Transport Processes in Ionized Gases

G. M. Kremer
Departamento de Física, Universidade Federal do Paraná
Caixa Postal 19081, 81531-990, Curitiba, SP, Brazil

Received October 1, 1996

Based on kinetic theory of gases and on the combined method of Chapman-Enskog and Grad, the laws of Ohm, Fourier and Navier-Stokes are derived for a non-relativistic fully ionized gas. Moreover, the combined method is applied to the BGK model of the relativistic Boltzmann equation and the Ohm's law is derived for a relativistic fully ionized gas.

I. Non-relativistic ionized gases

We may say that the objective of thermodynamics of a non-relativistic ionized gas is the determination of the five fields of mass density ϱ, velocity v_i and temperature T in all points of the fluid and at all times. To achieve this objective, we need five field equations that are based on the balance equations of mass density ϱ, momentum density ϱv_i and internal energy density $\varrho\varepsilon$, which read

$$\frac{\partial\varrho}{\partial t}+\frac{\partial\varrho v_i}{\partial x_i}=0,$$
$$\frac{\partial\varrho v_i}{\partial t}+\frac{\partial}{\partial x_j}(\varrho v_i v_j+p_{ij})=\sigma E_i^\star+(\mathbf{I}\times\mathbf{B})_i, \qquad (1)$$
$$\frac{\partial\varrho\varepsilon}{\partial t}+\frac{\partial}{\partial x_j}(\varrho\varepsilon v_j+q_j)+p_{ij}\frac{\partial v_i}{\partial x_j}=\mathbf{I}\cdot\mathbf{E}^\star.$$

In the above equations I_i is the electric current density, p_{ij} the pressure tensor and q_i the heat flux vector. Moreover, $E_i^\star=E_i+(\mathbf{v}\times\mathbf{B})_i$ with $\mathbf{E}$ denoting the external electric field, $\mathbf{B}$ the external magnetic flux density and σ the electrical charge density. The system of balance equations (1) is closed by considering I_i, p_{ij}, ε and q_i as constitutive quantities that are related to the values of the basic fields in a materially dependent manner through constitutives relations.

Here we are interested in a fully ideal ionized gas and we shall base on the kinetic theory of gases to determine the constitutive equations for I_i, $p_{\langle ij\rangle}=p_{ij}-\frac{1}{3}p_{rr}\delta_{ij}$ and q_i, since in this case ε and $p=\frac{1}{3}p_{rr}$ are known functions of (ϱ,T). In kinetic theory of gases the state of a fully non-relativistic ionized gas is characterized by the set of one-particle distribution functions $f_a(\mathbf{x},\mathbf{c},t)$ where $a=E,I$ denotes the constituents of a mixture of electrons and ions. The one-particle distribution function is such that $f(\mathbf{x},\mathbf{c}_a,t)d^3xd^3c_a$ gives at time t, the number of a particles in the volume element $d^3xd^3c_a$ around the particle position $\mathbf{x}$ and velocity $\mathbf{c}_a$. The one-particle distribution function of constituent a obeys the Boltzmann equation:

$$\frac{\partial f_a}{\partial t}+c_i^a\frac{\partial f_a}{\partial x_i}+\frac{e_a}{m_a}\left[E_i+(\mathbf{c}^a\times\mathbf{B})_i\right]\frac{\partial f_a}{\partial c_i^a}=\sum_{b=E}^{I}\int(f_a'f_b'-f_af_b)g^{ba}b\,db\,d\epsilon d^3c_b. \qquad (2)$$

In equation (2) e_a and m_a denote respectively the electric charge and the mass of a particle of constituent a, $\mathbf{g}^{ba}=\mathbf{c}^b-\mathbf{c}^a$ is the relative veloctiy of two particles before collision, b and ϵ are impact parameter, and the primes refer to after collision velocities, with f_b' denoting $f(\mathbf{x},\mathbf{c}_b',t)$ and so forth.

We characterize a macroscopic state of the ionized gas by the fields of v_i velocity of the mixture, ϱ_a partial mass density, $p_{\langle ij\rangle}^a$ partial pressure deviator, J_i^a partial diffusion flux, and q_i^a partial heat flux vector, which are defined by

$$\varrho_a=\int m_af_ad^3c_a, \qquad \text{with} \qquad \varrho=\sum_{a=E}^{I}\varrho_a, \qquad (3)$$

$$v_i=\frac{1}{\varrho}\sum_{a=E}^{I}\varrho_a v_i^a,\qquad\text{with}\qquad v_i^a=\frac{1}{\varrho_a}\int m_a c_i^a f_a d^3c_a,\tag{4}$$

$$T=\frac{2m_a}{3k\varrho_a}\int\frac{1}{2}m_aC_a^2f_ad^3c_a,\qquad p_{\langle ij\rangle}^a=\int m_aC_{\langle i}^aC_{j\rangle}^af_ad^3c_a,\tag{5}$$

$$J_i^a=\int m_a\xi_i^af_ad^3c_a,\qquad q_i^a=\int\frac{1}{2}m_aC_a^2C_i^af_ad^3c_a.\tag{6}$$

In the above equations ϱ is the mass density of the mixture, v_i^a the velocity of constituent a, k the Boltzmann constant, $C_i^a=c_i^a-v_i^a$ and $\xi_i^a=c_i^a-v_i$ partial peculiar velocities, and we have supposed that the constituents in the mixture are at the same temperature T.

The Grad distribution function [1] corresponding to the fields defined above is:

$$f_a=f_a^{(0)}\left\{1+\frac{m_a}{kT\varrho_a}\xi_i^aJ_i^a+\frac{m_a^2}{2(kT)^2\varrho_a}\left[\xi_i^a\xi_j^ap_{\langle ij\rangle}^a+\frac{4}{5}\xi_i^aq_i^a\left(\frac{m_a\xi_a^2}{2kT}-\frac{5}{2}\right)\right]\right\},\tag{7}$$

where

$$f_a^{(0)}=\frac{\varrho_a}{m_a}\left(\frac{m_a}{2kT\pi}\right)^{\frac{3}{2}}e^{-\frac{m_a\xi_a^2}{2kT}},$$

is the Maxwellian distribution function.

By applying the method of Chapman-Enskog [2] for the Grad distribution function (which is the scheme of the combined method of Chapman-Enskog and Grad [3]), it follows an equation that relates the fluxes (partial pressure deviator, partial diffusion flux, partial heat flux vector) with the forces (gradient of velocity, gradient of temperature and external electric field) that reads

$$f_a^{(0)}\left\{\frac{m_a\xi_k^a}{kT}\left[\frac{1}{\varrho_a}\frac{\partial p_a}{\partial x_k}-\frac{1}{\varrho}\frac{\partial p}{\partial x_k}+\left(\sum_{b=E}^{I}\frac{\varrho_be_b}{\varrho m_b}-\frac{e_a}{m_a}\right)E_k^\star\right]+\frac{1}{T}\frac{\partial T}{\partial x_k}\xi_k^a\left(\frac{m_a\xi_a^2}{2kT}-\frac{5}{2}\right)\right.$$
$$+\frac{m_a\xi_k^a\xi_l^a}{kT}\frac{\partial v_{\langle k}}{\partial x_{l\rangle}}+\frac{m_a}{kT\varrho_a}\xi_k^a\left[\frac{\varrho_a}{\varrho}\sum_{b=1}^{r}\frac{e_b}{m_b}(\mathbf{J}^b\times\mathbf{B})_k-\frac{e_a}{m_a}(\mathbf{J}^a\times\mathbf{B})_k\right]$$
$$\left.+\frac{e_a}{m_a}(\xi^a\times\mathbf{B})_k\frac{m_a^2}{(kT)^2\varrho_a}\left[\xi_l^ap_{\langle kl\rangle}^a+\frac{2}{5}q_k^a\left(\frac{m_a\xi_a^2}{2kT}-\frac{5}{2}\right)\right]\right\}=\mathcal{I}_a,\tag{8}$$

where

$$\mathcal{I}_a=\sum_{b=E}^{I}\left\{I_{ab}\left[\xi_k^a\xi_l^a\right]\frac{m_a^2}{2(kT)^2\varrho_a}p_{\langle kl\rangle}^a+I_{ab}\left[\xi_k^b\xi_l^b\right]\frac{m_b^2}{2(kT)^2\varrho_b}p_{\langle kl\rangle}^b\right.$$
$$+I_{ab}\left[\xi_k^a\right]\frac{m_a}{kT\varrho_a}J_k^a+I_{ab}\left[\xi_k^b\right]\frac{m_b}{kT\varrho_b}J_k^b$$
$$\left.+\frac{2}{5}I_{ab}\left[\left(\frac{m_a\xi_a^2}{2kT}-\frac{5}{2}\right)\xi_k^a\right]\frac{m_a^2}{(kT)^2\varrho_a}q_k^a+\frac{2}{5}I_{ab}\left[\left(\frac{m_b\xi_b^2}{2kT}-\frac{5}{2}\right)\xi_k^b\right]\frac{m_b^2}{(kT)^2\varrho_b}q_k^b\right\},\tag{9}$$

with

$$I_{ab}[\phi_a]=\int f_a^{(0)}f_b^{(0)}\left(\phi_a'-\phi_a\right)g^{ba}b\,db\,d\epsilon\,d^3c_b.\tag{10}$$

If we multiply equation (8) successively by $\frac{m_a}{kT\varrho_a}\xi_i^a$, $\frac{m_a^2}{(kT)^2\varrho_a}\left(\frac{m_a\xi_a^2}{2kT}-\frac{5}{2}\right)\xi_i^a$, $\frac{m_a}{kT}\xi_{\langle i}^a\xi_{j\rangle}^a$ and integrate the resulting equations over all values of ξ_a, we get a system of linear equations for $p_{\langle ij\rangle}^a$, J_i^a, q_i^a, which can be solved by inverting second- and fourth-order tensors. The constitutive equations for the total heat flux vector q_i, for the total electric current I_i and for the pressure deviator $p_{\langle ij\rangle}$ of the mixture in a linearized theory follow from:

$$q_i=\sum_{a=E}^{I}\left(q_i^a+\frac{5}{2}\frac{k}{m_a}TJ_i^a\right),\qquad I_i=\sum_{a=E}^{I}\frac{e_a}{m_a}J_i^a,\qquad p_{\langle ij\rangle}=\sum_{a=E}^{I}p_{\langle ij\rangle}^a.\tag{11}$$

They read

$$q_i = -\kappa_{ij}\frac{\partial T}{\partial x_j} + Q_{ij}\mathcal{E}_j, \qquad I_i = \sigma_{ij}\mathcal{E}_j - \frac{Q^\star_{ij}}{T}\frac{\partial T}{\partial x_j}, \tag{12}$$

$$p_{\langle ij\rangle} = -2\eta_{\langle ij\rangle\langle kl\rangle}\frac{\partial v_{\langle k}}{\partial x_{l\rangle}}, \tag{13}$$

where $\mathcal{E}_i = E^\star_i - \frac{m_E m_I T}{e_E m_I - e_I m_E}\frac{\partial}{\partial x_i}\left(\frac{\mu_E - \mu_I}{T}\right)$ is a combined electric field with μ_a denoting the chemical potential of constituent a. Equation $(12)_1$ represents the law of Fourier, while $(12)_2$ the law of Ohm. In these equations κ_{ij} is the thermal conductivity tensor, σ_{ij} is the electric conductivity tensor, while Q_{ij}, $Q^\star_{ij}$ are coefficients of cross effects. Equation (13) is the mathematical expression of the Navier-Stokes law, with $\eta_{\langle ij\rangle\langle kl\rangle}$ denoting the four-order tensor of shear viscosity. The general expression for the second-order tensors in terms of the magnetic flux density is:

$$\{\kappa_{ij}, \sigma_{ij}, Q_{ij}, Q^\star_{ij}\} = a_1\delta_{ij} + a_2\epsilon_{ijk}B_k + a_3 B_i B_j, \tag{14}$$

while that for the four-order tensor reads:

$$\begin{aligned}\eta_{\langle ij\rangle\langle kl\rangle} &= b_1(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl})\\ &+b_2(\varepsilon_{jlr}B_r\delta_{ik} + \varepsilon_{jkr}B_r\delta_{il} + \varepsilon_{ilr}B_r\delta_{jk} + \varepsilon_{ikr}B_r\delta_{jl}) + b_3(\delta_{ik}B_jB_l\\ &+\delta_{il}B_jB_k + \delta_{jk}B_iB_l + \delta_{jl}B_iB_k - \frac{4}{3}\delta_{kl}B_iB_j - \frac{4}{3}\delta_{ij}B_kB_l + \frac{4}{9}B^2\delta_{ij}\delta_{kl})\\ &+b_4(\varepsilon_{ikr}B_rB_jB_l + \varepsilon_{ilr}B_rB_jB_k + \varepsilon_{jkr}B_rB_iB_l + \varepsilon_{jlr}B_rB_iB_k)\\ &+b_5(B_iB_jB_kB_l - \frac{1}{3}B^2B_iB_j\delta_{kl} - \frac{1}{3}B^2B_kB_l\delta_{ij} + \frac{1}{9}B^4\delta_{ij}\delta_{kl}).\end{aligned} \tag{15}$$

In equations (14) and (15) a_1 through b_5 are scalar coefficients, the expressions for these coefficients in terms of the collision integrals of the Boltzmann equation are given in [3]. Moreover, the following Onsager reciprocity relations hold for the coefficients:

$$\sigma_{ij}(\mathbf{B}) = \sigma_{ji}(-\mathbf{B}), \qquad \kappa_{ij}(\mathbf{B}) = \kappa_{ji}(-\mathbf{B}), \tag{16}$$

$$Q_{ij}(\mathbf{B}) = Q^\star_{ji}(-\mathbf{B}) = Q^\star_{ij}(\mathbf{B}) = Q_{ji}(-\mathbf{B}). \tag{17}$$

II. Relativistic ionized gases

The objective of thermodynamics of relativistic ionized gases is the determination of the five fields of particle four-flow N^μ and temperature T in all events x^μ. To determine the five fields, we refer to the following balance equations for the particle four-flow N^μ and for the energy-momentum tensor $T^{\mu\nu}$:

$$\partial_\mu N^\mu = 0, \qquad \partial_\nu T^{\mu\nu} = -\frac{1}{c}F^{\mu\nu}J_\nu, \tag{18}$$

where $F^{\mu\nu}$ is the electromagnetic field tensor and J^μ the charge four-vector.

Using the four-velocity U^μ (such that $U^\mu U_\mu = c^2$) and the projector $\Delta^{\mu\nu} = g^{\mu\nu} - \frac{1}{c^2}U^\mu U^\nu$ (such that $U_\mu\Delta^{\mu\nu} = 0$), where $g^{\mu\nu}$ is the metric tensor with signature (1,-1,-1,-1), we introduce the following decompositions

$$N^\mu = nU^\mu, \qquad F^{\mu\nu} = \frac{1}{c}(U^\mu E^\nu - U^\nu E^\mu) + B^{\mu\nu}, \tag{19}$$

$$T^{\mu\nu} = p^{\langle\mu\nu\rangle} - (p + \varpi)\Delta^{\mu\nu} + \frac{1}{c^2}(U^\mu q^\nu + U^\nu q^\mu) + \frac{en}{c^2}U^\mu U^\nu. \tag{20}$$

where:

$$\begin{cases} n = \frac{1}{c^2}N^\mu U_\mu & \text{- particle number density,}\\ p^{\langle\mu\nu\rangle} = \left(\Delta^\mu_\sigma\Delta^\nu_\tau - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\sigma\tau}\right)T^{\sigma\tau} & \text{- pressure deviator,}\\ p + \varpi = -\frac{1}{3}\Delta^{\mu\nu}T_{\mu\nu} & \text{- pressure+dynamic pressure,}\\ q^\mu = \Delta^\mu_\sigma U_\tau T^{\sigma\tau} & \text{- heat flux,}\\ e = \frac{1}{nc^2}U_\mu T^{\mu\nu}U_\nu & \text{- internal energy per particle,}\\ B^{\mu\nu} = \Delta^\mu_\sigma F^{\sigma\tau}\Delta^\nu_\tau & \text{- magnetic flux tensor,}\\ E^\mu = -\frac{1}{c}F^{\mu\nu}U_\nu & \text{- electric field.}\end{cases} \tag{21}$$

To close the system of balance equations (18) we have to consider $p^{\langle\mu\nu\rangle}$, ϖ, q^μ and J^μ as constitutive quantities that are related to the basic fields through constitutive equations, since for ideal gases p and e are known functions of (n,T).

We shall base on the kinetic theory of gases to get the constitutive equations. The state of a relativistic fully ionized gas in kinetic theory is characterized by the set of one-particle distribution functions $f_a(\mathbf{x},\mathbf{p}_a,t)$ $(a=E,I)$ such that $f_a(\mathbf{x},\mathbf{p}_a,t)d^3x\,d^3p_a$ gives at time t, the number of a particles in the volume element d^3x about $\mathbf{x}$ and momenta in a range d^3p_a around $\mathbf{p}_a$. Here we are interested in a simple derivation of Ohm's law, and for that purpose we write down the BGK model [4] of the Boltzmann equation for the electrons, which reads:

$$p_E^\mu\frac{\partial f_E}{\partial x^\mu}-\frac{e_E}{c}p_{E\nu}F^{\mu\nu}\frac{\partial f_E}{\partial p_E^\mu}=-\frac{1}{\tau_E}(f_E-f_E^{(0)}),\tag{22}$$

where τ_E is a relaxation time, $(p_E^\mu)=(p_E^o,\mathbf{p}_E)$ denotes the momentum four-vector for the electrons and

$$f_E^{(0)}=\frac{n_Ec}{4\pi(m_Ec)^2K_2(\zeta_E)kT}\exp\left(-\frac{p_E^\mu U_\mu}{kT}\right),\qquad \zeta_E=\frac{m_Ec^2}{kT},\tag{23}$$

is the Maxwell-Jüttner distribution function, with $K_2=K_2(\zeta_E)$ denoting the modified Bessel function.

According to the kinetic theory of gases (see for example [5]), the electric current I^μ is defined in terms of the diffusion flux J_a^μ of constituent a by:

$$I^\mu=\sum_{a=E}^{I}e_aJ_a^\mu,\quad J_E^\mu=-J_I^\mu,\quad J_a^\mu=\Delta_\nu^\mu\int cp_a^\nu f_a\frac{d^3p_a}{p_a^o}.\tag{24}$$

The Grad distribution function for the constituent a by considering only the diffusion flux is given by:

$$f_a=f_a^{(0)}\left\{1+\frac{1}{p_a(\zeta_a+5G_a-\zeta_aG_a^2)}\left[-(\zeta_a+5G_a)J_a^\mu p_{a\mu}+\frac{G_a}{kT}J_a^\mu U^\nu p_{a\mu}p_{a\nu}\right]\right\},\tag{25}$$

where $G_a=K_3(\zeta_a)/K_2(\zeta_a)$.

Now by applying the combined method of Chapman-Enskog and Grad it follows

$$\begin{aligned}&f_E^{(0)}\left\{\left[\frac{1}{p_E}\partial_\mu p_E-\frac{h_E}{kT^2}\partial_\mu T\right]p_E^\mu+\frac{1}{kT^2}\left[U_\nu\partial_\mu T-T\partial_\mu U_\nu\right]p_E^\mu p_E^\nu\right\}\\&-\frac{e_E}{c}p_{E\nu}F^{\mu\nu}f_E^{(0)}\left\{-\frac{U_\mu}{kT}\left[1+\frac{1}{(\zeta_E+5G_E-\zeta_EG_E^2)p_E}\right.\right.\\&\times\left.\left(-(\zeta_E+5G_E)J_E^\sigma p_{E\sigma}+\frac{G_E}{kT}J_E^\sigma U^\tau p_{E\sigma}p_{E\tau}\right)\right]\\&+\left.\frac{1}{(\zeta_E+5G_E-\zeta_EG_E^2)p_E}\left[-(\zeta_E+5G_E)J_{E\mu}+\frac{G_E}{kT}(J_{E\mu}U_\sigma+J_{E\sigma}U_\mu)p_E^\sigma\right]\right\}\\&=\frac{-1}{\tau_E}f_E^{(0)}\left\{\frac{1}{p_E(\zeta_E+5G_E-\zeta_EG_E^2)}\left[-(\zeta_E+5G_E)J_E^\mu p_{E\mu}+\frac{G_E}{kT}J_E^\mu U^\nu p_{E\mu}p_{E\nu}\right]\right\},\end{aligned}\tag{26}$$

where h_E is the enthalpy per particle of the electrons.

If we multiply the above equation by p_E^ϵ, integrate the resulting equation over all values of d^3p_E/p_E^0 and eliminate the time derivative of the four-velocity by the use of the balance of linear momentum of the mixture for a gas where $p^{\langle\mu\nu\rangle}=0, q^\mu=0$ hold (which is the so-called Euler gas), i. e.

$$\sum_{a=E}^{I}m_an_aG_aDU^\mu=\nabla^\mu p+\sum_{a=E}^{I}e_an_aE^\mu-\frac{1}{c}B^{\mu\nu}I_\nu,\tag{27}$$

we get

$$\left(\frac{e_E}{m_E G_E} - \frac{e_I}{m_I G_I}\right)\mathcal{E}^\mu = \left[\frac{(m_I n_I G_I)^{-1} + (m_E n_E G_E)^{-1}}{\tau_E(e_E - e_I)} g^{\mu\nu}\right.$$
$$\left. + \frac{1}{c(e_E - e_I)}\left(\frac{e_E}{m_E n_E G_E} + \frac{e_I}{m_I n_I G_I}\right) B^{\mu\nu}\right] I_\nu . \tag{28}$$

In the above equations we have introduced the time derivative $D = U^\mu \partial_\mu$, the gradient $\nabla^\mu = \Delta^{\mu\nu}\partial_\nu$ and

$$\mathcal{E}^\mu = E^\mu + \left(\frac{e_E}{m_E G_E} - \frac{e_I}{m_I G_I}\right)^{-1}\left(\frac{1}{m_E n_E G_E}\nabla^\mu p_E - \frac{1}{m_I n_I G_I}\nabla^\mu p_I\right). \tag{29}$$

By solving equation (28) for the current four-vector, we get the mathematical expression of the relativistic Ohm's law

$$I^\mu = \sigma^{\mu\nu}\mathcal{E}_\nu , \tag{30}$$

where the electric conductivity tensor is given by

$$\sigma^{\mu\nu} = \frac{a}{b(b^2 - \frac{d^2}{2}B^{\epsilon\lambda}B_{\epsilon\lambda})}\left[\left(b^2 - \frac{d^2}{2}B^{\sigma\tau}B_{\sigma\tau}\right) g^{\mu\nu} - b\, d\, B^{\mu\nu} - d^2 B^{\mu\sigma}B_\sigma^\nu\right]. \tag{31}$$

In equation (31) the coefficients a, b and d read

$$a = \left(\frac{e_E}{m_E G_E} - \frac{e_I}{m_I G_I}\right), \quad b = \frac{(m_I n_I G_I)^{-1} + (m_E n_E G_E)^{-1}}{\tau_E(e_E - e_I)}, \tag{32}$$

$$d = \frac{1}{c(e_E - e_I)}\left(\frac{e_E}{m_E n_E G_E} + \frac{e_I}{m_I n_I G_I}\right). \tag{33}$$

Acknowledgements

The author gratefully acknowledges the support of the Conselho Nacional de Desenvolvimento Científico e Tecnológico of Brazil.

References

1. Grad, H., On the Kinetic Theory of Rarefied Gases, Commun. Pure Appl. Math., **2** (1949), 331-407.
2. Chapman, S., Cowling, T. G., The Mathematical Theory of Nonuniform Gases, Cambridge University Press, Cambridge, 1970.
3. Rodbard, M. G., Bezerra Jr., A. G., Kremer, G. M., A Combined Chapman-Enskog and Grad Method. II. Ionized Gases, Phys. Plasmas **2** (1995), 642-648.
4. Bhatnagar, P. L., Gross, E. P., Krook, M., A Model for Collision Processes in Charged and Neutral One-Component Systems, Phys. Rev. **94** (1954), 511-525.
5. de Groot, S. R., van Leeuwen, W. A., van Weert, Ch. G., Relativistic Kinetic Theory, North-Holland, Amsterdam, 1980.