation*}
$$

If the conditions (52) is fulfilled and assuming that $f_{i}^{D}(x)$ is orthogonal to the gradient of $P(x)$, that is

$$
\begin{equation*}
\sum_{i} f_{i}^{D}(x) \frac{\partial}{\partial x_{i}} P(x, t)=0 \tag{56}
\end{equation*}
$$

then the second term in equation (55) vanishes and what is left is a Fokker-Planck equation for a system described by a conservative force $f_{i}^{C}(x)=-\partial E / \partial x_{i}$. The solution is therefore

$$
\begin{equation*}
P(x)=A \exp \left\{-\frac{1}{\Theta} E(x)\right\} \tag{57}
\end{equation*}
$$

We can check from this solution that the assumption (56) is indeed correct. Using (57) we can check explicitly that $\Omega_{C}$ vanishes.

## VIII. Simple model

Let us consider a simple model of just two variables. The force is given by

$$
\begin{equation*}
\mathbf{f}=-K \mathbf{r}+A \widehat{z} \times \mathbf{r} \tag{58}
\end{equation*}
$$

where $K$ and $A$ are two parameters, and $\mathbf{r}$ and $\mathbf{f}$ are the two-dimensional vectors $\mathbf{r}=x_{1} \widehat{x}+x_{2} \widehat{y}$ and $\mathbf{f}=f_{1} \hat{x}+f_{2} \widehat{y}$. As long as $A \neq 0$ the force is nonconservative since $\nabla \times \mathbf{f}=-2 A \hat{z} \neq 0$. In this case

$$
\begin{equation*}
\Omega_{C}=\frac{1}{\alpha}\left\{K^{2}<\mathbf{r}^{2}>-2 \Theta K\right\} \tag{59}
\end{equation*}
$$

and

$$
\begin{equation*}
\Omega_{D}=\frac{1}{\alpha} A^{2}<\mathbf{r}^{2}>. \tag{60}
\end{equation*}
$$

The associate Fokker-Planck equation is

$$
\begin{equation*}
\alpha \frac{\partial P}{\partial t}=-\nabla \cdot(\mathbf{f} P)+\Theta \nabla^{2} P \tag{61}
\end{equation*}
$$

that is,

$$
\begin{equation*}
\alpha \frac{\partial P}{\partial t}=K \nabla \cdot(\mathbf{r} P)-A \nabla \cdot(\widehat{z} \times \mathbf{r} P)+\Theta \nabla^{2} P \tag{62}
\end{equation*}
$$

which can be written as

$$
\begin{equation*}
\alpha \frac{\partial P}{\partial t}=K \nabla \cdot(\mathbf{r} P)-A \widehat{z} \cdot \mathbf{r} \times \nabla P+\Theta \nabla^{2} P \tag{63}
\end{equation*}
$$

Although this equation does not have radial symmetry, it does possess a solution with radial symmetry. Indeed if we assume that $P$ is a function of $|\mathbf{r}|=r$ only then the vector product term in (63) vanishes and the equation becomes a Fokker-Planck equation for a conservative force given by $\mathbf{f}=-K \mathbf{r}$. The time dependent solution of such an equation is

$$
\begin{equation*}
P(r, t)=P_{0}(t) \exp \left\{-\frac{r^{2}}{2 B(t)}\right\} \tag{64}
\end{equation*}
$$

where

$$
\begin{align*}
B(t) & =B_{0} \exp \left\{-\frac{2 K t}{\alpha}\right\}+\frac{\Theta}{K} \\
P_{0}(t) & =\frac{1}{2 \pi B(t)} \tag{65}
\end{align*}
$$

For simplicity we used initial conditions such that $\langle\mathbf{r}\rangle=0$.

To obtain $\Omega_{C}$ and $\Omega_{D}$ we insert the result $<\mathbf{r}^{2}>=$ $2 B(t)$, obtained from (64), into equations (59) and (60) to get

$$
\begin{equation*}
\Omega_{C}=\frac{K^{2}}{\alpha} 2 B_{0} \exp \left\{-\frac{2 K t}{\alpha}\right\} \tag{66}
\end{equation*}
$$

and

$$
\begin{equation*}
\Omega_{D}=\frac{A^{2}}{\alpha} 2\left[B_{0} \exp \left\{-\frac{2 K t}{\alpha}\right\}+\frac{\Theta}{K}\right] \tag{67}
\end{equation*}
$$

Therefore, $\Omega_{C}$ and $\Omega_{D}$ decay exponentially to their stationary values. Notice that $\Omega_{D}$ is always nonnegative since $B(t)$ is nonnegative. The quantity $\Omega_{C}$ on the other hand can be positive or negative depending on the sign of $B_{0}$.

The stationary probability $P(\mathbf{r})$ is given by

$$
\begin{equation*}
P(\mathbf{r})=P_{0} \exp \left\{-\frac{1}{\Theta} E(\mathbf{r})\right\}=P_{0} \exp \left\{-\frac{K}{2 \Theta} r^{2}\right\} \tag{68}
\end{equation*}
$$

where the normalization factor is $P_{0}=K / 2 \pi \Theta$, is obtained by taking the limit $t \rightarrow \infty$ in equation (64). The stationary values $\Omega_{C}$ and $\Omega_{D}$ are

$$
\begin{equation*}
\Omega_{C}=0 \tag{69}
\end{equation*}
$$

and

$$
\begin{equation*}
\Omega_{D}=\frac{2 \Theta}{K \alpha} A^{2} \tag{70}
\end{equation*}
$$

so that the rate of dissipation of energy $\Omega_{D}$ is proportional to $A^{2}$ and, of course, it vanishes when $A$ goes to zero, that is, in the equilibrium case.

We may also calculate the stationary current. From its definition

$$
\begin{equation*}
\mathbf{J}(x)=\mathbf{f}(x) P(x)-\Theta \nabla P(x) \tag{71}
\end{equation*}
$$

we obtain the result

$$
\begin{equation*}
\mathbf{J}(x)=A(\widehat{z} \times \mathbf{r}) P_{0} \exp \left\{-\frac{K}{2 \Theta} r^{2}\right\} \tag{72}
\end{equation*}
$$

Hence, the stationary current is proportional to $A$ and also vanishes in the equilibrium state, as expected.

## IX. Conclusion

We have studied a nonequilibrium classical system of interacting particles by a stochastic approach whose
starting point is a Langevin equation of motion in position space. From this equation we have obtained expressions for the rate in which energy is exchanged as well as the rate in which energy is dissipated. The exchanged and dissipated power are related to the irrotational and solenoidal parts of the forces, respectively. We have applied the results to the case where these two parts are orthogonal and solved exactly a model in which the forces are linear functions of positions.

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[^0]:    *This paper is dedicated to Prof. Roberto Luzzi in his 60th anniversary.

