PPN Constraints for a Static Spherical Solution with Naked Singularity

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A spherical static exact solution for gravity coupled non-trivially to two scalar fields exhibits a naked singularity when no ordinary matter is present. A PPN analysis of this solution shows that observational constraints gives an upper bound on the strength of the scalar fields. These fields are constant when the post Newtonian limit of General Relativity is assumed to hold true. We claim, therefore, that the original theory may describe gravitation at large scale only, but not at local scale.

Introduction

In a recent paper[1], it was found a class of static symmetrical spherical solution exhibiting a naked singularity. This exact solution has a singularity for $r = 0$, but there is no horizon event covering it; a light ray emitted from the origin can attain any point of the space in a finite time. The theory from which this solution was obtained is a five-dimensional coupling of gravity to a Maxwellian-type field where the fifth dimension is time-like. When it is performed a dimensional reduction to four dimensions, we obtain an effective Lagrangian where gravity is coupled to two scalar fields, one coming from the fifth dimension, and other from the Maxwellian-type field. These scalar fields couple non-trivially between themselves.

There are several possibilities to discuss the relevance of such a theory depending on the scale (cosmological or astronomical) where it is assumed to be meaningfull. The important question of the stability will be studied in a subsequent article. In the present paper, we study the observational constraints arising from the three classical tests when the theory is considered to be relevant at the astronomical scale.

In [1] it was determined cosmological and static spherical symmetric exact solutions. Can this spherical static solution, with its non-standard properties, describe the physics at local scale, for example, at the scale of the solar system? Can we give a general answer to the previous question in the general case when no ordinary matter is present? How can we impose observational limits on the strength of the scalar fields (and as a consequence on the introduction of extra time-like dimensions)? These are the question we want to answer here.

We will consider the three classical tests of General Relativity in the context of the above theory. We will use the Parametrized Post-Newtonian (PPN) approach. This paper is organized as follow. In the next section, we exhibit the exact solution for a spherical symmetrical static metric in the realm of the model described above. In section III, we give the PPN expansion of this exact solution, showing that it leads to a remarkable negative answer (although the curvature is not zero the relativitic PPN effects for the three tests are absent). In section IV, we generalize the previous result: we show that the model admits no solution satisfying the three tests except in the trivial case when the scalar fields are constant. While in the previous sections the scalar fields were the only sources of the gravitational field, in section V we consider that "ordinary" matter is present as well. We prove that the trivial solution, with constant scalar fields, is the only solution which has the same post-Newtonian limit as General Relativity. In the conclusion we discuss the implications of these results for the domain of applicability of this kind of theory.

II. The Naked singularity solution

The effective Lagrangian which comes from the dimensional reduction of the Einstein-Maxwell theory in five dimensions can be written as [1]

$$L = \sqrt{-g}\left(\Phi R + \frac{\Psi_{;\rho}\Psi^{;\rho}}{\Phi}\right) \ , \tag{1}$$

which leads to the following field equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{3}{2\Phi^2}\left(\Psi_{;\mu}\Psi_{;\nu} - \frac{1}{2}g_{\mu\nu}\Psi_{;\rho}\Psi^{;\rho}\right) + \frac{1}{\Phi}(\Phi_{;\mu;\nu} - g_{\mu\nu}\Box\Phi); \tag{2}$$

$$\Box\Phi - \frac{\Psi_{;\rho}\Psi^{;\rho}}{\Phi} = 0; \tag{3}$$

$$\Box\Psi - \frac{\Phi_{;\rho}\Psi^{;\rho}}{\Phi} = 0. \tag{4}$$

The static spherical symmetric metric has the form

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + \sin\theta d\phi^2). \tag{5}$$

With this choice for the metric, the field equations become

$$\frac{1}{2}\frac{B''}{B} - \frac{1}{4}\frac{B'}{B}\left(\frac{B'}{B} + \frac{A'}{A}\right) - \frac{1}{r}\left(\frac{A'}{A} + 2\frac{\Phi'}{\Phi}\right) = \frac{1}{2}\frac{B'}{B}\frac{\Phi'}{\Phi} \ ; \tag{6}$$

$$\frac{1}{2}\frac{B''}{B} - \frac{1}{4}\frac{B'}{B}\left(\frac{B'}{B} + \frac{A'}{A}\right) + \frac{1}{r}\frac{B'}{B} = -\frac{1}{2}\left(\frac{\Psi'}{\Phi}\right)^2 - \frac{1}{2}\frac{B'}{B}\frac{\Phi'}{\Phi} \ ; \tag{7}$$

$$\Phi'' + \left(-\frac{1}{2}\frac{A'}{A} + \frac{1}{2}\frac{B'}{B} + \frac{2}{r}\right)\Phi' = \frac{{\Psi'}^2}{\Phi} \ ; \tag{8}$$

$$\Psi'' + \left(-\frac{1}{2}\frac{A'}{A} + \frac{1}{2}\frac{B'}{B} + \frac{2}{r}\right)\Psi' = \frac{\Phi'}{\Phi}\Psi' \ ; \tag{9}$$

$$1 - \frac{r}{2A}\left(-\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{A} = \frac{r^2}{2A}\left(\frac{\Psi'}{\Phi}\right)^2 + \frac{r}{A}\frac{\Phi'}{\Phi} \ . \tag{10}$$

Equation (9) admits the first integral,

$$\Psi' = \Psi_0 u \, \Phi \quad , \quad u = \left(\frac{A}{B}\right)^{\frac{1}{2}}\frac{1}{r^2} \quad , \quad \Psi_0 = cte. \tag{11}$$

With the aid of (11) we can rewrite (8) as

$$\Phi' = \Psi_0 u \Psi + a'\Psi_0 u \quad , \quad a' = cte. \tag{12}$$

If we denote F as a primitive function of $u(F' = u)$, we have

$$\Phi = a e^{\Psi_0 F} + b e^{-\Psi_0 F} \ , \tag{13}$$

$$\Psi = a e^{\Psi_0 F} - b e^{-\Psi_0 F} - a' \ . \tag{14}$$

The solution found in [1] corresponds to the case where $a = 0$, $k^2 = \Psi_0$ and and $B\Phi = 1$. It leads to the expressions

$$A = \frac{x^2}{1+x^2}\left(\frac{x}{1+\sqrt{1+x^2}}\right)^{\pm 2} \ , \tag{15}$$

$$B = \left(\frac{x}{1+\sqrt{1+x^2}}\right)^{\pm 2} \ , \tag{16}$$

where $x = r/k$.

When $x \to 0$, the functions (15,16) go to zero, with the upper sign, and to infinity with the lower sign. However, in this limit we are inside the source and these solutions are no longer valid. So, we should, in order to complete the analysis of the possible existence of a singularity, work out the interior problem. This will be discussed in section V.

III. PPN development of the spherical symmetric metric

In general, the PPN parameters are obtained by developing the static spherical symmetric metric in its isotropic form,

$$ds^2 = \bar{B}(r)dt^2 - \bar{A}(r)(d\bar{r}^2 + \bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2\theta d\phi^2) \ . \tag{17}$$

The central mass field is described, in the post-Newtonian approximation, by means of a development in $\bar{B}$ and $\bar{A}$ [2]:

$$\bar{B} = 1 - 2\alpha\frac{GM}{\bar{r}c^2} + 2\beta\frac{G^2M^2}{\bar{r}^2c^4} + \ldots \ ; \tag{18}$$

$$\bar{A} = 1 + 2\gamma\frac{GM}{\bar{r}c^2} + \ldots \ ; \tag{19}$$

One can also use the standard coordinates

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + r^2\sin^2\theta d\phi^2) \ . \tag{20}$$

In this case, the PPN development takes the form,

$$B = 1 - 2\alpha\frac{GM}{rc^2} + 2(\beta - \alpha\gamma)\frac{G^2M^2}{r^2c^4} + \ldots \ ; \tag{21}$$

$$A = 1 + 2\gamma\frac{GM}{rc^2} + \ldots \ . \tag{22}$$

The experimental values are $\alpha = \beta = \gamma \simeq 1$:

- $\alpha = 1$ permits to find the Newtonian theory in the low speed limit;
- $\beta = 1$ leads to acceptable values for the advance of the perihelion of the planets;
- $\gamma = 1$ is compatible with the observed deviations of light rays by the Sun and with the Shapiro's effect.

Among the Eddington and Robertson parameters (α, β and γ), β is experimentally known with the lower accuracy; however, its uncertainty remains smaller than 1%. The General Relativity theory predicts $\alpha = \beta = \gamma = 1$ [3].

We develop now the functions B and A given by (15.16). We get,

$$B = 1 - \frac{n}{x} + \frac{n^2}{2}\frac{1}{x^2} + \ldots, \tag{23}$$

$$A = 1 - \frac{n}{x} + \ldots, \tag{24}$$

where $n = \pm 2$ and $\frac{1}{x} = \frac{k}{r}$. The condition $\alpha = 1$ leads to $nk = 2\frac{GM}{c^2}$. So,

$$B = 1 - 2\frac{GM}{rc^2} + 2\frac{G^2M^2}{r^2c^4} + \ldots, \tag{25}$$

$$A = 1 - 2\frac{GM}{rc^2} + \ldots, \tag{26}$$

Identifying this development to (21,22) we obtain $\gamma = -1$ and $\beta = 0$. As consequence, the deviation of light rays $\Delta = 2\frac{GM}{r_0c^2}(1+\gamma)$ and the advance of the perihelion of the planets $\Delta\phi = 6\frac{GM}{L}\left(\frac{2-\beta+2\gamma}{3}\right)$ are zero.

Remarkably, we have a null result for the gravitational effects of a space-time with spherical symmetry described by (15,16), in spite of the fact that this space-time is strongly curved. This suggests that the considered solutions may not describe the field of a central mass.

IV. Central mass field

The field equations (6,7,8,9,10) admit as a spherical symmetric static solution $\Phi = cte$ and $\Psi = cte$, leading to

$$ds^2 = \left(1 - 2\frac{GM}{rc^2}\right) dt^2 - \left(1 - 2\frac{GM}{rc^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \ , \quad (27)$$

that is, the Schwarzschild metric in the standard coordinates. The theory under consideration can describe the field of a central mass. We can ask if this "trivial solution" is unique.

To answer this, we consider the following development for B and A:

$$B = 1 - \frac{\alpha}{x} + \frac{\beta}{x^2} + ..., \quad (28)$$

$$A = 1 - \frac{\gamma}{x} + \frac{\delta}{x^2} + ..., \quad (29)$$

where $\frac{1}{x} = 2\frac{GM}{rc^2}$. Using this relation we can rewrite the field equations (6,7,8,9,10) employing the substitutions

$$\frac{d}{dr}() = ()' \rightarrow \frac{d}{dx}() = ()' \ ; \quad (30)$$

$$r \rightarrow x \ . \quad (31)$$

Moreover, the substitution $\Psi_0 \rightarrow 2\Psi_0\frac{GM}{c^2}$ implies

$$\frac{d\Psi}{dx} = \Psi' = \Psi_0 \left(\frac{A}{B}\right)^{1/2} \frac{1}{x^2}\Phi \ . \quad (32)$$

Then, we find the following developments for the terms of interest to us:

$$B = 1 - \frac{\alpha}{x} + \frac{b}{x^2} + O_3 \ \ , \ \ A = 1 + \frac{\gamma}{x} + \frac{\delta}{x^2} + O_3 \ ; \quad (33)$$

$$\frac{B'}{B} = \frac{\alpha}{x^2} + \frac{\alpha^2 - 2b}{x^3} + O_4 \ \ , \ \ \frac{A'}{A} = \frac{\gamma}{x^2} + \frac{\gamma^2 - 2\delta}{x^3} + O_4 \ ; \quad (34)$$

$$\frac{B''}{B} = -2\frac{\alpha}{x^3} + \frac{6b - 2\alpha^2}{x^4} + O_5 \ \ , \ \ \left(\frac{\Psi'}{\Phi}\right)^2 = \frac{\Psi_0^2}{x^4} + O_5 \ . \quad (35)$$

Using (10) we calculate Φ'/Φ:

$$\frac{\Phi'}{\Phi} = \frac{\gamma - \alpha}{2}\frac{1}{x^2} + \frac{\gamma^2 - \alpha^2 + 2b - \Psi_0^2}{2}\frac{1}{x^3} + O_4 \ . \quad (36)$$

Equation (7) leads to

$$\left(2b - \frac{1}{2}\alpha(\alpha - \gamma) + \Psi_0^2\right)\frac{1}{x^4} + O_5 = 0 \quad (37)$$

implying $2b - \frac{1}{2}\alpha(\alpha-\gamma) + \Psi_0^2 = 0$. If we now impose $2b - \frac{1}{2}\alpha(\alpha-\gamma) = 0$ as in General Relativity, we have $\Psi_0 = 0$. So, we find $\Phi' = 0$ and $\Psi' = 0$. We must conclude that the only solution for the theory considered that has the same post-Newtonian limit as General Relativity is the trivial one. In the order of approximation we have worked here, the only solution that is consistent with observations is that given by General Relativity.

V. Observational limits

Let us now consider equation (2). Within the framework of General Relativity, the right hand side is interpreted as a source term for gravitation. We write

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \Theta_{\mu\nu} \quad (38)$$

with obvious notations.

When "ordinary matter" is present, Einstein's equations reads

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \Theta_{\mu\nu} + T_{\mu\nu} \quad (39)$$

where $T_{\mu\nu}$ is the energy-momentum tensor of "ordinary matter". This equation defines what we consider to be "ordinary matter".

If $T_{\mu\nu}$ is the ususal energy momentum tensor in General Relativity, the corresponding term in $\mathcal{L}$ is $\sqrt{-g}\Phi L$ where L is the usual Lagrangian, i.e, $T_{\mu\nu}$ is the variational derivative of $\int \sqrt{-g} L d^4x$ relatively to $g_{\mu\nu}$.

We consider the static case with spherical symmetry, the origin being in the region where $T_{\mu\nu} \neq 0$. The equations inside the matter are modified (the singularity disappears). Outside the matter, the field equations remain equations (2, 7, 8); therefore equations (6) to (10) still hold true.

The developments of the previous section remain unchanged. Thus the validity of post-Newtonian expansions of General Relativity implies that $\Psi_0 \neq 0$. On another hand, one can assume that $\Psi_0 = 0$. Then

$$2b - \frac{1}{2}\alpha(\alpha-\gamma) = \beta - \frac{1}{2}\alpha(\alpha-\gamma) = -\frac{1}{2}\Psi_0^2 < 0 \quad (40)$$

The present accuracy of α, β and γ gives the limit

$$|\Psi_0| \ll 1 \quad (41)$$

Due to the high accuracy of Eddington and Robertson's parameters, the upper limit of $|\Psi_0|$ remains presently rather low.

VI. Conclusion

We have studied the exact spherical static symmetric solution found in [1]. This solution was obtained in the context of a model where gravity is coupled non-trivially to two scalar fields. This model was obtained from a fundamental theory where gravity is coupled to a Maxwellian-type field in five dimensions, the fifth dimension being time-like. The solution studied exhibits a naked singularity: it diverges for $r = 0$ but there is no horizon event covering this singularity.

We used the Parametrized Post-Newtonian approach, in order to establish possible limits on the application of such a model to local physics (at the astronomical scale). We found a remarkable negative result: despite the fact that this solution describes a Riemannian geometry with a curved space-time, its gravitational effect, in the first order in the PPN expansion, is exactly the same as of a newtonian theory.

Then, we returned back to the field equations, and look for a possible case, besides the quoted exact solution, where there could be detectable effects. We proved that the only case where we can have consistent Newtonian and post-Newtonian limits is the trivial case of General Relativity where the scalar fields Φ and Ψ are constants. This result holds true even when ordinary matter is present.

If we believe that post Newtonian limit of General Relativity is valid with the same conviction that we trust the validity of Newtonian limit, the previous results suggest that such a model may only have cosmological applications. It could for example describe the Universe at large scale, or in its beginning, but it could not be used at local scale. If we remember that the extra dimension of the original theory is time-like (leading to two time coordinates), this would imply that this new time-like coordinate has collapsed, attaining a stable small value. In this sense, the results here obtained give a restriction on models employing extra time-like dimensions.

References

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