Evaluation of Asymptotic Series of the Semiclassical Scattering Theory

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The asymptotic series of an integral representation of the form

$$\int d\lambda G(\lambda)\exp[iF(\lambda)]$$

is discussed in the context of the semiclassical formalism for the elastic scattering amplitude in which $G(\lambda)$ is not necessarily a slowly varying function. By using techniques of resummation of late terms of asymptotic expansions, it is shown how the series associated to a stationary point of the phase $F(\lambda)$ can be modified, to deal with the situation in which the function $G(\lambda)$ has a simple pole. The effect of the pole which physically may be interpreted as a diffraction phenomenon appears in the asymptotic series as a Stokes discontinuity. It is shown that the formalism obtained gives a very accurate description of two typical heavy-ion elasic scattering data.

I. Introduction

In a quantum description of a collision process the amplitude for elastic scattering is given by the partial wave sum

$$f(\theta)=\frac{1}{2ik}\sum_{\ell=0}^{\infty}(2\ell+1)(S_\ell-1)P_\ell(\cos\theta) \qquad (1)$$

where ℓ is the angular momentum, k is the wave number, S_ℓ is the S-matrix and P_ℓ is the Legendre polynomial. Under semiclassical conditions the contributions to the sum come from partial waves with large angular momenta, the sum may then be transformed into an integral and by using a convenient asymptotic expression for the Legendre polynomials one can approximate (1) as [3]

$$f(\theta)=\frac{1}{ik\sqrt{2\pi i\sin\theta}}I(\theta)$$

with

$$I(\theta)=\int_{\frac{1}{2}}^{\infty}d\lambda G(\lambda)\exp[iF(\lambda)] \qquad (2)$$

where $\lambda=\ell+\frac{1}{2}$ is the semiclassical angular momentum and, after separating the scattering function $S(\lambda)$ into its modulus and argument, the functions $G(\lambda)=\lambda^{\frac{1}{2}}|S(\lambda)|$ and $F(\lambda)=2\delta(\lambda)-\lambda\theta$ have been introduced. Usually $|S(\lambda)|=1$, however in nuclear physics due to the presence of reactive processes we can have an attenuation of the flux of the low partial waves and as a consequence more generally $|S(\lambda)|\leq 1$. The scattering function can in principle be expressed in terms of an underline complex potential but I am here more interested in the direct approach in which its modulus and phase are appropriately parametrized. The next and last step in the construction of the semiclassical formalism consists in evaluating the above integral using the stationary phase method. As its integrand is

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a rapidly oscillating function of λ, the contribution to the integral will come from the region around a stationary point λ_s, defined as a point where the derivative of the phase $F'(\lambda) = 2\delta'(\lambda) - \theta$ vanishes, namely a root of the equation $\Theta(\lambda_s) = 2\delta'(\lambda) = \theta$. Then by expanding $F(\lambda)$ up to second order terms and taking out of the integral the - in principle- slowly varying function $G(\lambda)$ calculated at the point $\lambda = \lambda_s$ we obtain

$$I(\theta) \simeq \sqrt{\frac{2\pi}{-iF_2}}\lambda_s^{\frac{1}{2}}\,|S(\lambda_s)|\exp i\left[F(\lambda_s)\right] \ , \qquad (3)$$

which leads to the classical cross-section with an attenuation factor that takes care of the reduction of the flux at the elastic channel.

However, if the physical process is characterized by an strong absorption such that there is a suppression of the low partial waves with $|S(\lambda)|$ varying from one to zero in a relatively small region of width Δ around a critical angular momentum Λ, this region will also contribute to the integral. This contribution may be estimated by expanding the phase up to first order around Λ. After taking the slowly varying factor $\lambda^{\frac{1}{2}}$ out of the integral with $\lambda = \Lambda$ the resulting integral becomes the Fourier transform of the modulus of the scattering function and we find

$$I(\theta) \simeq i\frac{A(\theta_\Lambda - \theta)}{\theta_\Lambda - \theta}\Lambda^{\frac{1}{2}}\exp i\left[2\delta(\Lambda) - \Lambda\theta\right] \ , \qquad (4)$$

where $A(\theta_\Lambda - \theta)$ is the transform of the derivative of $|S(\lambda)|$ and $\theta_\Lambda = \Theta(\Lambda)$ is the critical angle.

Finally if these two contributions come from regions well separated in the domain of integration that is in the case of scattering to an angle far from the critical angle, we can suppose the elastic amplitude to be given by the sum of these two contributions[9]. Considering for example the usual parametrization

$$|S(\lambda)| = \frac{1}{\exp\left(\frac{\Lambda-\lambda}{\Delta}\right) + 1} \qquad (5)$$

for which $A(\theta_\Lambda - \theta) = \frac{\pi(\theta_\Lambda - \theta)}{\sinh[\pi(\theta_\Lambda - \theta)]}$ we can write

$$I(\theta) \simeq \sqrt{\frac{\pi}{-iF_2}}\lambda_s^{\frac{1}{2}}\,|S(\lambda_s)|\exp\left[iF(\lambda_s)\right] \pm i\pi\Delta\Lambda^{\frac{1}{2}}\exp\left[iF(\Lambda) - \pi\Delta\,|\theta_\Lambda - \theta|\right] \ . \qquad (6)$$

This kind of scattering situation was interpreted as a Fresnel diffraction phenomenon[6] and an explicit closed expression for the scattering amplitude was derived by Frahn and Gross[7]. In their derivation, it was assumed that we are near to the sharp cut-off case ($\Delta = 0$) in which all the angular momenta below the critical angular momentum Λ are suppressed. What I am interested here is the similarity of the above expression with the phenomenon of Stokes's discontinuities in which there are a dominant and a subdominant exponential contribution to an asymptotic series[2]. It is the purpose of this paper to show that an asymptotic expression may be obtained applying to the asymptotic series whose first term is given by Eq. (3) the technique of Borel summation of late terms of an asymptotic expansion as developed by Dingle[5] in which Stokes's discontinuities appear naturally. Actually, we have here a singularity different from those envisaged by Dingle who was concerned with the case of others possible saddle-points. His argument to discard poles of $G(\lambda)$ is that if this function varies rapidly due to the presence of poles it, or the singular part of it, should be included in the phase (Dingle, p.143). Nevertheless for physical reasons it is interesting to treat the effect of a pole which is a diffractive effect separately from that of a stationary phase that is a refractive one. Besides this, from a practical point of view it is convenient to look for real stationary points instead of complex ones. So we are going to show that his method can be extended to treat the case of poles and that it provides a useful description of diffractive effects in the semiclassical asymptotic series.

II. Resumming the asymptotic series

Let us start deriving the complete asymptotic series assuming that the phase has only one stationary phase point. This means that as a function of the angular momentum it can be mapped into a parabola and, if we suppose its extremum to be a maximum we can put

$$F(\lambda) = F(\lambda_s) - t^2 \ . \tag{7}$$

Changing variables from λ to t the integral becomes

$$I(\theta) = \exp\left[iF(\lambda_s)\right] \int_{-\infty}^{\infty} dt \frac{d\lambda(t)}{dt} G(t) \exp\left(-it^2\right) \ , \tag{8}$$

where the lower limit of integration has been extended to $-\infty$. By expanding $\frac{d\lambda}{dt}G$ in a power series about the stationary point $t = 0$

$$\frac{d\lambda}{dt} G = \sum_{r=0}^{\infty} c_r t^r$$

we obtain the asymptotic series

$$I(\theta) = \exp\left[iF(\lambda_s)\right] \sum_{r=0}^{\infty} \frac{c_{2r}}{i^{r+\frac{1}{2}}} \Gamma\left(r + \frac{1}{2}\right) \ , \tag{9}$$

where $\Gamma(x)$ is the gamma function. This series is more conveniently written in the standard form

$$I(\theta) = \sqrt{\frac{2\pi}{-iF_2}} \exp\left[iF(\lambda_s)\right] \sum_{0}^{\infty} \frac{Q_{2r}}{i^r} \tag{10}$$

where the Q_r have been tabulated (Dingle,p.119) up to $r = 8$ as a function of the derivatives of the phase and of the function G at the stationary point. As a reference we have the first two terms we need

$$Q_0 = G_0$$

that leads immediately to (3) and

$$Q_2 = \frac{1}{24F_2^3}\left[G_0\left(5F_3^2 - 3F_2F_4\right) - 12G_1F_2F_3 + 12G_2F_2^2\right] \ ,$$

where the subscripts denote the order of derivatives.

The above series is an asymptotic one, so one should expect the Q_{2r} to decrease initially up to a certain $r = n$ and then to increase in such way that the whole series turns out to be divergent. In the case of strong absorption this divergence will be caused by nonanaliticity of the modulus of the S-matrix. In fact, assuming for example the usual parametrization, Eq. (5), there will be an infinite number of pairs of simple poles in the complex plane, symmetric with respect to the real axis. Dingle's assumption is that the information about the singularity is carried by the late terms of the asymptotic series. In order to decode this information the theorem of Darboux is used to express the coefficients of the power series expansion about $t = 0$ in terms of the coefficients of the expansion about the singularity. Let t_i with $i = 1, 2, ...$ be simple poles of the function $G[\lambda(t)]$ in the complex t-plane, then about one of these points we have the Taylor-MacLaurin expansion

$$\frac{d\lambda}{dt} G = \frac{a_{-1}}{t - t_i} + \sum_{k=0}^{\infty} a_k (t - t_i)^k \ , \tag{11}$$

where the rhs can be arranged into a power series of t. Equating the coefficients of t^n in the two expansions we obtain the coefficients we are looking for. In fact, here it is enough to restrict the calculation only to the contribution that comes from the first term, so we put

$$c_{2r} \simeq -\frac{a_{-1}}{t_i^{2r+1}} \ .$$

Considering the first n terms of the asymptotic series with n such that we are near the least term we replace then for $r \geq n$ the coefficients by their above estimation. Using then the integral representation of the gamma function we have

$$I(\theta) = \sqrt{\frac{2\pi}{-iF_2}} \exp\left[iF(\lambda_s)\right] \sum_{0}^{n-1} \frac{Q_{2r}}{i^r} - \frac{a_0 \exp\left[iF(\lambda_s)\right]}{\sqrt{i\mathcal{F}_i}} \sum_{r=n}^{\infty} \int_0^{\infty} ds s^{-\frac{1}{2}} \exp(-s) \left(\frac{s}{i\mathcal{F}_i}\right)^n \tag{12}$$

where the quantity $\mathcal{F}_i = t_i^2 = F(\lambda_s) - F(\lambda_i)$ was introduced and $i\mathcal{F}_i$ is the so called singulant in Dingle's terminology. Interchanging now the order of the sum and the integration, we can write

$$I(\theta) = I_n(\theta) - \frac{a_0 \exp[iF(\lambda_s)]}{(i\mathcal{F}_i)^{n+\frac{1}{2}}} \Gamma\left(n + \frac{1}{2}\right) \Lambda_{n-\frac{1}{2}}(-i\mathcal{F}_i) \ , \tag{13}$$

where I_n stands for the first n terms of the series and

$$\Lambda_{n-\frac{1}{2}}(-i\mathcal{F}_i) = \frac{1}{\Gamma\left(n+\frac{1}{2}\right)} \int_0^\infty \frac{dt t^{n-\frac{1}{2}} \exp(-t)}{1 - \frac{t}{i\mathcal{F}_i}} ... \tag{14}$$

defines what Dingle calls the terminant of the series. To calculate it we use the absolutely convergent expansion (Dingle, p.416)

$$\Lambda_s(x) = \frac{x}{s}\left[1 - \frac{x}{s-1} + \frac{x^2}{(s-1)(s-2)} - \cdots\right] - \frac{\pi x^{s+1} \exp(x)}{\Gamma(s+1) \sin \pi s} \tag{15}$$

from which we can derive

$$\Lambda_{n-\frac{1}{2}}(-i\mathcal{F}_i) = \overline{\Lambda}_{n-\frac{1}{2}}(-i\mathcal{F}_i) \mp \frac{2\pi (i\mathcal{F}_i)^{n+\frac{1}{2}} \exp(-i\mathcal{F}_i)}{\Gamma\left(n+\frac{1}{2}\right)}, \tag{16}$$

where

$$\overline{\Lambda}_{n-\frac{1}{2}}(-i\mathcal{F}_i) = -\frac{i\mathcal{F}_i}{s}\left[1 + \frac{i\mathcal{F}_i}{s-1} + \frac{(i\mathcal{F}_i)^2}{(s-1)(s-2)} - \cdots\right] + \frac{\pi (i\mathcal{F}_i)^{n+\frac{1}{2}} \exp(-i\mathcal{F}_i)}{\Gamma\left(n+\frac{1}{2}\right) \tan \pi s} \ . \tag{17}$$

In (16) a factor of two has been introduced to give the correct boundary condition of the diffracted wave. The $-(+)$ sign correspond to a pole in the upper (lower) part of the complex λ-plane Thus we see that if we assume the parametrization Eq. (5) for which $a_0 = \Delta\sqrt{\lambda_i}$ we find asymptotically a result which coincides for $\Delta \ll \Lambda$ with the diffracted waves as they are given by Eq. (6).

III. Applications

We apply now the formalism to two examples extracted from Frahn's book on diffractive effects in heavy-ion reactions[8]. For simplicity we will consider two cases in which the real part of the phase shift is purely coulombian and the absorption is parametrized as in Eq.5. The scattering is then completely determined by four parameters: the wave number, the Sommerfeld parameter η , related to the strength of the electrostatic potential, the angular momentum Λ and the diffusivity Δ. The coulomb phase shift may be written asymptotically, that is, for $\eta \gg 1$ and $\lambda \gg 1$ as

$$2i\delta_c = (\lambda + i\eta)\ln(\lambda + i\eta) - (\lambda - i\eta)\ln(\lambda - i\eta)$$

from which we derive the deflection function $\Theta(\lambda) = 2\arctan\left(\frac{\eta}{\lambda}\right)$ and the correspondent stationary point $\lambda_s = \eta\cot\left(\frac{\theta}{2}\right)$. All the other quantities are easily calculated, thus we have

$$F_2 = -\frac{2\eta}{\eta^2 + \lambda_s^2} \ ,$$

$$F_3 = \frac{\lambda_s}{\eta} F_2^2 \ ,$$

$$F_4 = \frac{1}{\eta} F_2^2 + \frac{2\lambda_s}{\eta} F_2 F_3 \ ,$$

and for the absorption factor

$$|S(\lambda_s)|' = \frac{1}{\Delta} |S(\lambda_s)| \left[1 - |S(\lambda_s)|\right] \ ,$$

$$|S(\lambda_s)|'' = \frac{1}{\Delta} |S(\lambda_s)|' \left[1 - 2\,|S(\lambda_s)|\right] \ .$$

Higher order derivatives are easily derived in terms of the precedent ones. Besides that we have

$$G_0 = \lambda_s^{\frac{1}{2}} \left|S(\lambda_s)\right|$$

$$G_1 = \frac{1}{2\lambda_s^{\frac{1}{2}}} \left|S(\lambda_s)\right| + \lambda_s^{\frac{1}{2}} \left|S(\lambda_s)\right|'$$

and

$$G_2 = -\frac{1}{4\lambda_s^{\frac{3}{2}}} \left|S(\lambda_s)\right| + \frac{1}{\lambda_s^{\frac{1}{2}}} \left|S(\lambda_s)\right| + \lambda_s^{\frac{1}{2}} \left|S(\lambda_s)\right|''.$$

Let's pass now to the applications.

III.1 ^{16}O+^{28}Si at 35.0 Mev

For this system $\Lambda = 26.21$, $\Delta = 1.0$ and $\eta = 9.51$. Let us start showing in Fig. 1 the first three terms of the semiclassical series as a function of the scattering angle θ. It can be seen that in the region around the critical angle $\theta_\Lambda = 39.9°$ Q_2 and Q_4 become very large and the first term Q_0 is the least term of the series. So, in the numerical calculation $n = 2$ until $\theta = 37°$, $n = 1$ from this angle up to $\theta = \theta_\Lambda$ and for $\theta > \theta_\Lambda$, $n = 2$. In Fig. 2 it can be seen how the nearest pair of singulants $i\mathcal{F}_1^{\pm}$, with $\pm$ denoting the upper (lower) pole, behave in the complex plane. The one associated to the pole located at $\lambda^+ = \Lambda + i\pi\Delta$ moves clockwise and crosses the Stokes's line defined by the points such that $Im\left(i\mathcal{F}_1^+\right) = 0$ for $\theta = 37°$ while the other comes from the lower to upper part of the complex plane crossing the positive real axis for $\theta = 44°$. Finally in Fig. 3 it is plotted the ratio of the cross-section to the Rutherford cross-section and for comparison the result of an exact calculation of the partial wave sum. It can be seen that the method gives an excellent representation of the exact amplitude. It is important to remark that in order to obtain the nice fit seen in the figure it was necessary to sum the contributions of all upper(lower) poles, i.e., $\lambda_i^{\pm} = \Lambda \pm (2i+1)\,\pi\Delta$, $i = 0, 1, 2...$, and it was found numerically that the best point to switch from upper to lower poles is when the singulant $i\mathcal{F}_1^+$ crosses the Stokes's line, i.e., at $\theta = 37°$.

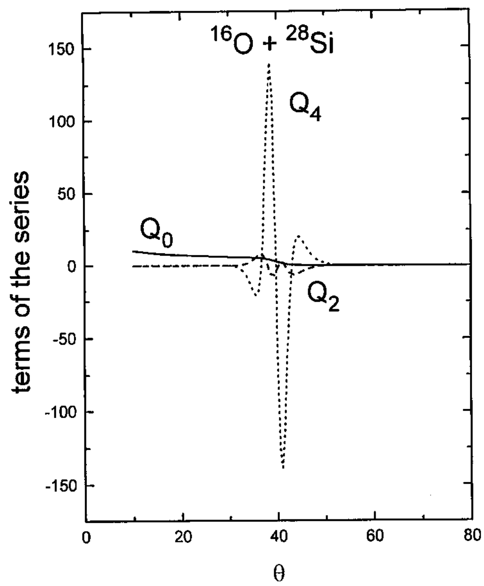

Figure 1. The first three terms of the semiclassical asymptotic series, Eq. 10 as a function of the angle for the system $^{16}O +^{28} Si$.

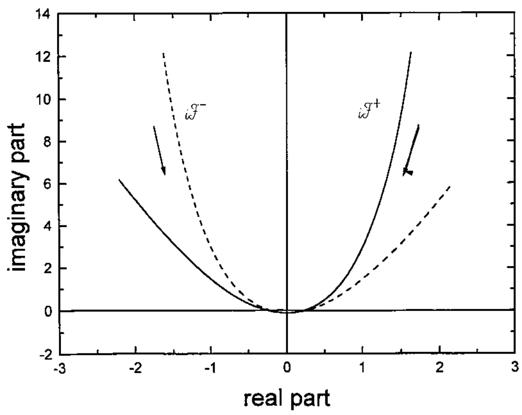

Figure 2. The behaviour of singulants associated to the nearest pair of poles in the complex plane for system $^{16}O +^{28} Si$.

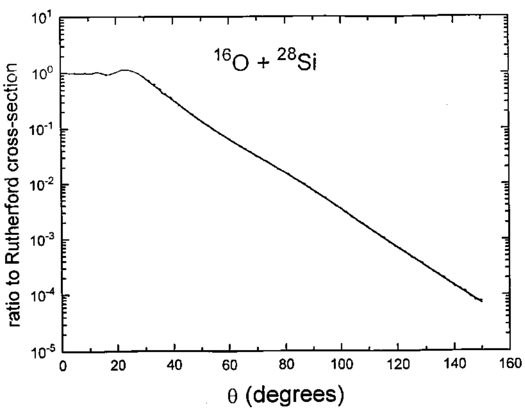

Figure 3. The ratio of the cross-section to the Rutherford cross-section for $^{16}O +^{28} Si$. The solid line is the asymptotic calculation and the dotted line the exact partial wave sum.

III.2 ^{16}O+^{60}Ni at 48.5 Mev

For this system $\Lambda = 31.75$ $\Delta = 1.34$ and $\eta = 18.0$. Again, we start showing the first three terms of the semiclassical series as a function of the scattering angle θ, in Fig. 4. It can be seen that in the region around the critical angle $\theta_\Lambda = 59.13°$ Q_2 and Q_4 become very large. So, in the numerical calculation, $n = 2$ up to $\theta = 51°$ and $n = 1$ up to $\theta = 52.7°$ and $n = 0$ for larger angles. In Fig. 5 it is shown how the pair of nearest singulants $i\mathcal{F}_i$ behave in the complex plane. Again we have the same behaviour of the previous system with the one associated to the pole $\lambda^+ = \Lambda + i\pi\Delta$ moving clockwise and crossing the Stokes's line for $\theta = 44°$ while the other moving in the opposite sense crosses for $\theta = 54°$. Finally, in Fig. 6 it is plotted the ratio of the cross-section to the Rutherford cross-section together with the exact calculation of the partial wave sum. It can be seen that the method gives an excellent representation of the exact amplitude. Again it was found that the best fit was obtained switching from upper to lower poles at the Stokes's line associated to the first upper pole.

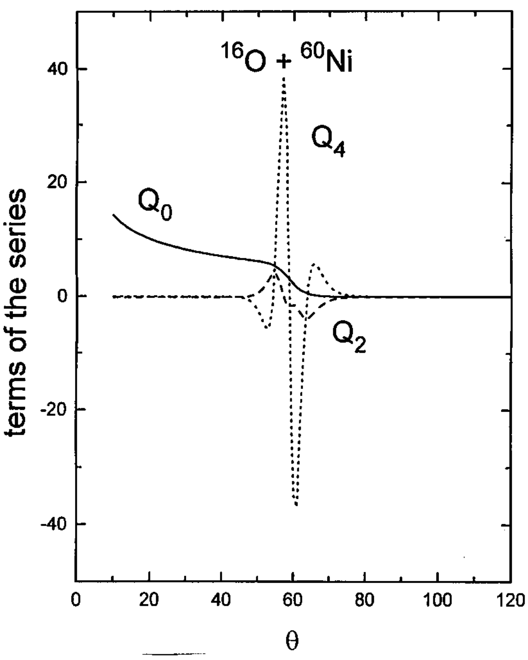

Figure 4. The same as in Fig. 1 for system $^{16}O +^{60} Ni$.

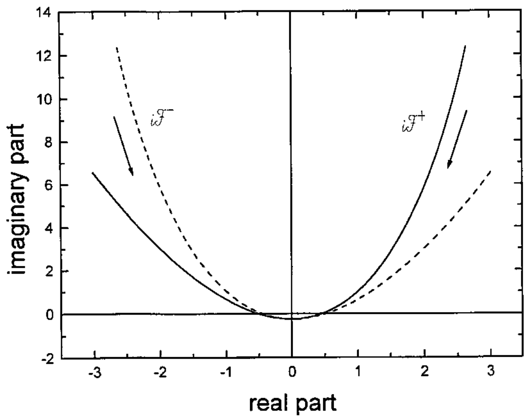

Figure 5. The same as in Fig. 2 for system $^{16}O +^{60} Ni$.

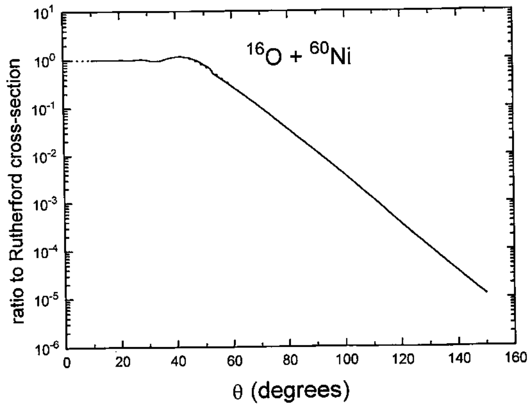

Figure 6. The same as in Fig. 3 for system $^{16}O +^{60} Ni$.

IV. Concluding remarks

A resurgence formula for the asymptotic series representation of the semiclassical elastic scattering amplitude was obtained using Dingle's method of Borel summation of late terms of asymptotic expansions. In order to do this, it was necessary to extend his method to treat the case in which the form of the late terms is determined by the presence of poles in the integrand. The formalism has proved to give a very accurate result in two typical heavy-ion collision cases in which the cross-section is dominated by diffractive effects. The advantage of this treatment over the previous formalism for this kind of scattering situation[7] is that while that was constructed for the specific case of strong absorption, here we have a representation which works in principle uniformly for any situation (any value of the diffusivity Δ). What varies from one collision case to the other is only the number of terms to be considered in the asymptotic series and this is fixed by the least term in the expansion. The formalism has however a limitation since it was derived assuming that we have only one stationary phase point, i.e., a monotonic deflection function. This leaves out the very important case of rainbow scattering in which more than one semiclassical trajectory contribute to the cross-section at a given angle[10]. It would be very important if the method developed here could be extended to include the case of the uniform semiclasical expansion for the rainbow scattering[1].

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