The Image Method for the Casimir Effect of a Massive Scalar Field

L. C. de Albuquerque*
Departamento de Física Matemática - Universidade de São Paulo
Caixa Postal 66.318 - CEP 05389-970, São Paulo, Brazil

C. Farina†
Instituto de Física - Universidade Federal do Rio de Janeiro
Caixa Postal 68528 - CEP 21945-970, Rio de Janeiro, Brazil

L. G. A. Theodoro
Grupo de Materiais - Instituto de Pesquisa da Marinha
R. Ipiru 2 - CEP 21931-090, Rio de Janeiro, Brazil

Received May 8, 1997

We use the Green function method to compute the Casimir energy of a massive scalar particle. In order to evaluate the Green function we employ the image method which had been used before only for the massless case.

Using perturbative QED calculations, Casimir and Polder[1] showed in 1948 that the dispersive van der Waals forces between two electrically polarizable bodies at large distances (so that retardation effects in the electromagnetic field propagation must be taken into account) can be obtained from the total energy of the system in interaction with the quantized electromagnetic field. However, for infinitely polarizable materials, as for example for two parallel perfectly conducting plates, a much simpler calculation can be done. Basically, for this case, the force between the two plates can be obtained only from the energy of the quantized electromagnetic field. The interaction with matter (the conducting plates) is simulated by Dirichlet boundary conditions at each plate. This result was shown in that same year of 1948 by Casimir [2]. We could even say that the importance of Casimir's paper lies on the method that he used and not on the physical result itself of a macroscopic attractive force between the plates, since this kind of attraction had already been explained for two polarizable atoms in 1930 by London [3]. Casimir's method consists basically in computing how the sum of the zero-point energies of the field modes is affected by the conducting plates. In this quite simple method the attention is focused in the field itself.

Since then, the number of methods used to compute the Casimir energy have remarkably increased. It is not a surprise that particular attention was given to procedures which use directly the relevant Green functions, since physicists are quite familiar with them [4]. For the standard electromagnetic Casimir effect, Lowell S. Brown and G. Jordan Maclay [5] were the first to use an approach based on Green functions. With the aid of the image method these authors were able to compute the complete electromagnetic stress-energy tensor of Casimir's problem. An excellent review on the Casimir effect and related topics, which includes a discussion of the image method, can be found in ref.[6].

In this brief report we shall generalize the image method solution to the Casimir effect of a massive scalar field. The geometry to be considered here is the

*e-mail: claudio@fma1.if.usp.br
†e-mail: farina@if.ufrj.br

standard one, namely, two parallel perfectly conducting plates, located at $x^3 = 0$ and $x^3 = a$. However, instead of computing the stress-energy tensor as in ref.[5], which contains much more information than the necessary for our purposes (recall that only the component T^{00} is needed), we shall compute directly the Casimir energy and for this we shall employ a common procedure in effective action calculations.

First we define the vacuum persistence amplitude in the presence of the plates $\langle 0_+|0_-\rangle^{\rm plates}$. Since we are dealing with a static problem, this amplitude is simply $\langle 0_+|0_-\rangle^{\rm plates} = e^{-\mathcal{E}_{\rm vac}(a)T}$, where $\mathcal{E}_{\rm vac}(a)$ is the vacuum energy shift caused by the presence of the plates and T can be interpreted as the time interval of measurement. Once the effective action is defined such that $\langle 0_+|0_-\rangle^{\rm plates} = e^{W^{(1)}}$, we have [7]

$$\mathcal{E}_{\rm vac}(a) = -\frac{W^{(1)}}{T}\ . \tag{1}$$

The connection between $W^{(1)}$ and Green functions is well known [8], so for the case at hand we can write

$$\frac{\partial W^{(1)}}{\partial m^2} = \frac{i}{2} TrG\,, \tag{2}$$

where $G = (P^2 + m^2)^{-1}$ and the trace must be computed assuming Dirichlet boundary conditions at the plates.

In fact, once we are interested in the energy shift caused by the presence of the plates, we can subtract from the r.h.s. of (2) the TrG computed without boundary conditions (in other words, with the plates infinitely far apart), that is

$$\frac{\partial W^{(1)}}{\partial m^2} = \frac{i}{2}\left[TrG|_{\substack{\rm with\\ \rm plates}} - TrG|_{\substack{\rm no\\ \rm plates}}\right]\ . \tag{3}$$

In order to compute the trace above, we shall first evaluate the Green function $G(x,y) = \langle x|G|y\rangle$ subjected to Dirichlet boundary conditions at the plates. A very convenient way of obtaining an explicit expression for $G(x,y)$ is to employ the image method (see ref.[5] for the massless case).

Basically one sums over all the possible propagations between x and y including those which involve an arbitrary number of reflections on the plates. This is equivalent to sum over the propagations from all possible images of y (the choice of x works as well) as if the plates were perfect mirrors. Besides, there is an extra minus sign associated to the terms involving an odd number of reflections.

Hence, the desired Green function takes the form

$$G(x,y) = \sum_{n=-\infty}^{\infty} G^0(x-y+2anu) - \sum_{n=-\infty}^{\infty} G^0(x+y+2anu)\ , \tag{4}$$

where u is a unit four vector in the OX^3 direction, given by $(0,0,0,1)$ and $G^0(x,y)$ is the Feynman propagator for a relativistic massive scalar field (see the Appendix).

From equations (4) and (3), we get

$$\frac{\partial W^{(1)}}{\partial m^2} = \frac{i}{2}\left(I_1 - I_2\right)\ , \tag{5}$$

where

$$I_1 = \int d^4x \sum_{n=-\infty}^{\infty}{}' G^0(2anu)\ , \tag{6}$$

$$I_2 = \int d^4x \sum_{n=-\infty}^{\infty} G^0(2x+2anu) \tag{7}$$

and the prime in the sum of (6) means that $n = 0$ is excluded from this sum.

Although not apparent, I_2 can be disregarded, since it is indeed independent of the distance a between the plates. This can be shown as follows: defining $\xi = x^3 + an$, eq.(7) takes the form

$$I_2 \;=\; \int dx^0dx^1dx^2 \sum_{n=-\infty}^{\infty}\int_{na}^{(n+1)a} d\xi\, G^0(2x^0,2x^1,2x^2,2\xi)$$

$$= \int dx^0 dx^1 dx^2 \int_{-\infty}^{\infty} G^0(2x^0, 2x^1, 2x^2, 2\xi) . \quad (8)$$

The previous expression is clearly independent of a and has no physical relevance. From equations (6), (5) and the explicit form of the Green function for a massive scalar field given in the Appendix (see (5)), our task is simply to integrate the following equation

$$\frac{\partial W^{(1)}}{\partial m^2} = -\frac{TA}{8\pi^2} \sum_{n=1}^{\infty} \frac{m}{n} K_1(2amn) , \quad (9)$$

where $T = \int dx^0$ and A is the area of the plates.

Using that [9]

$$\frac{m}{n} K_1(2amn) = \frac{\partial}{\partial m^2} \left[-\frac{m^2}{an^2} K_2(2amn) \right] , \quad (10)$$

integration of (9) yields

$$W^{(1)} = \frac{TAm^2}{8\pi^2 a} \sum_{n=1}^{\infty} \frac{1}{n^2} K_2(2amn) + C , \quad (11)$$

where C is a m-independent integration constant to be determined.

Therefore, from (1), the Casimir energy per unit area is given by

$$\frac{1}{A} \mathcal{E}_{\rm vac}(a, m) = -\frac{1}{8\pi^2} \frac{m^2}{a} \sum_{n=1}^{\infty} \frac{1}{n^2} K_2(2amn) + \frac{C}{TA} . \quad (12)$$

In order to fix C we must impose that the expression for $\mathcal{E}_{\rm vac}(a, m)$ must coincide with a particular (already known) value for a given m. The simplest and most convenient choice is $m = \infty$. Recall that in this limit the quantum fluctuations of any relativistic field must disappear and so does the Casimir effect. In other words, $m = \infty$ corresponds to the classical limit and the Casimir effect is a genuine quantum effect. Imposing, then, that $\mathcal{E}_{\rm vac}(a, m = \infty) = 0$, and having in mind the properties of Bessel functions, it is straightforward to show that $C = 0$. Hence, the final result is given by eq.(12), in complete agreement with previous calculations[4].

In this report we used an effective action technique based on Green functions to compute the Casimir energy of a massive scalar field. The relevant Green function was constructed explicitly with the aid of the image method. An advantage of this method is that once we know the Green function without boundary conditions, the image method readily provides the desired Green function (satisfying Dirichlet boundary conditions).

As mentioned before, the image method had already been applied for the standard electromagnetic Casimir effect (plane geometry). For this case the electric modes and magnetic modes decouple and can be treated separately. Besides, it was shown by W. Lukosz[10] that due to different boundary conditions obeyed by the electric and magnetic fields (Dirichlet and Neuman conditions, respectively) the contributions corresponding to an odd number of reflections coming from the E modes cancel exactly those coming from the B modes. The contributions coming from an even number of reflections are exactly the same and this explains in the light of the image method why the standard electromagnetic Casimir energy is twice the result obtained for the Casimir energy of a massless scalar field (this will be true only if we restrict ourselves to plane geometry, but it fails if we go for instance to setups containing spherical conductors).

In our case (massive scalar field) there is no cancellation (since there are no electric and magnetic modes as in the electromagnetic case), but still here the contributions coming from an odd number of reflections disappear from the final result. As we showed explicitly, the reason for that is because these contributions do not contain any dependence on the distance a between the plates.

The authors are indebted with A. Tort, M. V. Cougo-Pinto, H. Boschi-Filho and A. J. Segui-Santonja for enlightening discussions. One of us (C.F.) thanks also A. Das for helpful conversations. LCA aknowledges A. J. da Silva and M. Gomes for many discussions. This work was partially supported FAPESP and CNPq (The National Research Council of Brazil).

Appendix

In this appendix we show an extremely simple derivation of the relativistic Feynman propagator $G^0(x,y)$ for a free massive scalar field which satisfies the following differential equation (métrica (-+++)).

$$(\partial^2 - m^2 + i\varepsilon)G^0(x,y) = -\delta^{(4)}(x-y)\ . \tag{1}$$

Defining the operator $\mathbf{G} = (P^2 + m^2 + i\varepsilon)^{-1}$, we may write

$$\begin{aligned} G^0(x,y) &= \langle x|(P^2+m^2-i\varepsilon)^{-1}|y\rangle \\ &= i\int_0^\infty ds\, e^{-is(m^2-i\varepsilon)}\, \langle x|e^{-isP^2}|y\rangle\ . \end{aligned} \tag{2}$$

Recalling that

$$\langle x|e^{-isP^2}|y\rangle = \frac{-i}{16\pi^2}\exp\{i\frac{(\Delta x)^2}{4s}\}\ , \tag{3}$$

and using the integral representation for the modified Bessel function of second kind [9]

$$\int_0^\infty dx\, x^{\alpha-1}e^{-\gamma x-\beta/x} = 2(\beta/\gamma)^{\alpha/2}K_{\alpha-1}(2\sqrt{\beta\gamma})\ , \tag{4}$$

we finally obtain

$$G^0(x,y) = \frac{i}{4\pi^2}\frac{m}{|x-y|}K_1(m|x-y|)\ , \tag{5}$$

where we have used that $K_{-\nu}(z) = K_\nu(z)$.

References

[1] H. B. G. Casimir and D. Polder, Phys. Rev. **73**, 360 (1948).

[2] H. B. G. Casimir, Koninkl. Ned. Akad. Wetenschap. Proc. **51**, 793 (1948).

[3] F. London, Z. Phys. **63**, 245 (1930).

[4] G. Plunien, B. Muller and W. Greiner, Phys. Rep. **134**, 87 (1986) and references therein.

[5] Lowell S. Brown and G. Jordan Maclay, Phys. Rev. **184** (1969), 1272.

[6] V. M. Mostepanenko and N. N. Trunov, *The Casimir Effect and Its Applications*, Clarendon Press, Oxford, 1997.

[7] W. Greiner, B. Müller and J. Rafelski; *Quantum Electrodynamics of Strong Fields* (Springer-Verlag, New York, 1985).

[8] M. R. Brown and M. J. Duff, Phys. Rev. **D11**, 2124 (1975).

[9] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1980).

[10] W. Lukosz, Z. Physik **258**, 99 (1973).