The Lorentz-Dirac Equation for Linear and Nonlinear Potentials

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Received January 31, 1997

We consider the solution of the Lorentz-Dirac equation of Classical Electrodynamics in the two cases of a harmonic force and a constant magnetic field. We compare the spectrum and linewidth to the ones predicted by the quantum theory and find that they agree for large quantum numbers. We review some predictions of the Lorentz-Dirac equation for the isolated hydrogen atom.

I. Introduction

Historically, the understanding of the Electrodynamics of a charged particle interacting with its own electromagnetic field[1, 2, 3, 4, 5, 6], came very late. A classical solution to the self-interaction of a charged particle of very small radius is described by the Lorentz-Dirac equation of motion, henceforth called LDE. The derivation of this equation was first presented in a lecture by Lorentz in 1906, and first published in 1909[1, 4]. Lorentz's theory had many difficulties[6], and a major progress came only in 1938, when Dirac[2] produced a covariant derivation without mention to the structure of the particle. Dirac was also the first to recognize and understand the runaway solutions to the LDE. Dirac's derivation still suffered from an arbitrary mass term[4]. In 1948, the regularization approach, invented to treat the Lamb shift, offered a satisfactory solution to the divergent mass renormalization[3]. Since its derivation up to today this equation has been the subject of numerous publications dealing with its theoretical peculiarities. For example, non-uniqueness of solutions [7], pre-acceleration[6] and runaway, to cite a few. It is interesting to mention that runaway-like behavior has recently been observed in plasma physics[8]. There are many other alternative proposals of radiation reaction forces in the literature[11, 12, 13]. However, some of these alternative forces do not agree with quantum electrodynamics (QED) for the prediction of linewidths[11]. In part I of this letter we show that the LDE predicts a linewidth in agreement with QED for the two cases of a harmonic force and a constant magnetic field. In part II we discuss the predictions of the Lorentz-Dirac equation for the impossible isolated hydrogen atom.

Part I

Linear Potentials

The usual way to use the LDE in most physical situations is that one considers only non-runaway solutions [5], unless there is an external plasma to feed energy for a runaway. Since the LDE is a third-order ordinary differential equation, the asymptotic non-runaway condition makes it necessary to specify only the initial position and velocity, as usual with Newtonian mechanics. The non-relativistic version of the LDE equation for an electron of charge e can be written as

$$\frac{2e^2}{3c^3}\,\dddot{\mathbf{x}}_e = m_e\ddot{\mathbf{x}}_e - \mathbf{F}_{ext}, \qquad (1)$$

where $\mathbf{x}_e$ is the position of the electron, m_e is the renormalized[4] electronic mass, $\mathbf{F}_{ext}$ is the external force acting on the electron, and c is the speed of light. For an external linear force given by $\mathbf{F}_{ext} = -m_e\omega_o^2\mathbf{x}_e$, equation (1) is a third-order linear differential equation with a general solution given by

$$\mathbf{x}_e = \mathbf{A}\exp(\lambda t) + \exp(-\gamma t)(\mathbf{B}\cos(\omega_o t) + \mathbf{C}\sin(\omega_o t)), \qquad (2)$$

where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are constant vectors, λ is the positive real root of $\frac{2e^2}{3c^3}\lambda^3 = m_e\lambda^2 + \omega_o^2$, and γ is given by

$$\gamma = \frac{e^2\omega_o^2}{3m_e c^3}. \quad (3)$$

The physical condition that the solution must not be a runaway implies $\mathbf{A} = 0$. Notice that because $\gamma \ll \omega_o$ for physical values of ω_o (e.g., atomic physics), this solution describes a harmonic oscillation with slowly decreasing amplitude. According to the above solution, a gas of "classical" harmonic oscillators would emit light of frequency ω_o with a linewidth given by γ. The total energy of the electron, kinetic plus harmonic potential, decreases approximately according to

$$E(t) = E(0)\exp(-2\gamma t), \quad (4)$$

for the non-runaway solution.

In the quantum description of this system, the electron originally in a highly excited eigenstate of the harmonic oscillator, of quantum number n, can spontaneously decay because of interaction with the electromagnetic vacuum. The probability per unit time of decaying to the lower level $f = n - 1$ is given by [14]

$$\nu = \frac{4\alpha\omega_o^3}{3c^2}| < f|\mathbf{r}|n > |^2, \quad (5)$$

where $< f|\mathbf{r}|n >$ is the matrix element of the position operator $\mathbf{r}$ and $\alpha = e^2/\hbar c$ is the fine-structure constant. The above matrix element is easily evaluated to $| < n-1|\mathbf{r}|n > |^2 = \hbar n/(2m_e\omega_o)$ [14]. Of course, from the level $n-1$ the electron decays to $n-2$ and so on until it reaches the ground state of the harmonic oscillator. This relaxation process is controlled by the probability of *not decaying* to the lower level after a time t, which for small times is[15]

$$p = \exp(-nDt) \approx 1 - \nu t, \quad (6)$$

where $D \equiv \nu/n = 2\alpha\hbar\omega_o^2/(3m_e c^3)$. According to the statistical treatment of the interaction with the radiation, done in reference[15], the probability of decaying from the level m to the ground state after a time t is given by

$$P_{m\to 0} = (1 - \exp(-Dt))^m. \quad (7)$$

From the above we can define the average time to decay from the level m to the ground state by

$$t_w = \int_o^\infty t\frac{dP}{dt}dt, \quad (8)$$

which evaluates to $\bar{t} = (1/D)\sum_1^m(1/k)$. For large values of m this sum can be approximated by [16]

$$t_w = (1/D)(C + \ln(m)), \quad (9)$$

where $C = 0.5771\ldots$ is Euler's constant. The energy of the mth level of the harmonic oscillator is given by $E_m = (m + \frac{1}{2})\hbar\omega_o$, so that we can write

$$\ln(E_m/E_o) = \ln(m + \frac{1}{2}) + \ln(2), \quad (10)$$

where E_o is the ground-state energy. Substituting (10) into equation (9) we obtain

$$t_w = (1/D)\ln(E_m/E_o) - (1/D)(\frac{1 + 0.24m}{2m}). \quad (11)$$

To compare this with equation (4), we must remember that (11) is the time to arrive at the ground state energy $E_o = \hbar\omega_o/2$. Of course classicaly the energy will decrease to zero, wich is a difference between the theories. With that in mind, if we substitute $E(t_w) = E_o$ and $E(0) = E_m$ into equation (4) we obtain

$$t_w = (1/2\gamma)\ln(E_m/E_o). \quad (12)$$

Both results agree for large values of m because $D = 2\gamma$. Notice that $D = 2\alpha\hbar\omega_o^2/(3m_e c^3)$ depends only on the combination $\alpha\hbar = e^2/c$ and, therefore, our results follow independently of the value of $\hbar$ (or α, if one prefers). The agreement is exact only for large values of m, but even at a low value as $m = 11$ the agreement is within 5%.

We now turn to another simple linear problem of an electron in a constant external magnetic field. The equation of motion for an electron in a constant magnetic field along the $\hat{\mathbf{z}}$ direction is

$$\frac{2e^2}{3c^3}\dddot{\mathbf{x}} = m_e\ddot{\mathbf{x}} + \frac{eB}{c}\dot{\mathbf{x}} \wedge \hat{\mathbf{z}}, \quad (13)$$

where B is the magnitude of the magnetic field. The general non-runaway solution to (13) has a translation with constant velocity along the $\hat{\mathbf{z}}$ direction, which separates out from the dynamics on the xy plane. We disregard this trivial motion and look for a solution on the xy plane of the form $\dot{x} = v_x(0)\exp\lambda t$ and

$\dot{y} = v_y(0)\exp\lambda t$. Substituting the above solution into equation (13) we obtain the homogeneous system

$$\begin{pmatrix} \lambda^2\tau - \lambda & -\omega_c \\ \omega_c & \lambda^2\tau - \lambda \end{pmatrix} \begin{pmatrix} v_x(0) \\ v_y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (14)$$

where $\omega_c = eB/(m_e c)$ and $\tau = 2e^2/(3c^3)$. The above equation has nontrivial solutions only if the determinantal condition $(\tau\lambda^2 - \lambda)^2 + \omega_c^2 = 0$ is satisfied. Together with the non-runaway condition the solution for λ is

$$\lambda = \pm i\omega_c - \frac{1}{2}\tau\omega_c^2, \quad (15)$$

which describes an electron slowly spiraling to the center. The kinetic energy of the electron decays as

$$E(t) = E(0)\exp(-\tau\omega_c^2 t), \quad (16)$$

which is radiated in the form of light of frequency ω_c, and linewidth $\tau\omega_c^2 = 2e^2\omega_c^2/(3m_e c^3)$. For the quantum description of an electron in a constant magnetic field we use the usual Hamiltonian[17]

$$H = \frac{1}{2m_e}|\mathbf{p} - \frac{e}{c}\mathbf{A}|^2, \quad (17)$$

where $\mathbf{A} = -\frac{B}{2}(\mathbf{x} \wedge \hat{\mathbf{z}})$ is the usual vector potential in the Coulomb gauge and $\mathbf{p}$ is the 2-dimensional momentum vector operator conjugate to the 2-dimensional position operator. It is interesting to mention that there are many different hamiltonians for this problem, which produce different results upon quantization[18, 19]. In this letter we will keep to the Hamiltonian of equation (17). This Hamiltonian can be written as

$$H = \frac{1}{2m_e}(p_x^2 + p_y^2) + \frac{\omega_c}{2}L_z + \frac{\omega_c^2}{8}(X^2 + Y^2), \quad (18)$$

which describes a 2-dimensional harmonic oscillator with an extra angular momentum coupling term. Now we follow reference [17] and define the mixed annihilation operators

$$a_d = \frac{1}{\sqrt{2}}(\beta x + \frac{ip_x}{\hbar\beta} - i\beta y + \frac{p_y}{\hbar\beta}) \quad (19)$$

$$a_g = \frac{1}{\sqrt{2}}(\beta x - \frac{ip_x}{\hbar\beta} + i\beta y - \frac{p_y}{\hbar\beta}), \quad (20)$$

where $\beta = \sqrt{\mu\omega_c/2\hbar}$ and x, y and p_x, p_y are the position and momentum operators respectively. It is easy to check that the operators a_d and a_g commute and satisfy the usual bosonic commutation relations

$$[a_d, a_d^\dagger] = 1, \quad (21)$$

$$[a_g, a_g^\dagger] = 1. \quad (22)$$

The Hamiltonian of equation (18), written in terms of the a and $a^\dagger$ operators, is

$$H = \hbar\omega_c(a_d^\dagger a_d + \frac{1}{2}), \quad (23)$$

which is the usual Landau quantization of the levels in the magnetic field. The position operator x can be written in terms of the a and $a^\dagger$ in the usual way

$$x = \frac{1}{2\beta}(a_d + a_d^\dagger + a_g + a_g^\dagger). \quad (24)$$

From the above we can calculate the matrix element to enter in formula (5),

$$< n_d|x|n_d - 1 >= n_d\sqrt{\frac{\hbar}{2\mu\omega_c}}, \quad (25)$$

and the probability to decay to the level $n_d - 1$ per unit time calculated from equation (5) is

$$\nu = \frac{2n_d\alpha\hbar\omega_o^2}{3m_e c^3}. \quad (26)$$

Again, the theory for the decay is the same [15], as outlined bellow equation (6) and formula (9) holds for the quantum decay with $D = 2e^2\omega_c^2/(3m_e c^3)$ and $m = n_d$. The quantum linewidth corresponding to equation (11) agrees with equation (16) in the same way.

Since there are so many alternative proposals for classical radiation reaction forces, it is of interest to check that they agree with QED in the large quantum number limit. For example the proposal of reference [11] does not agree with QED in this limit exactly because of the nonlinearity of the proposed radiation-reaction force. It would be of interest to do the same comparison for nonlinear potentials, but we are not aware of nonlinear cases where solutions to the LDE are known.

Part II

LDE and the Hydrogen Atom

In the following we discuss some predictions of the LDE for the case of a hydrogen atom. First, let us mention that for a hydrogen atom the linewidth of spontaneous emission, calculated from the dynamical decay

of the semi-classical orbits with the radiation damping, agrees well with the quantum values. The time for the electron to spiral down from the classical circular orbit corresponding to the nth Bohr radius to the one of the $(n-1)th$ radius, is calculated in reference[20]. The linewidth agrees with QED, even down to the smallest quantum numbers. Despite this success, in the case of hydrogen it is well known that the electromagnetic model fails in explaining how is it that only one frequency is emitted by the isolated atom. As the electron spirals down, it continuously oscillates with all the intermediate oscillation frequencies, which is the generic case in a nonlinear potential. Of course, if the atom is to emit a single frequency we must have the electron oscillating with the same frequency for the time corresponding to the linewidth (typically 10^6 turns for a hydrogen atom). As it was first argued by Bohr[21], a single *isolated* hydrogen atom can not emit a sharp frequency because radiation losses produce dramatic changes in the frequency over this time. To examine this in detail, let us write the Coulomb force as $\mathbf{F} = -\partial V/\partial \mathbf{x}_e$ and multiply the LDE equation (1) by the electron's velocity. The result can be arranged as

$$\frac{d}{dt}[\frac{m_e}{2}|\dot{\mathbf{x}}_e|^2 + V(\mathbf{x}_e) - \frac{2e^2}{3c^3}\dot{\mathbf{x}}_e \cdot \ddot{\mathbf{x}}_e] = -\frac{2e^2}{3c^3}|\ddot{\mathbf{x}}_e|^2. \quad (27)$$

If a periodic solution to the LDE equation existed, the left side of (27)(which is an exact differential) would integrate to zero over the period of this periodic orbit $\mathbf{x}_e(t)$. This is seen to be impossible because the right side would integrate to a negative number. We can also show that a quasi-periodic orbit, with a finite number of incommensurable sharp frequencies, is impossible: Just integrate (27) over a time that is an approximate minimum multiple of all the periods. Because the periods are not rationally related, one needs to use increasingly better rational approximations to the periods. The above arguments can be made relativistic by using the relativistic version of equation (27)[4]. Let us now review the dynamics predicted by the LDE in the case of a single hydrogen atom with an infinitely heavy proton. For one-dimensional motion, (zero angular momentum) we have the counter-intuitive result that the electron will always runaway from the proton, even at distances much larger than the classical electronic radius (Eliezer's Theorem)[23, 24]. Since runaway solutions are not physical, one can not have physical solutions of the LDE for one-dimensional motion in a hydrogen atom. One is then naturally led to consider the case of nonzero angular momentum. For this nonzero angular momentum case, recent results on singular perturbation theory of scattering[22], (as described by the LDE with a Coulomb potential), predict a minimum angular momentum for the existence of "quasi-mechanical" motions. This result suggests that below a critical angular momentum the electron will also display runaway-like behavior for 2-dimensional motion in a hydrogen atom, according to the LDE. The value of this critical angular momentum, found in reference[22], is $6e^2/c$, some twenty times smaller than $\hbar$.

One could conjecture that the electron in an isolated hydrogen atom would ultimately fall onto the proton because of the radiative self-interaction. What can actually be proved from equation (27) is that a non-runaway orbit staying inside a *bounded* region of space (and avoiding the origin) for all times is impossible. Again, this is done by integrating equation (27) over an arbitrarily large time interval: the left side would integrate to a bounded result while the right would be an arbitrarily large negative number (because there would be a minimum value for the acceleration). Notice that it is incorrect to conclude from this that the electron falls onto the proton, since spatially unbounded runaway-like orbits also satisfy (27). It is interesting to notice that Bohr could not know about the LDE and the possibility of runaway-like motions, and in his work of 1913[21] he mentions only that an accelerated charge should continually loose energy by "Larmor losses". In any case, as we know now, a satisfactory understanding of the problem came only in 1938. Eliezer's theorem is of 1948. A word of caution should be said about the name "runaway-like" for orbits: a better name would be "non-mechanical" orbits, describing the fact that they are not small perturbations of mechanical orbits. There are many modifications that could make these low-angular momentum orbits more "physical". To cite a few, a finite radius in the electron[25], and a finite mass in the proton. We refer to chapter seven of [3] for an excellent discussion of the large literature of electron models and chapter six for Lorentz-Dirac theory and references to the enormous and ever growing literature. After a hundred years of research, no electromagnetic

model is available that will have bound states at the correct atomic magnitude for the isolated hydrogen atom. The reason for this is that the Coulombic orbits always have a nonzero dipole that is constantly producing radiation (unless the electron is so far from the proton that the acceleration vanishes). Differently than hydrogen, a two-electron atom like helium can have orbits with a zero dipole that do not radiate, and classical electrodynamics will produce interesting results in this case (and sharp spectroscopic lines), but we will not develop this here (this is also possible for the hydrogen molecule).

The full runaway is the result in the idealized case of an isolated atom. For the case of a hydrogen gas, because of interatomic interaction, the orbit might start as runaway-like and then be captured by the neighboring atom. This atom has now two electrons, the naked proton is then attracted and the hydrogen molecule is formed. As any chemist knows, hydrogen is very reactive and is found in nature in the form of hydrogen molecule. It has been only 16 years since atomic hydrogen has been stabilized in the laboratory, at cold temperatures and strong magnetic fields[26]. It is found experimentally the tendency to form hydrogen molecules at highly excited states[26].

I acknowledge discussions with my colleagues G. L. Ferreira, R. Napolitano, S. Mizrahi, V. V. Dodonov and F. Alcaraz. I also had a Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) fellowship during the completion of this work.

References

[1] H. A. Lorentz, *The Theory of Electrons and Its Applications to the Phenomena of Light and Radiant Heat*, Second edition, Dover Publications, Inc., New York (1952).

[2] P. A. M. Dirac, *Proc. R. Soc. London* **A167**, 148 (1938); *Phys. Rev.* **74**, 817 (1948).

[3] S. Coleman (chapter 6) and P. Pearle (chapter seven) in *Electromagnetism, Paths to Research*, edited by D. Teplitz, Plenum Press, New York (1982).

[4] F. Rohrlich, *Classical Charged Particles*, Addison-Wesley Publishing (1965).

[5] G. N. Plass, *Reviews of Modern Physics* **33**, 37 (1961).

[6] J. D. Jackson, *Classical Electrodynamics*, Wiley, New York (1975).

[7] R. Blanco, *Phys. Rev. E* **51**, 680 (1995).

[8] B. Kurzan, K.-H. Steuer, and G. Fussmann, *Phys. Rev. Lett.* **75**, 4626 (1995).

[9] L. Page, *Phys. Rev.* **11**, 376 (1918), see also G. A. Schott, *Electromagnetic Radiation*, Cambridge (1912).

[10] A. O. Rahilly, *Electromagnetic Theory*, vol.1, Dover (1965).

[11] F. V. Hartemann and N. C. Luhmann, *Phys. Rev. Lett* **74**, 1107 (1995).

[12] L. Herrera, *Phys. Lett. A* **145**, 14 (1990).

[13] G. W. Ford and R. F. O'Connell, *Phys. Lett. A* **174**, 182 (1993).

[14] E. Merzbacher, *Quantum Mechanics*, Second Edition, p. 563, John Wiley & Sons, Inc. (1970).

[15] V. V. Dodonov and S. S. Mizrahi, *Physica A* **214**, 619 (1995).

[16] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, p. xxxviii, Academic Press, New York (1965).

[17] C. Cohen-Tannoudji, B. Diu and F. Laloë, *Quantum Mechanics*, Vol. 2, p. 733, Hermann Publishers, Paris (1977).

[18] V. V. Dodonov, V. I. Man'ko, and V. D. Skarzhinsky, *Hadronic Journal*, **4**, 1734 (1981).

[19] I. Anderson and G. Thompson, *Memoirs of the American Mathematical Society*, vol. **98**, number 473 (1992).

[20] M. Seidl and P. O. Lipas, *Eur. J. Phys.* **17**, 25 (1996).

[21] N. Bohr, *Philosophical Magazine*, **26**, 1 (1913) and *Philosophical Magazine*, **26**, 476 (1913).

[22] A. Carati and L. Galgani, *Nonlinearity* **6**, 905 (1993).

[23] W. E. Baylis and J. Huschilt, *Physical Review D*, **13**, 3262 (1976).

[24] S. Parrott, *Foundations of Physics* **23**, 1093 (1993).

[25] E. Moniz and D. Sharp, Phys. Rev. D **15**, 2850 (1977).

[26] E. Meyer, Z. Zhao, J. Mester and I. Silveira, *Phys. Rev. B*, **50**, 9339 (1994).