On the Correspondence Principle for Rotations*

S. G. Mokarzel[1], M. C. Nemes[2] and A.F.R. de Toledo Piza[1]

[1]Instituto de Física, Departamento de Física-Matemática
Universidade de São Paulo, Cx. Postal 66318, 05315-970, São Paulo, SP, Brasil

[2] Departamento de Física, Universidade de Minas Gerais,
Cx. Postal 702, 31270-970, Belo Horizonte, MG, Brasil

Received October 20, 1996

In the present work we calculate the radiation emitted from a charged rotating ellipsoid in the context of Classical Electrodynamics. The results are compared with its Quantum Mechanical version. An explicit derivation of the correspondence principle is given, showing that for angular momenta of the order of $60\hbar$ the classical regime is reached within 12% accuracy.

I. Introduction

The correspondence principle states that the results of Classical Physics should be contained in the Quantum Mechanical results as limiting cases. The limit should be reached for "large quantum numbers". This idea has been a powerful guide for theoretical conjectures and the construction of the Quantum theory.

Historically the most important and successful investigation based on this Principle is the spatial structure of the hydrogen atom. Many other applications in several contexts followed to our knowledge. Not as much attention has been devoted to the Correspondence Principle in the context of the radiation emitted by rotating charged bodies.

In what follows we set up a classical model for the emitted radiations of a rotating charged ellipsoid. We compare the results with the quantum mechanical version of the model and answer the following question: What is the order of magnitude of the angular momentum of a rotating rigid body for which the classical regime has been reached?

In section II we perform the classical calculations of the emmitted radiation of a rotating ellipsoid. In section III we construct its quantum version and compare the results. Conclusions are given in section IV.

II. Classical calculation

The vector field associated to the quadrupole radiation can be written as

$$\mathbf{A}_{\rm quadr} = \frac{1}{6c^2 r}\ddot{\mathbf{D}} \tag{1}$$

where r is the distance between some origin O and the observation point, c is the light velocity and

$$\mathbf{D} = \sum_{\alpha,\beta} D_{\alpha,\beta} n_\beta \quad \mathbf{n} = \frac{\mathbf{r}}{r} \tag{2}$$

with

$$D_{\alpha,\beta} = \int_{V'} [3x'_\alpha x'_\beta - r'^2 \delta_{\alpha,\beta}]\rho(\mathbf{x}')d^3x' \tag{3}$$

$\rho(\mathbf{x}')$ being the charge density which generates the electromagnetic fields. It is enough to calculate the magnetic field,

$$\mathbf{H} = \frac{1}{6c^3 r}\dddot{\mathbf{D}} \times \mathbf{n} \tag{4}$$

in order to obtain Poynting's vector

$$\vec{\sigma} = \frac{c}{4\pi}H^2\mathbf{n} \tag{5}$$

and the radiation intensity in the solid angle $d\Omega$

$$\begin{aligned} dI &= |\vec{\sigma}|dS & (6)\\ &= \frac{c}{4\pi}H^2 r^2 d\Omega\ . & (7)\end{aligned}$$

The integrated intensity is thus given as

*Work partially supported by CNPq.

$$I = \frac{1}{180c^5}\sum_{\alpha,\beta}(\dddot{D}_{\alpha,\beta})^2 \ . \qquad (8)$$

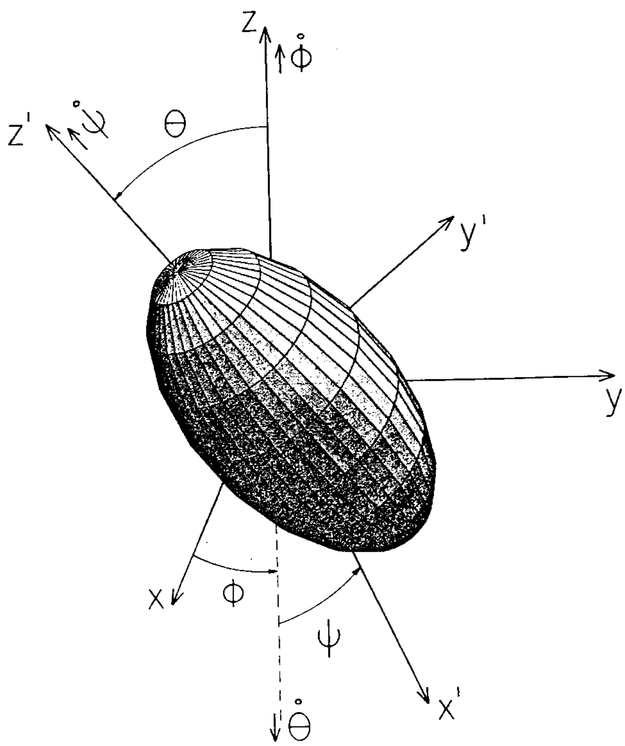

Figure 1. Classical Model.

Now consider the ellipsoid in Fig. 1, which rotates with constant angular velocity $\vec{\omega}$ around an arbitrary axis. In the body's rotating frame the quadrupole tensor $D_{\alpha,\beta}$ is given by

$$D'_{\alpha,\beta} = \frac{ze}{5}(a^2-b^2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad (9)$$

where ze corresponds to the total charge. In order to obtain $D_{\alpha,\beta}$, ie, eq.(9) in the laboratory system, one needs to calculate

$$D_{\gamma,\delta} = T_{\gamma,\alpha}T_{\delta,\beta}D'_{\alpha,\beta} = T_{\gamma,\alpha}D'_{\alpha,\beta}T_{\beta,\delta} \qquad (10)$$

where the matrix T effects such transformation and can be written as

$$T_{\gamma,\alpha} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \qquad (11)$$

where

$$\begin{aligned}
T_{11} &= \cos\psi\,\cos\phi - \cos\theta\,\sin\phi\,\sin\psi \\
T_{12} &= -\sin\psi\,\cos\phi - \cos\theta\,\sin\phi\,\cos\psi \\
T_{13} &= \sin\theta\,\sin\phi \\
T_{21} &= \cos\psi\,\sin\phi + \cos\theta\,\cos\phi\,\sin\psi \\
T_{22} &= -\sin\psi\,\sin\phi + \cos\theta\,\cos\psi\,\cos\phi \\
T_{23} &= -\sin\theta\,\cos\phi \\
T_{31} &= \sin\theta\,\sin\psi \\
T_{32} &= \sin\theta\,\cos\psi \\
T_{33} &= \cos\theta
\end{aligned}$$

Using the above expressions we get

$$D_{\gamma,\delta} = \frac{ze}{5}(a^2-b^2)\times\begin{pmatrix} 1-3\sin^2\theta\,\sin^2\phi & 3\sin^2\theta\,\sin\phi\,\cos\phi & -3\sin\theta\,\cos\theta\,\sin\phi \\ 3\sin^2\theta\,\sin\phi\,\cos\phi & 1-3\sin^2\theta\,\cos^2\phi & 3\sin\theta\,\cos\theta\,\sin\phi \\ -3\sin^2\theta\,\cos\theta\,\sin\phi & 3\sin\theta\,\cos\theta\,\cos\phi & 3\sin^2\theta-2 \end{pmatrix} \qquad (12)$$

It is an interesting exercise in classical mechanics to obtain for the present case, from Euler's equations the following important relations

$$\theta = cte \quad \cos\theta = \frac{K}{J} \quad \phi = \frac{J}{\jmath}t \qquad (13)$$

where K is the projection of the angular moment on the rotor axis,

J is the total angular moment, and

$\jmath$ is the moment of inertia

It is now a simple matter to obtain

$$\sum_{\alpha,\gamma}(\dddot{D}_{\gamma,\delta})^2 = \left[\frac{ze}{5}(a^2-b^2)3\dot{\phi}^3\right]^2(32-62\cos^2\theta+30\cos^4\theta)$$
$$= \frac{3}{25}[ze(a^2-b^2)]^2\left(\frac{J}{\jmath}\right)^6\left(96-186\frac{K^2}{J^2}+90\frac{K^4}{J^4}\right) \quad (14)$$

Inserting this expression into (8) we get

$$I=\frac{1}{180c^5}\frac{3}{25}[ze(a^2-b^2)]^2\left(\frac{J}{\jmath}\right)^6\left(96-186\frac{K^2}{J^2}+90\frac{K^4}{J^4}\right) \quad (15)$$

In order to compare with the quantal result we rewrite the above expressions in terms of the average radius R_0 and deformation parameter β according to (ref. [1])

$$R_{MAX}=a=R_0\left[1+\sqrt{\frac{5}{16\pi}}(2\beta)\right]$$

$$R_{min}=b=R_0\left[1+\sqrt{\frac{5}{16\pi}}(-\beta)\right] \quad (16)$$

Assuming β small (not an essential hypothesis) we get

$$(a^2-b^2)\cong\frac{3}{2}\sqrt{\frac{5}{\pi}}R_0^2\beta \quad (17)$$

according to which the classical expression for the emitted radiation is given by

$$I_{clas}=\frac{\pi}{75c^5}\left(\frac{3}{4\pi}zeR_0^2\beta\right)^2\left(\frac{J}{\jmath}\right)^6\left(96-186\frac{K^2}{J^2}+90\frac{K^4}{J^4}\right) \quad (18)$$

III. Quantum calculation

The essential ingredients for the corresponding quantum calculation are the following:

a) the system's hamiltonian

$$H=H_R+H_{rad}+H_{int} \quad (19)$$

where H_R describes the nucleus and models a rigid rotor with axial symmetry

$$H=\frac{1}{2\jmath}\mathbf{J}^2+J_3^2\left(\frac{1}{2\jmath_3}-\frac{1}{2\jmath}\right) \quad (20)$$

and $\jmath$ stands for the moment of inertia in $x--y$ directions and $\jmath_3$ for the z direction. $\mathbf{J}$ is the angular momentum operator and J_3 its z–projection on the proper axis. The eigenfunctions and eigenvalues of H_R are well known (refs. [2-4])

$$H_R D^J_{MK}(\alpha\beta\gamma)=\left[\frac{\hbar^2 J(J+1)}{2\jmath}+\hbar^2K^2\left(\frac{1}{2\jmath_3}\frac{1}{2\jmath}\right)\right]D^J_{MK}(\alpha\beta\gamma) \quad (21)$$

where

$$D^J_{MK}(\alpha\beta\gamma)=\langle JmK|e^{-i/\hbar\ \alpha\hat{J}_z}e^{-i/\hbar\ \beta\hat{J}_y}e^{-i/\hbar\ \gamma\hat{J}_z}|JMK\rangle$$

b) The Quantized Radiation Field H_{rad} (refs. [5])

$$H_{rad} = \frac{1}{8\pi} \int \left[\frac{1}{c^2} \dot{\mathbf{A}}^2 + (\nabla \times \mathbf{A})^2 \right] d^3r \tag{22}$$

with

$$\mathbf{A}(\mathbf{r}, t) = \sqrt{\frac{\hbar c^2}{2\pi^2}} \sum_\lambda \int \frac{d\mathbf{k}}{\sqrt{2w_k}} [\mathbf{e}^*_{k\lambda} b^\dagger_\lambda(\mathbf{k}) e^{-(\mathbf{k}\cdot r - w_k t)} \mathbf{e}_{k\lambda} b_\lambda(\mathbf{k}) e^{(\mathbf{k}\cdot r - w_k t)}] \tag{23}$$

where λ describes the possible polarization of the field and $b^\dagger$, b are the usual photon creation and annihilation operator (see ref. [5])

c) The interaction hamiltonian H_{int},

$$H_{int} = -\frac{1}{c} \int \mathbf{j} \cdot \mathbf{A} d\mathbf{r} \tag{24}$$

where $\mathbf{j}$ stands for the nuclear current operator.

d) Fermi's Golden rule which gives the transition probability per unit time from an initial state $|i\rangle$, in our case the rotor in a given excited state and zero photons, to a final state $|f\rangle$ where the nucleus is de-excited by means of emitting a photon

$$dw(i \to f\mathbf{k}\sigma) = \frac{2\pi}{\hbar} \frac{k^2}{\hbar c} |M_{fi}|^2 d\Omega \tag{25}$$

with

$$M_{fi}(\mathbf{k}, \sigma) \equiv \langle f; \mathbf{k}\sigma | H_{int} \ i; 00 \rangle \tag{26}$$

The calculation now is standard and contained in many text books. For details the reader is referred to (ref. [5]) whose notation we follow. There the calculation is performed step by step until the final result

$$w(Ej : i \to f) = 8\pi c \frac{e^2}{\hbar c} \frac{j+1}{j[(2j+1)!!]^2} k^{2j+1} B(Ej : i \to f) \tag{27}$$

with

$$B(Ej : i \to f) = \frac{1}{[2j_i + 1]e^2} (J_f || Q_j^{(E)} || J_i)^2 \tag{28}$$

In eq.(28) $(J_f||Q_j^{(E)}||J_i)$ stands for the reduced matrix element of the electric multipole operator, defined as:

$$Q_{jm}^{(E)} \equiv \int d\mathbf{r} r^j Y^*_{jm} \rho \tag{29}$$

In the above expressions the dynamics of the transition is contained in $B(E2 : i \to f)$, in particular in the reduced matrix element $(J_f||Q_j^{(E)}||J_i)$. The factors involving the photon angular momentum j and k^{2j+1} come from the usual expansion of the electromagnetic field for large wavelengths as compared to the system's sizes involved.

The calculation of $B(E2)$ for a rigid rotor can be found in ref([1]) with techniques of ([2]) and is given by

$$B(E2 : J+1 \to J) = \frac{1}{e^2} \left(\frac{3}{4\pi} zeR_0^2\beta \right)^2 \times \frac{6}{4} \frac{(J-K+1)(J+K+1)}{\left(J+\frac{1}{2}\right)(J+2)(J+1)J} \left(\frac{2J+1}{2J+3} \right)^2 \tag{30}$$

$$B(E2 : J+2 \to J) = \frac{1}{e^2} \left(\frac{3}{4\pi} zeR_0^2\beta \right)^2 \times \frac{3}{8} \frac{(J-K+2)(J+K+2)(J-K+1)(J+K+1)}{\left(J+\frac{5}{2}\right)\left(J+\frac{3}{2}\right)(J+2)(J+1)} \tag{31}$$

which gives for the transition probability

$$T(E_2)_{J+1\to J} = \frac{\pi}{75c^5}\left(\frac{3}{4\pi}zeR_0^2\beta\right)^2\left[\frac{\hbar}{J}(J+1)\right]^6 K^2\times 6\frac{(J-K+1)(J+K+1)}{(J+2)\left(J+\frac{1}{2}\right)(J+1)J}\left(\frac{2J+1}{2J+3}\right)^2 \quad (32)$$

$$T(E_2)_{J+2\to J} = \frac{\pi}{75c^5}\left(\frac{3}{4\pi}zeR_0^2\beta\right)^2\left[\frac{\hbar}{J}(2J+3)\right]^6 \times \frac{3}{2}\frac{(J-K+2)(J+K+2)(J-K+1)(J+K+1)}{\left(J+\frac{5}{2}\right)\left(J+\frac{3}{2}\right)(J+2)(J+1)} \quad (33)$$

We are now in a position to investigate the Correspondence Principle, i.e, to compare Quantum and Classical results in the limit of total large angular momenta

First Case: $K=0$

In this case there will be no $J+1\to J$ transition, since it proportional to K. It becomes then a simple matter to verify that eq.(33) tends, in this limit, to

$$T(E_2)_{J+2\to J} \to \frac{\pi}{75c^5}\left(\frac{3}{4\pi}zeR_0^2\beta\right)^2\left(\frac{J}{J}\right)^6 96 = T^{cl}(E_2)_{J+2\to J} \quad (34)$$

which corresponds precisely to the classical expression eq.(18) when $\theta=\frac{\pi}{2}$, ie the rotating system is contained in the $x-y$ plane (see Fig 1).

Second Case: $K\neq 0$

This corresponds to having arbitrary θ. In this case there will be a contribution from the transition $J+1\to J$. when performing the classical limit in the eqs. (32 and 33) we shall consider K finite, $K\gg 1$ and $\frac{K}{J}\ll 1$. We find

$$T(E_2)_{J+1\to J} \to T^{cl}(E_2)_{J+1\to J}\left(\frac{3}{4\pi}zeR_0^2\beta\right)^2\left(\frac{J}{J}\right)^6\left[6\left(\frac{K}{J}\right)^2 - 6\left(\frac{K}{J}\right)^4\right] \quad (35)$$

$$T(E_2)_{J+2\to J} \to T^{cl}(E_2)_{J+2\to J}\left(\frac{3}{4\pi}zeR_0^2\beta\right)^2\left(\frac{J}{J}\right)^6\left[96 - 192\left(\frac{K}{J}\right)^2 + 96\left(\frac{K}{J}\right)^4\right] \quad (36)$$

Finally, the Correspondence Principle in this case can be written as

$$T^{cl}(E_2)_{J+1\to J} + T^{cl}(E_2)_{J+2\to J} = I_{clas} \quad (37)$$

where I_{clas} as given in eq.(18)

The accuracy of the classical limit for $J\cong 60\hbar$ can be studied by comparing the classical and quantal radiation intensity as function of J, for the case, $K=0$. In Fig. 2 we show the percentage of discrepancy $\frac{|T^{cl}(E_2)_{J+2\to J} - I_{clas}|}{I_{clas}}100$ as a function of J.

We see that when $J\cong 60\hbar$ this discrepancy is around 12%.

IV. Conclusions

In the present work we have investigated the correspondence principle in the context of the radiation emitted by a rotating ellipsoid. We found that the classical limit is reached for angular momenta of the order $60\hbar$ within 12% accuracy. In fact angular momenta of the order $60\hbar$ have recently been found experimentally in superdeformed nuclei produced by means of the fusion of several combinations of nuclei. Their masses lie in the range 160 - 170 and their spectra are found to be in excellent agreement with those predicted by the prolate rigid rotor with axis ratio very close to 2:1. Such structure covers a spin interval from $34\hbar$ to $58\hbar$ (refs. [6-8]).

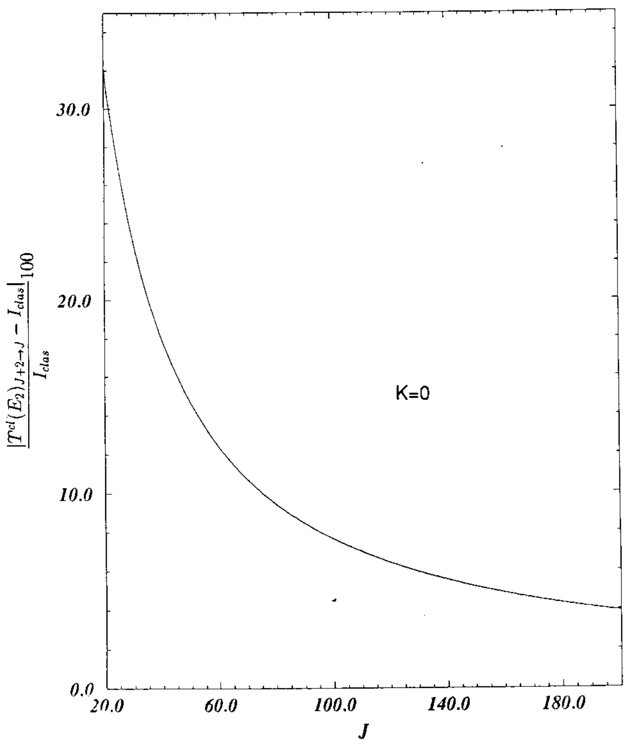

Figure 2. Percentage of discrepancy between quantum and classical result as a function of the total angular momentum.

References

1. A. Bohr and B.R. Mottelson; *Nuclear Structure* V.I e II N. Y., Benjamim (1969).
2. D. M. Brink and G. R. Satchler; *Angular Momentum Oxford,* Clarendon (1971).
3. M. E. Rose; *Elementary Theory of Angular Momentum* N. Y., Wiley (1957).
4. A. R. Edmonds; *Angular Momentum in Quantum Mechanics* Princeton, Princeton Universithy Press, (1960).
5. A. de Shalit and H. Feshbach; *Theoretical Nuclear Physics* Volume I: Nuclear Structure N. Y., John Wiley and Sons (1974).
6. R. Chapman, J.C. Lisle, J.N. Mo, E. Paul, A. Simcock, J C. Willmott, J.R. Leslie, H.G. Prince, P.M. Walter, J.C. Bacelar, J.D. Garrett, G.B. Hagemann, B. Herskind, A. Holm and P.J. Nolan; Phys. Rev. Lett. **51**, 2265 (1983).
7. J.C. Bacelar, M. Diebel, C. Ellegaard, J.D. Garrett, G.B. Hagemann, B. Herskind, A. Holm, C.X. Yang, J.Y. Zhang, P.O. Tjom, J.C. Lisle; Nucl. Phys. **A 442**, 509 (1985).
8. J. Recht, Y.K. Agarwal, K.P. Blume, M. Guttormsen, H. Hübel, H. Kluge, K.H. Maier, A. Maj, N. Roy, D.J. Decman, J. Dudek, and W. Nazarewicz; Nucl. Phys. **A 440**, 366 (1985).