A Note on Moments of Gaussian Grassmann Multivariable Integrals

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Using the anticommuting nature of the generators of the Grassmann algebra, we show that the moments of gaussian Grassmann multivariable integral are related to the cofactors of the matrix of the gaussian exponential.

The path integral method has played a central role in many branches in Physics. Among the physycal quantities that can be written as a path integral, we have the grand canonical partition function for self–interacting fermionic system, whose path integral expression is[1]:

$$\mathcal{Z}(\beta,\mu)=\int \mathcal{D}\psi(\vec{\mathbf{x}},\tau)\mathcal{D}\bar{\psi}(\vec{\mathbf{x}},\tau)e^{-\int d\vec{\mathbf{x}}\int_0^\beta d\tau\,\bar{\psi}(\vec{\mathbf{x}},\tau)\partial_\tau\psi(\vec{\mathbf{x}},\tau)}e^{-\int_0^\beta d\tau\,\mathbf{K}}, \tag{1}$$

where $\psi(\vec{\mathbf{x}},\tau)$ and $\bar{\psi}(\vec{\mathbf{x}},\tau)$ are anticommuting variables with continuous labels $\vec{\mathbf{x}}$ and τ, satisfing anti–periodic boundary conditions in the temperature parameter τ. β is the inverse of temperature ($\beta=\frac{1}{kT}$), and,

$$\mathbf{K}=\mathbf{H}-\mu\mathbf{N}, \tag{2}$$

$\mathbf{H}$ is the hamiltonian of the fermionic system, μ the chemical potencial and $\mathbf{N}$ the total number of particles operator. The variables of the funcional integral, $\psi(\vec{\mathbf{x}},\tau)$ and $\bar{\psi}(\vec{\mathbf{x}},\tau)$, are generators of a Grassmann algebra.

The interaction part of $\mathbf{H}$, introduce in the exponential of expression (1), at least, one power term of the variables of degree bigger than two. Due to our inability to calculate integrals beyond the gaussian approximation, the contribution from the interacting terms of hamiltonian to the r.h.s. of eq.(1) is obtained using perturbation theory. The terms in the perturbation serie of eq.(1) corresponds to the moments of the gaussian Grassmann integral. Formally, these integrals are calculated by introducing in eq.(1) an external Grassmann current and taking functional derivatives with respect to it.

Recently, we studied a lattice version of expression (1). The grand canonical partition function of self–interacting fermionic models, in the high temperature limit, was calculated writing eq.(1) as a multivariable Grassmann integral[2]. For doing that, it was not enough to have formal results for the moments of gaussian grassmannian multivariable integrals, we needed their explicity expressions.

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Following references [2], we consider in this note a regularized version of the fermionic system. It is considered on a lattice with a finite number of space sites. In this case, the moments of gaussian fermionic path integral is regularized and becomes a multivariable grassmannian integral, where the integral variables are generators of a Grassmann algebra. Here, we derive a closed form for the moments of gaussian Grassmann integrals, for a Grassmann algebra of dimension 2^{2N}, where N is any positive integer.

For a Grassmann algebra of dimension 2^{2N}, whose generators are: $\{\eta_1, \cdots, \eta_N;$ $\bar{\eta}_1, \cdots, \bar{\eta}_N\}$, it is a known result that[3]:

$$\int \prod_{i=1}^{\mathrm{N}} d\eta_i d\bar{\eta}_i \; e^{\sum_{i,j=1}^{\mathrm{N}} \bar{\eta}_i A_{ij} \eta_j} = det(\mathbf{A}), \tag{3}$$

where the entries A_{ij} of matrix $\mathbf{A}$ are commuting quantities. The matrix $\mathbf{A}$ does not need to have an inverse.

We will show in this note that the moments of integral (3) are cofactors of matrix $\mathbf{A}$.

We first consider the case where there is only one product $\bar{\eta}_l \eta_k$ in the integrand of the gaussian integral (3), that is,

$$M(l,k) \equiv \int \prod_{i=1}^{\mathrm{N}} d\eta_i d\bar{\eta}_i \; \bar{\eta}_l \eta_k \; e^{\sum_{i,j=1}^{\mathrm{N}} \bar{\eta}_i A_{ij} \eta_j} , \tag{4}$$

where l, k are fixed and $1 \leq l, k \leq N$.

Due to the definition of integration in the Grassmann algebra[3] and that for all generators of the algebra we have: $\bar{\eta}_i^2 = \eta_i^2 = 0, i = 1, \cdots, N$, the only non–null terms in eq.(4) are the ones where the integrand has N products of the form: $\bar{\eta}_i \eta_j$. Expanding the exponential on the r.h.s. of eq.(4) and only keeping the non–null terms, $M(l,k)$ is written as:

$$\begin{aligned} M(l,k) &= \int \prod_{i=1}^{N} d\eta_i d\bar{\eta}_i \; \bar{\eta}_l \eta_k \frac{1}{(N-1)!} \sum_{\substack{i_1,\cdots,i_{N-1}=1 \\ j_1,\cdots,j_{N-1}=1}}^{N} A_{i_1 j_1} \cdots A_{i_{N-1} j_{N-1}} \times \\ &\times \; \bar{\eta}_{i_1} \eta_{j_1} \, \bar{\eta}_{i_2} \eta_{j_2} \, \cdots \bar{\eta}_{i_{N-1}} \eta_{j_{N-1}}, \end{aligned} \tag{5}$$

where the indices are such that $i_n \neq l, n = 1, \cdots, N-1$, and $j_n \neq k, n = 1, \cdots, N-1$. Once the product $\bar{\eta}_{i_n} \eta_{j_n}$ is a commutative quantity, each term in the sum of eq.(5) appears $(N-1)!$ times. Those equal terms, in the sum, correspond to the $(N-1)!$ permutations of a given configuration:

$$A_{i_1 j_1} \, A_{i_2 j_2} \cdots A_{i_{N-1} j_{N-1}}.$$

The $(N-1)!$ distinct terms in eq.(4) can be generated by fixing one configuration for $\{i_1, i_2, \cdots, i_{N-1}\}$, for example, we choose: $\{i_1 = 1, \cdots, i_{l-1} = l-1, i_l = l+1, \cdots, i_{N-1} = N\}$, and, taking all the terms coming from the sum over the indices $j_n, n = 1, \cdots, N-1$. Therefore, $M(l,k)$ becomes

$$\begin{aligned} M(l,k) &= \int \prod_{i=1}^{N} d\eta_i d\bar{\eta}_i \; \bar{\eta}_l \eta_k \sum_{\substack{j_1,\cdots,j_{N-1}=1 \\ j_n \neq k}}^{N} A_{1 j_1} \cdots A_{l-1, j_{l-1}} A_{l+1, j_l} \cdots A_{N j_{N-1}} \times \\ &\times \; \bar{\eta}_1 \eta_{j_1} \, \bar{\eta}_2 \eta_{j_2} \, \cdots \bar{\eta}_{l-1} \eta_{j_{l-1}} \bar{\eta}_{l+1} \eta_{j_l} \cdots \bar{\eta}_N \eta_{j_{N-1}}. \end{aligned} \tag{6}$$

Renaming the variables: $j_l \to j_{l+1}, j_{l+1} \to j_{l+2}, \cdots, j_{N-1} \to j_N$, we have that:

$$M(l,k) = \int \prod_{i=1}^{N} d\eta_i d\bar{\eta}_i \, \bar{\eta}_l \eta_k \sum_{\substack{j_1,\cdots,j_{l-1}=1 \\ j_{l+1},\cdots,j_N=1}}^{N} A_{1j_1} \cdots A_{l-1,j_{l-1}} A_{l+1,j_{l+1}} \cdots A_{Nj_N} \times$$
$$\times \; \bar{\eta}_1 \eta_{j_1} \, \bar{\eta}_2 \eta_{j_2} \cdots \bar{\eta}_{l-1} \eta_{j_{l-1}} \bar{\eta}_{l+1} \eta_{j_{l+1}} \cdots \bar{\eta}_N \eta_{j_N}. \quad (7)$$

Even though the restriction $j_n \neq k$ is not written explicitly, it is guaranteed by the presence of η_k in the integrand of eq.(7).

Defining the matrix $\mathbf{B}(l,k)$ as:

$$B_{ij}(l,k) = \begin{cases} A_{ij}, & \text{if } i \neq l \text{ and } j \neq k \\ \delta_{il}\delta_{jk}, & \text{if } i = l \text{ or } j = k \end{cases}, \quad (8)$$

and $i,j = 1,2,\cdots,N$, we have that

$$\bar{\eta}_l \eta_k = \sum_{j_l=1}^{N} B_{lj_l}(l,k) \, \bar{\eta}_l \eta_{j_l}. \quad (9)$$

Substituting eq.(9) in expression (7), and remembering that in this expression the indices j_n never assume the value k, A_{nj_n} can be replaced by B_{nj_n} in eq.(7). The expression of $M(l,k)$ is written as:

$$M(l,k) = \int \prod_{i=1}^{N} d\eta_i d\bar{\eta}_i \sum_{j_1,\cdots,j_N=1}^{N} B_{1j_1} \cdots B_{l-1,j_{l-1}} B_{l,j_l} \cdots B_{Nj_N} \times$$
$$\times \; \bar{\eta}_1 \eta_{j_1} \cdots \bar{\eta}_l \eta_{j_l} \cdots \bar{\eta}_N \eta_{j_N}. \quad (10)$$

Integrating over $\bar{\eta}_i$, we get

$$M(l,k) = \sum_{j_1,\cdots,j_N=1}^{N} B_{1j_1} \cdots B_{l-1,j_{l-1}} B_{l,j_l} \cdots B_{Nj_N} \int d\eta_N \cdots d\eta_1 \, \eta_{j_1} \eta_{j_2} \cdots \eta_{j_N}. \quad (11)$$

Using the definition of determinant[4,5], we finally have that

$$\begin{aligned} M(l,k) &= det\mathbf{B} \quad (12) \\ &= (-1)^{l+k} A(l,k), \end{aligned}$$

where $A(l,k)$ is the minor determinant of matrix $\mathbf{A}$, when the line l and the column k are cut. $M(l,k)$ is the cofactor of matrix $\mathbf{A}$.

Using an analogous procedure, we now consider the case of moments of the gaussian Grassmann multivariable integral when we have m products:

$$\bar{\eta}_{l_1} \eta_{k_1} \, \bar{\eta}_{l_2} \eta_{k_2} \cdots \bar{\eta}_{l_m} \eta_{k_m}$$

in the integrand of eq.(3), where $m \leq N$.

We represent the fixed sets of indices as: $L = \{l_1, l_2, \cdots, l_m\}$ and $K = \{k_1, k_2, \cdots, k_m\}$. In analogy to eq.(4), we write $M(L,K)$ as

$$M(L,K) \equiv \int \prod_{i=1}^{\mathrm{N}} d\eta_i d\bar{\eta}_i \, \bar{\eta}_{l_1} \eta_{k_1} \cdots \bar{\eta}_{l_m} \eta_{k_m} \; e^{\sum_{i,j=1}^{\mathrm{N}} \bar{\eta}_i A_{ij} \eta_j}, \quad (13)$$

where the products are ordered such that: $l_1 < l_2 < \cdots < l_m$ and $k_1 < k_2 < \cdots < k_m$.

Using an analogous reasoning, as before, and keeping only the distinct terms in eq.(13), it becomes

$$M(L,K) = \int \prod_{i=1}^{N} d\eta_i d\bar{\eta}_i \, \bar{\eta}_{l_1}\eta_{k_1}\cdots\bar{\eta}_{l_m}\eta_{k_m}\times$$
$$\times \sum_{\substack{j_1,\cdots,j_{l_1-1}=1 \\ j_{l_1+1},\cdots,j_{l_2-1}=1 \\ \vdots \\ j_{l_m+1},\cdots,j_N=1}}^{N} A_{1j_1}\cdots A_{l_1-1,j_{l_1-1}}A_{l_1+1,j_{l_1+1}}\times\cdots\times A_{l_2-1,j_{l_2-1}}A_{l_2+1,j_{l_2+1}}\times$$
$$\cdots\times A_{l_m-1,j_{l_m-1}}A_{l_m+1,j_{l_m+1}}\cdots A_{Nj_N}\times\ \bar{\eta}_1\eta_{j_1}\ \cdots\bar{\eta}_{l_1-1}\eta_{j_{l_1-1}}\ \bar{\eta}_{l_1+1}\eta_{j_{l_1+1}}\ \cdots$$
$$\bar{\eta}_{l_2-1}\eta_{j_{l_2-1}}\ \bar{\eta}_{l_2+1}\eta_{j_{l_2+1}}\cdots\bar{\eta}_{l_m-1}\eta_{j_{l_m-1}}\ \bar{\eta}_{l_m+1}\eta_{j_{l_m+1}}\cdots\bar{\eta}_N\eta_{j_N}. \quad (14)$$

Due to the presence of: $\bar{\eta}_{l_1}, \bar{\eta}_{l_2}, \cdots \bar{\eta}_{l_m}$ and $\eta_{k_1}, \eta_{k_2}, \cdots, \eta_{k_m}$ in the integrand of eq.(14), the lines $l_1, l_2, \cdots, l_m$ and columns $k_1, k_2, \cdots, k_m$ of matrix $\mathbf{A}$ do not contribute to $M(L,K)$.

Defining the matrix $\mathbf{B}(L,K)$ as

$$B_{ij}(L,K) = \begin{cases} A_{ij}, & \text{if } i \neq l_1,\cdots,l_n \text{ and } j \neq k_1,\cdots k_n \\ \delta_{il_1}\delta_{jk_1}, & \text{if } i = l_1 \text{ or } j = k_1 \\ \vdots & \\ \delta_{il_m}\delta_{jk_m}, & \text{if } i = l_m \text{ or } j = k_m, \end{cases} \quad (15)$$

and $i, j = 1, 2, \cdots, N$, we can write that

$$\begin{aligned} \bar{\eta}_{l_1}\eta_{k_1} &= \sum_{j_{l_1}=1}^{N} B_{l_1 j_{l_1}}(L,K)\,\bar{\eta}_{l_1}\eta_{j_{l_1}}, \\ \bar{\eta}_{l_2}\eta_{k_2} &= \sum_{j_{l_2}=1}^{N} B_{l_2 j_{l_2}}(L,K)\,\bar{\eta}_{l_2}\eta_{j_{l_2}}, \\ &\vdots \\ \bar{\eta}_{l_m}\eta_{k_m} &= \sum_{j_m=1}^{N} B_{l_m j_{l_m}}(L,K)\,\bar{\eta}_{l_m}\eta_{j_{l_m}}. \end{aligned} \quad (16)$$

Once in expression (14) the lines: $l_1, \cdots, l_m$, and, the columns: $k_1, k_2, \cdots, k_m$ of matrix $\mathbf{A}$ do not contribute, then the elements A_{ij} can be replaced by $B_{ij}(L,K)$. The expression $M(L,K)$ is rewritten as

$$\begin{aligned} M(L,K) &= \int \prod_{i=1}^{N} d\eta_i d\bar{\eta}_i \sum_{j_1,\cdots,j_N=1}^{N} B_{1j_1}\cdots B_{l-1,j_{l-1}}B_{l,j_l}\cdots B_{Nj_N}\times \\ &\times\ \bar{\eta}_1\eta_{j_1}\ \cdots\bar{\eta}_l\eta_{j_l}\ \cdots\bar{\eta}_N\eta_{j_N} = det\mathbf{B}(L,K), \end{aligned} \quad (17)$$

since the r.h.s. of eq.(17) is equal to the r.h.s. of eq.(10).

From the definition of $\mathbf{B}(L,K)$ (eq.(15)) we finally have

$$M(L,K) = (-1)^{(l_1+l_2+\cdots+l_m)+(k_1+k_2+\cdots+k_m)}A(L,K), \quad (18)$$

where $A(K, L)$ is the determinant of the matrix obtained from matrix $\mathbf{A}$ by cutting the lines: $\{l_1, l_2, \cdots, l_n\}$, and, the columns: $\{k_1, k_2, \cdots, k_n\}$.

In summary, we can say that the effect of the presence of a product $\bar{\eta}_l \eta_k$ in the integrand of the gaussian integral (3), is to replace the line l of matrix $\mathbf{A}$, A_{lj}, by δ_{jk}, and its column k, A_{ik}, by δ_{il}. The results (12) and (18) are valids even if the matrix $\mathbf{A}$ does not have an inverse.

The determinant of matrices $\mathbf{B}(L, K)$ (eq.(15)) are easily written in terms of a determinat of smaller dimension. We conclude that the presence of products of Grassmann generators in the integrand of the gaussian grassmannian integrals reduce the dimension of the matrix which we have to calcule the determinant.

The results derived here are easily applied to a self–interacting fermionic model regularized on a lattice[6].

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