δ-Expansion at Finite Temperature

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We apply the δ-expansion perturbation scheme to the $\lambda\phi^4$ self-interacting scalar field theory in 3+1 D at finite temperature. In the δ-expansion the interaction term is written as $\lambda(\phi^2)^{1+\delta}$ and δ is considered as the perturbation parameter. We compute within this perturbative approach the renormalized mass at finite temperature at a finite order in δ. The results are compared with the usual loop-expansion at finite temperature.

I. Introduction

The study of field theories at finite temperatures has long been an important issue in high energy physics (for a general review see [1]). However, in many situations, when working with field theories at finite temperatures, usual perturbations schemes break down due to the appearance of infrared divergences (for example, close to critical temperatures, in field theories with symmetry breaking and for massless field theories, like QCD, or for small values of "effective masses"). In those situations, we must usually perform a resummation procedure to take in to account relevant contributions in the infrared region (see for example [2]) or make use of nonperturbative approaches for studying the theory in the infrared region, as a renormalization group study or by the ϵ-expansion technique, for example.

Recently a new perturbation scheme in field theory was proposed, known as the δ-expansion [3, 4]. In this novel perturbation scheme, instead of using Lagrangian parameters for the expansion, like an expansion in the interaction coupling constant λ in the $\lambda\phi^4$ theory (regarding λ as a weak-coupling constant) or the usual loop-expansion (in powers of $\hbar$) the δ-expansion makes use of an artificial parameter (δ).

In the usual $\lambda\phi^4$ theory in 3+1D, the interaction term is rewritten as $\lambda M^4 (M^{-2}\phi^2)^{1+\delta}$, where M is an arbitrary mass parameter introduced to make the cou-

pling constant λ dimensionless. δ is regarded as a small positive parameter that can be used as a perturbative parameter in the theory, for example when Green's functions are computed.

If one expands the interaction term in powers of δ we get $\lambda\phi^4 \rightarrow \lambda M^2\phi^2 + \lambda M^2\phi^2 \sum_{n=1}^{\infty} \frac{\delta^n}{n!} \left[\ln(M^{-2}\phi^2)\right]^n$. Therefore, the δ-expansion generates a mass term which can not only make the behavior of the theory in the infrared region better, but also introduces nonperturbative effects in the coupling λ once M is fixed according to an appropriate procedure, as described below.

We recover the original interaction term for $\delta = 1$ and the dependence on the arbitrary mass parameter M goes away. However, in this paper, we will be interested exactly in what happens when we keep the δ-expansion up to a finite order in δ and when the results carry a dependence on M. We will be particularly interested in computing the renormalized mass m_R, at finite temperature, in the $\lambda\phi^4$ scalar model at some finite order in δ. Since in this case m_R is dependent on M, we must choose an optimization scheme to fix the value of M. Here we choose the Principle of Minimal Sensitivity (PMS) [5], where the quantities we are interested in are required to be stationary with respect to M. We are going to show that, in the evaluation of the renormalized mass at finite temperature, by fixing the mass parameter M through this variational method, we obtain a gap equation for the effective mass at finite tempera-

ture without having to use the usual resummation of diagrams (see, for instance, [2]).

The paper in organized in the following way: In Section II, we give a brief review of the δ-expansion technique and show how to compute the vertex Green's functions at finite temperature. In Section III, we demonstrate the procedure by computing the effective mass at finite temperature and obtain the gap equation. In Section IV, we have our conclusions and comments on further applications of the method.

II. The δ-expansion approach: computing

Green's functions at finite temperature

We begin by giving a short review of the δ-expansion approach. The $\lambda\phi^4$ Lagrangian density for a scalar field ϕ, in 3+1D, given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4 \tag{1}$$

is rewritten as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4}M^4(M^{-2}\phi^2)^{1+\delta} \ . \tag{2}$$

Expanding (2) in powers of δ we get

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}(\mu^2 + \frac{\lambda}{4}2M^2)\phi^2 - \frac{\lambda}{4}M^2\phi^2 \sum_{n=1}^{\infty} \frac{\delta^n}{n!} \left[\ln(M^{-2}\phi^2)\right]^n \ . \tag{3}$$

If one uses that

$$\sum_{n=0}^{\infty} \frac{\delta^n}{n!} \left[\ln(M^{-2}\phi^2)\right]^n = \sum_{n=0}^{\infty} \frac{\delta^n}{n!}\frac{d^n}{dk^n}(M^{-2}\phi^2)^k |_{k=0} = e^{\delta\partial_k}(M^{-2}\phi^2)^k |_{k=0} \ , \tag{4}$$

then (3) can be written as [4]

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}(\mu^2 + \frac{\lambda}{4}2M^2)\phi^2 - D_k \phi^{2k+2}|_{k=0} \ , \tag{5}$$

where, from the relation (4), D_k is a derivative operator given by

$$D_k = \frac{\lambda}{4}M^2 \left(e^{\delta\partial_k} - 1\right)\left(M^{-2}\right)^k \ . \tag{6}$$

In Ref. [4] it was shown that the n-point Green's function $G^{(n)}(x_1, x_2, \ldots, x_n)$ can be written as[1]

$$G^{(n)}(x_1, x_2, \ldots, x_n) = \prod_{p=0}^{\infty} \frac{1}{p!} \int d^4y_1 d^4y_2 \ldots d^4y_p \langle 0|T\phi(x_1)\phi(x_2)\ldots\phi(x_n)\times$$

$$D_{k_1}D_{k_2}\ldots D_{k_p}\left[\phi^2(y_1)\right]^{k_1+1}\left[\phi^2(y_2)\right]^{k_2+1}\ldots\left[\phi^2(y_p)\right]^{k_p+1}|0\rangle_c|_{k=0} \ , \tag{7}$$

which can be computed, as shown in Ref. [4], by first considering the k_i's as integers with the same value such that we can draw all diagrams coming from (7). From (6), if the k's are integers then D_k can be regarded as small and (7) can be computed by ordinary diagrammatic perturbation. At the end, considering the k's as continuous with $k_i \neq k_j$, $i \neq j$, we apply the derivative operators D_{k_i} and finally we make all k's igual to zero.

Once we know how to compute the Green's functions, we can obtain the renormalized mass m_R, the renormalized coupling constant λ_R and the wave-function renormalization constant Z from the usual definitions:

$$m_R^2 = Z \left[G_c^{(2)}(p^2)\right]^{-1}|_{p^2=0} \ , \tag{8}$$

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[1] In Ref. [3] the Green's functions are defined differently but the final results are completely analogous.

$$\lambda_R = -Z^2 G_c^{(4)}(0,0,0,0) \,, \qquad (9)$$

and

$$Z^{-1} = 1 + \frac{d}{dp^2} \left[G_c^{(2)}(p^2) \right]^{-1} |_{p^2=0} \,, \qquad (10)$$

where $G_c^{(2)}(p^2)$ and $G_c^{(4)}(p_1, p_2, p_3, p_4)$ are the connected two-point and four-point Euclidean Green's functions, in momentum space, respectively.

At lowest order (λ), the two-point Green's function $G^{(2)}$ is given by an one-vertex diagram as [4]

$$G_{(1v)}^{(2)} = -D_{k_1} \frac{(2k_1+2)!}{2^{k_1} k_1!} [I(m)]^{k_1} |_{k_1=0} \,, \qquad (11)$$

where D_{k_1} is given by (6) and $I(m)$ is the usual loop integral, at $T \neq 0$, given by [2]

$$I(m) = \frac{1}{\beta} \sum_{n=-\infty}^{n=+\infty} \int \frac{d^3 q}{(2\pi)^3} \frac{1}{\omega_n^2 + q^2 + m^2} \,, \qquad (12)$$

where, from (3), $m^2 = \mu^2 + \frac{\lambda}{4} 2M^2$, $\beta = T^{-1}$ is the inverse of the temperature and $\omega_n = \frac{2\pi n}{\beta}$.

Subtracting the zero temperature divergent contribution[2] of (12), one can write the following expansion [6] for $I(m)$ in powers of $m^2 \beta^2$:

$$I(m) = \frac{T^2}{12} - \frac{mT}{4\pi} - \frac{m^2}{8\pi^2} \left(\ln(\frac{m}{4\pi T}) + \gamma - \frac{1}{2} \right) + \frac{m^4}{T^2} \frac{\xi(3)}{2^7 \pi^4} + \mathcal{O}\left(m^4 \beta^4\right) \,. \qquad (13)$$

A consistent evaluation of the quantities (8)-(10) at an order higher than δ must also include the evaluation of Green's functions of an equivalent order in the number of vertices.

The two-vertex Green's function, $G_{(2v)}^{(2)}$, from (7), would be given by (including symmetry factors)

$$
\begin{aligned}
G_{(2v)}^{(2)} &= \frac{1}{2!} \sum_{n=0}^{2} \frac{2!}{(2-n)! n!} D_{k_1} D_{k_2} \sum_{l=2}^{+\infty} \frac{(2k_1+2)!}{2^{k_1 + \frac{(n-l)}{2}} \left(k_1 + \frac{(n-l)}{2} \right)!} [I(m)]^{k_1 + \frac{(n-l)}{2}} \times \\
&\times \frac{(2k_2+2)!}{2^{k_2 + 1 - \frac{(n+l)}{2}} \left(k_2 + 1 - \frac{(n+l)}{2} \right)!} [I(m)]^{k_2 + 1 - \frac{(n+l)}{2}} \frac{J_l(\tilde{p})}{l!} |_{k_1, k_2 = 0} \,,
\end{aligned} \qquad (14)
$$

where, (14) must be evaluated subject to the following constraint in the positive integer numbers n and l: $n+l$ must be even in order to have an even number of field lines leaving each vertex. In (14), $J_l(\tilde{p})$ represents internal propagators (between vertices) that at $T \neq 0$ are given by

$$J_l(\vec{p}, \omega_n) = \prod_{i=1}^{l} \frac{1}{\beta} \sum_{n_i} \int \frac{d^3 q_i}{(2\pi)^3} \frac{\delta^3(\vec{p} - \sum_{j=1}^{l} \vec{q}_j) \delta_{n, \sum_j n_j}}{\omega_{n_i}^2 + \vec{q}_i^2 + m^2} \qquad (15)$$

where $\omega_n = \frac{2\pi n}{\beta}$, $\omega_{n_i} = \frac{2\pi n_i}{\beta}$ are the Matsubara frequencies (n, $n_i = 0, \pm 1, \pm 2, \ldots$). $\tilde{p}$ (the external mo-

mentum) is defined by:

$$\tilde{p} = \begin{cases} 0 \,, & \text{for } n = 0 \text{ or } n = 2; \\ p \,, & \text{for } n = 1 \,. \end{cases} \qquad (16)$$

Green's functions of higher vertices can be equivalently defined, using the prescriptions given before. However, for the purposes of this paper, equations (11) and (14) will suffice for us.

Using Eq. (6) for the derivative operator D_k in (11) and (14) and using that $e^{\delta \partial_k} - 1$ can be expanded as $\delta \frac{\partial}{\partial k} + \frac{\delta^2}{2!} \frac{\partial^2}{\partial k^2} + \ldots$, we obtain for the two-point Green's function, up to two vertices, the expression

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[2] From the expressions we will obtain later, it is straightforward to show that renormalization by the introduction of counterterms in the Lagrangian density is enough. Thus we are going to refer only to the finite ($T \neq 0$) contributions. For a discussion of the renormalization up to order δ^2, at $T = 0$, see [4] and Bender and Jones in [3].

$$
\begin{aligned}
G^{(2)} &= G^{(2)}_{(1v)} + G^{(2)}_{(2v)} + \ldots = D_{k_1}\tilde{G}^{(2)}_{(1v)}(k_1)|_{k_1=0} + D_{k_1}D_{k_2}\tilde{G}^{(2)}_{(2v)}(k_1,k_2)|_{k_1,k_2=0} + \ldots = \\
&= \frac{\lambda}{4}M^2\left(\delta\frac{\partial}{\partial k_1} + \frac{\delta^2}{2!}\frac{\partial^2}{\partial k_1^2} + \ldots\right)\left(M^{-2}\right)^{k_1}\tilde{G}^{(2)}_{(1v)}(k_1)|_{k_1=0} + \\
&+ \frac{\lambda^2}{16}M^4\left(\delta^2\frac{\partial^2}{\partial k_1\partial k_2} + \ldots\right)\left(M^{-2}\right)^{k_1+k_2}\tilde{G}^{(2)}_{(2v)}(k_1,k_2)|_{k_1,k_2=0} + \ldots,
\end{aligned}
\tag{17}
$$

where we have explicited the terms up to order δ^2. As an example of the evaluation procedure, we compute below the effective mass at finite temperature and at order δ^2.

III. Computing the Effective Mass at $T \neq 0$

First, let us comment about expanding up to order δ^2 in the limit of high temperatures, $m\beta \ll 1$. When evaluating quantities like effective masses up to order δ^2, for consistency we must also include contributions coming from the two-vertex Green's function, since its leading term in δ is of order δ^2 (see Eq. (17)). However, if we restrict our analysis in the limit of high tempera-

ture and extending the results of [4] for $G^{(2)}_{(2v)}$ up to order δ^2 (evaluated in [4] at zero temperature), a rather analogous evaluation at finite temperature allows one to show that $G^{(2)}_{(2v)}$, when compared with $G^{(2)}_{(1v)}$, contributes with subleading corrections[3] to the effective mass $m_{\mathrm{eff}}(T)$ at finite temperature, when we restrict the evaluations in the high temperature limit $m\beta \ll 1$. For the same reason above, since the wave function Z^{-1}, at order δ^2, receives contributions only from $G^{(2)}_{(2v)}$, in the high temperature limit we can write $Z^{-1} \overset{m\beta \ll 1}{\simeq} 1$.

We may, therefore, restrict just to (11), for $m\beta \ll 1$. Up to second order in δ, we get the following expression for the two-point Green's function given by (11):

$$
\begin{aligned}
G^{(2)}_{(1v)} &= -\frac{\lambda}{4}2M^2\left\{\delta\left[\ln\left(\frac{I(m)}{2M^2}\right) + 2\psi(3) - \psi(1)\right] + \frac{\delta^2}{2!}\left[\left[\ln\left(\frac{I(m)}{2M^2}\right) + 2\psi(3) - \psi(1)\right]^2\right.\right. \\
&+ \left.\left. 4\psi'(3) - \psi'(1)\right]\right\} + \mathcal{O}(\delta^3),
\end{aligned}
\tag{18}
$$

where $\psi(x)$ and $\psi'(x)$ are the psi-function and its first derivative [8], respectively.

Substituting (18) in (8), we get the following expression for the effective mass up to second order in δ, within the one-vertex two-point Green's function:

$$
m_{\mathrm{eff}}^2 = \mu^2 + \frac{\lambda}{4}2M^2 - G^{(2)}_{(1v)},
\tag{19}
$$

with $G^{(2)}_{(1v)}$ given by (18).

The whole dependence of (19) on the arbitrary mass parameter M can be removed by requiring that[5, 7]

$$
\frac{\partial m_{\mathrm{eff}}^2}{\partial M^2} = 0,
\tag{20}
$$

at each order in the δ-expansion. The condition (20) fixes the value of the mass parameter M as being the one that leaves m_{eff}^2 stationary (the PMS condition).

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[3] An explicitly evaluation shows that $G^{(2)}_{(2v)} \overset{m\beta \ll 1}{\longrightarrow} \frac{\mathrm{constant}}{\mathrm{T}^2}$, while, from Eq. (18) and (13), $G^{(2)}_{(1v)} \overset{m\beta \ll 1}{\longrightarrow}$ constant $\times (\ln T)^2$.

Using the variational procedure above, we get the following expressions for the mass parameter M at each

order in δ, for $\delta = 1$, and in the high temperature limit ($I(m) \simeq \frac{T^2}{12}$, in Eq. (13)):

$$2M^2 = \begin{cases} \frac{T^2}{12} \exp\left[2\psi(3) - \psi(1)\right] , & \text{up to order } \delta \\ \frac{T^2}{12} \exp\left[2\psi(3) - \psi(1) - \sqrt{\psi'(1) - 4\psi'(3)} \right] , & \text{up to order } \delta^2 \end{cases} , \qquad (21)$$

Using (21) for M^2 back in (19), we get the following expression for the effective mass at finite temperature, up to orders δ and δ^2, respectively:

$$m_{\text{eff}}^2(T) = \begin{cases} \mu^2 + \frac{\lambda}{4}\frac{T^2}{12} \exp\left[2\psi(3) - \psi(1)\right] , \\ \mu^2 + \frac{\lambda}{4}\frac{T^2}{12} \exp\left[2\psi(3) - \psi(1) - \sqrt{\psi'(1) - 4\psi'(3)} \right] \left(1 + \sqrt{\psi'(1) - 4\psi'(3)} \right) . \end{cases} \qquad (22)$$

It is easy to show that (22) must converge to the usual 1-loop approximation for the finite temperature effective mass [2]. From (11), we can write

$$m_{\text{eff}}^2 = \mu^2 + \frac{\lambda}{4}M^2 \frac{(2\delta + 2)!}{2^\delta \delta!} \left[M^{-2}I(m) \right]^\delta , \qquad (23)$$

such that, in the high temperature limit, for $I(m) \simeq \frac{T^2}{12}$ and for $\delta = 1$, we get the usual result, $m_{\text{eff}}^2(T) \simeq$

$\mu^2 + \lambda\frac{T^2}{4}$. However it is remarkable that even for $\delta = 1$ and at lowest order, the expansion (19) is still consistent with the usual result obtained via loop expansion. From (22), at first order in δ (the first term in the right hand side in (22)), using the numerical values for the ψ-functions [8], we get $m_{\text{eff}}^2(T) \simeq \mu^2 + \lambda'\frac{T^2}{4}$, where $\lambda' = \frac{\lambda}{12}\exp[2\psi(3) - \psi(1)] \simeq 0.94\lambda$. At second order in δ, we have: $m_{\text{eff}}^2(T) \simeq \mu^2 + \lambda''\frac{T^2}{4}$, where, from (22),

$$\lambda'' = \frac{\lambda}{12} \exp\left[2\psi(3) - \psi(1) - \sqrt{\psi'(1) - 4\psi'(3)} \right] \left(1 + \sqrt{\psi'(1) - 4\psi'(3)} \right) \simeq 0.91\lambda .$$

With the consistent evaluation of terms of higher order in δ with the introduction of higher order-vertex terms, it is expected that $\lambda'(\lambda'') \to \lambda$.

It is interesting to see that, from (12) and (13), the loop integrals are written with propagators carrying the extra factor $\frac{\lambda}{4}2M^2$. However the variational condition, Eq. (20), used to fix the value of M, makes it possible to express the propagators with a finite temperature mass. At first order in δ, using (18) in (19) we obtain that $2M^2 = I(m) \exp[2\psi(3) - \psi(1)]$, where $m^2 = \mu^2 + \frac{\lambda}{4}2M^2$. If one redefines the coupling λ as $\lambda' = \frac{\lambda}{12}\exp[2\psi(3) - \psi(1)] \simeq 0.94\lambda$ (or $\lambda'' \simeq 0.91\lambda$, at order δ^2), we then get, from the first order in δ term of (18) substituted in (19) and for $\delta = 1$,

$$m_{\text{eff}}^2 = \mu^2 + 3\lambda' I(m_{\text{eff}}) , \qquad (24)$$

where the $I(m_{\text{eff}})$ term can be expanded as in (13). We recognize Eq. (24) as a gap equation. Eq. (24) is similar to the gap equation in the $\lambda\phi^4$ model, obtained by incorporating, in the loop expansion, the largest infrared divergences, summing up the so called daisy (or superdaisy) diagrams [2, 9].

If one expands (24) in the high temperature limit, we can obtain an approximate equation for m_{eff}

$$m_{\text{eff}}^2 = \mu^2 + \frac{\lambda'}{4}T^2 - 3\frac{\lambda'}{4\pi}Tm_{\text{eff}} , \qquad (25)$$

from which we obtain, assuming $m_{\text{eff}} \geq 0$, the solution

(valid up to $\mathcal{O}(\frac{\lambda^2 T^2}{m_{\mathrm{eff}}^2})$)

$$m_{\mathrm{eff}} = -3\frac{\lambda'}{8\pi}T + \sqrt{\left(\frac{3\lambda'T}{8\pi}\right)^2 + \mu^2 + \frac{\lambda'}{4}T^2} , \quad (26)$$

which is in accordance with the result obtained in [9], by considering the contribution of superdaisy diagrams in the gap equation, taking into account the leading infrared contributions at high temperatures.

IV. Conclusions

The use of the δ-expansion, as shown in [7], for the particular case of the massless scalar $\lambda(\phi^4)_4$ theory with $O(N)$ global symmetry, at finite temperature, reproduces quite well the exact result ($N \rightarrow \infty$ limit) for the gap equation, up to order δ^2. These results confirm our conclusions in that they show that the use of the PMS condition in the δ-expansion is self-consistent and is able to lead to nonperturbative results.

It would also be interesting to use the δ-expansion for evaluating higher order corrections for effective potentials at finite temperature, in connection with the program of resummation, which has been an important problem in the recent literature (see, for example, [9] and references therein). Work in this direction is in progress.

The δ-expansion has also been employed in the evaluation of critical exponents [10], using exactly its ability of exploring the infrared region, for T close to T_c, the critical temperature in spontaneously broken theories. A variant of the δ-expansion used here, called the linear δ-expansion [11], has proven to be a powerful tool for studying vacuum contributions on self-energies and in energy densities of very different field theories (for an example, see for instance [12]). The version of the δ-expansion used in this paper, usually called the non-linear or logarithmic δ-expansion, shares many properties with the linear one, representing, therefore, a promising method for getting vacuum fluctuation contributions, not only quantum but also thermal contributions, as we have briefly demonstrated in this paper.

Acknowledgements

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