A Symmetry Property of the Space-Time of General Relativity in Terms of the Space-Matter Tensor

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The concept of matter collineation has been introduced. The conditions under which a space-time may admit such collineation have been given. The electromagnetic and the pure radiation fields have also been considered.

I. Introduction

In the general theory of relativity the curvature tensor describing the gravitational field consists of two parts viz, the matter part and the free gravitational part. The interaction between these two parts is described through Bianchi identities. For a given distribution of matter, the construction of gravitational potentials satisfying Einstein's field equations is the principal aim of all investigations in gravitation physics, and this has been often achieved by imposing symmetries on the geometry compatible with the dynamics of the chosen distribution of matter. The geometrical symmetries of the space-time are expressible through the vanishing of the Lie derivative of certain tensors with respect to a vector. This vector may be time-like, space-like or null.

In a series of papers[1−6] Katzin, Levinle, Davis and their collaborators have identified 16 symmetries for the gravitational field with their interrelationships and have obtained the corresponding weak conservation laws as the integrals of the geodesic equations and also shown that in the absence of free gravitational field, a non-Einstein space with a non-zero scalar curvature does not admit a proper curvature collineation (CC) because it degenerates to motion (M). In these conformally flat spaces the CC and the special conformal motion (S Conf M) are equivalent. For type N gravitational fields, it has been shown by Collinson[7] that these fields admit CC which are not conformal motion. Collinson and French[8] have shown that the conformal motion admitted by Petrov type N space-time with twist-free geodesic rays is not homothetic. The only type N fields that admit conformal motion are pp-waves[9] and an example of a pp-wave admitting a particular homothetic Killing vector is given by McIntosh[10]. Type N vacuum spaces which admit an expanding and/or twisting congruences and a homothetic motion are investigated by Halford[11]. Tariq and Tupper[12], McIntosh and Halford[13] and Hall[14], among many others, have considered the symmetries of the null electromagnetic and gravitational fields.

In an attempt to study the geometric and physical properties of the electromagnetic fields, Ahsan[15] and Ahsan and Husain[16] have investigated different types of collineations and, along with many other interesting results, it is seen that for a null electromagnetic field a motion does not imply Maxwell collineation (MC) and conversely. Considering the important role played by Nijenhuis tensor in the study of electromagnetic fields (c.f. [16]), the concept of torsion collineation (TC) in terms of the Lie derivative of the Nijenhuis tensor has been introduced by Ahsan[17] and the conditions are given under which a null electromagnetic field may admit TC along the propagation and polarization vectors of the electromagnetic field.

Radhakrishna and his collaborators[18,19] have transcribed the known geometric symmetries of the space-time in the language of spin-coefficient formalism[20] and have obtained interesting relationships between null electromagnetic and null gravitational fields. Recently, using the Newman-Penrose formalism[20], it is shown that[21] a null electromagnetic field always admit Maxwell Collineation (MC) and existence of a Ricci collineation (RC) and M are possible only under certain geometric conditions on the spin- coefficients.

Different types of matter distributions compatible with geometrical symmetries have been the subject of interest of several investigators for quite some time and in this respect, Oliver and Davis[22], for the space-times filled with perfect fluid, have studied the time-like symmetries with special reference to conformal motion and family of contracted Ricci collineations (FCRC). The perfect fluid space-times including electromagnetic field which admit symmetry mapping belonging to FCRC have been studied by Norris et al.[23]. The role of metric symmetries in the study of fluid space-times, with the main emphasis on conformal collineation, has been explored by Duggal[24] alongwith the applications to astrophysics.

The quest for finding the different types of symmetries of the space-time under different assumptions is ON with results of elegance - and the list of workers in this particular field of interest is very long, we have mentioned here only a few.

Motivated by the role of symmetries in general theory of relativity, in this paper, we have defined yet another symmetry of the space-time. This is defined in terms of the vanishing of the Lie derivative of the 'Space-matter tensor'. In the next section, the space-matter tensor and its properties are given. The concept of matter collineation (MTC) has been introduced and the necessary and sufficient conditions are obtained under which a space-time including electromagnetic fields may admit a MTC.

II. Matter Collineation

In 1969, Petrov[25] introduced the so called space-matter tensor which satisfies all the algebraic properties of the Rieman curvature tensor and is more general than the Weyl conformal curvature tensor. The algebraic properties, the classification and the spinor equivalent of the space-matter tensor have been studied by Ahsan in a series of papers [26], [27], [28].

Let the Einstein field equations be

$$R_{ab} - \frac{1}{2} R g_{ab} = \lambda T_{ab} \tag{1}$$

where λ is a constant and T_{ab} is the energy-momentum tensor. On contraction (1) yields

$$\lambda T = -R \ . \tag{2}$$

Define a fourth order tensor A_{abcd} [25] as

$$A_{abcd} = \frac{\lambda}{2}(g_{ac} - T_{bd} + g_{bd}T_{ac} - g_{ad}T_{bc} - g_{bc}T_{ad}) \tag{3}$$

which has the following properties:

$$A_{abcd} = -A_{bacd} = -A_{abdc} = +A_{cdab}, \ A_{abcd} + A_{acdb} + A_{adbc} = 0 \ . \tag{4}$$

Contraction of (3) over b and d yields

$$A_{ac} = \lambda T_{ac} + \frac{\lambda}{2} T g_{ac} = \lambda T_{ac} - \frac{R}{2} g_{ac} \ . \tag{5}$$

The space-matter tensor P_{abcd} is defined as [25]

$$P_{abcd} = R_{abcd} A_{abcd} + \sigma(g_{ac}g_{bd} - g_{ad}g_{bc}) \ . \tag{6}$$

where σ is a constant (a scalar). The first part of this tensor represents the curvature of the space and the second part the distribution and motion of the matter. This tensor has the following properties:

(i) $P_{abcd} = -P_{bacd} = -P_{abdc} = P_{cdab}$, $P_{abcd} + P_{acdb} + P_{adbc} = 0$.

(ii) $P_{ac} = R_{ac} - \lambda T_{ac} + \frac{R}{2} g_{ac} + 3\sigma g_{ac} = (R + 3\sigma) g_{ac}$.

(iii) If the distribution and motion of the matter, i.e., T_{ab} and the space-matter tensor P_{abcd} are given, then R_{abcd}, the curvature of the space is determined to within the scalar σ.

(iv) If $T_{ab} = 0$ and $\sigma = 0$, then P_{abcd} is the curvature of the empty space-time.

(v) If g_{ab}, the metric tensor, σ, the scalar and P_{abcd} are known, then T_{ab} can be determined uniquely.

It is known that [29] the Riemann curvature tensor may be decomposed as

$$R_{abcd} = C_{abcd} + E_{abcd} + G_{abcd} \tag{7}$$

where C_{abcd} is the Weyl tensor, E_{abcd} is the Einstein tensor defined by

$$E_{abcd} = -\frac{1}{2}(g_{ac}S_{bd} + g_{bd}S_{ac} - g_{ad}S_{bc} - g_{bc}S_{ad}) \tag{8}$$

where

$$S_{ab} = R_{ab} - \frac{1}{4} R g_{ab} \tag{9}$$

being the traceless Ricci tensor, and G_{abcd} is defined by

$$G_{abcd} = -\frac{R}{12}(g_{ac}g_{bd} - g_{ad}g_{bc}) \ . \tag{10}$$

Using the above equations, it has been shown by Ahsan[27] that the space-matter tensor P_{abcd} may be decomposed as

$$P_{abcd} = C_{abcd} + (g_{ad}R_{bc} + g_{bc}R_{ad} - g_{ac}R_{bd} - g_{bd}R_{ac}) + \left(\frac{2}{3}R + \sigma\right)(g_{ac}g_{bd} - g_{ad}g_{bc}), \tag{11}$$

which can also be written as

$$P^h_{bcd} = C^h_{bcd} + (\delta^h_d R_{bc} - \delta^h_c R_{bd} + g_{bc}R^h_d + g_{bd}R^h_c) + \left(\frac{2}{3}R + \sigma\right)(\delta^h_c g_{bd} - \delta^h_d g_{bc}). \tag{12}$$

We now have

Definition 1. A matter collineation (MTC) is defined to be a point transformation $x^i \to x^i + \xi^i dt$ leaving the form of the space-matter tensor P^h_{bcd} given by equation (12) invariant, that is

$$\mathcal{L}_\xi P^h_{bcd} = 0 \tag{13}$$

where $\mathcal{L}_\xi$ denotes the Lie derivatives along the vector ξ.

Since every motion in a V_n is a W Conf C [1], we thus have from equation (12) and (13)

Theorem 1. A V_n admits MTC if it admits M, RC and $\sigma = 0$.

Now consider a V_n for which $R_{ij} = 0 = \sigma$ and denote this space as V_n^0. Equations (12) and (13) thus yield

Theorem 2. In a V_n^0 every motion is a MTC.

The energy momentum tensor T_{ab} of an electromagnetic field is defined by

$$T_{ab} = F_{ak}F_b^k - \frac{1}{4}g_{ab}F_{ij}F^{ij} \tag{14}$$

From (14), equations (3), (7) and (6) yield, after a lengthy calculation, the following representation of space-matter tensor in case of a non-null electromagnetic field

$$P_{abcd} = C_{abcd} - g_{ac}F_{bk}F_d^k - g_{bd}F_{ap}F_c^p + g_{ad}F_{bt}F_c^t + g_{bc}F_{ax}F_d^x + \left(\sigma + \frac{1}{2}F_{ij}F^{ij}\right)(g_{ac}g_{bd} - g_{ad}g_{bc}) \tag{15}$$

from which we have

$$P^h_{bcd} = C^h_{bcd} - \delta^h_c F_{bk}F_d^k - g_{bd}F^h_p F^p_c + \delta^h_d F_{bt}F^t_c + g_{bc}F^h_x F^x_d + \left(\sigma + \frac{1}{2}F_{ij}F^{ij}\right)(\delta^h_c g_{bd} - \delta^h_d g_{bc}) \; . \tag{16}$$

It is known [30], for non-null electromagnetic fields, that $\mathcal{L}_\xi g_{ij} = 0 \Longrightarrow \mathcal{L}F_{ij} = 0$, thus taking the Lie derivative of equation (16), we have

Theorem 3. A non-null electromagnetic field admits MTC if and only if it admits motion and $\sigma = 0$.

The energy-momentum tensor for a null electromagnetic field is given by

$$T_{ab} = F_{ac}F^c_b \tag{17}$$

where $F_{ac} = s_a t_c - t_a s_c$ and $s_a s^a = s_a t^a = 0$, $t_a t^a = 1$, vectors s and t are the propagation and polarization vectors, respectively.

Consider now equation (17) and use the same technique as was used earlier in obtaining equation (16), we have

$$P^h_{bcd} = C^h_{bcd} + 2(\delta^h_d F_{bk}F^k_c - \delta^h_c F_{bt}F^t_d + g_{bc}F^h_p F^p_d - g_{bd}F^h_f F^f_c) \tag{18}$$

which is the representation of space-matter tensor for a null electromagnetic field.

Following[15], it is easy to prove the following:

Theorem 4. A null electromagnetic field admits MTC along the vector ξ (propagation/polarization) if ξ is Killing and expansion-free.

As we are working with the null electromagnetic field, it is therefore natural to expect that the Lichnerowicz conditions for total radiation[16] are satisfied and we have

$$T_{ab} = \phi^2 k_a k_b \tag{19}$$

where k_a is the tangent vector. We now have

Definition 2. A null electromagnetic field admits a total radiation collineation (TRC) if $\mathcal{L}_\xi T_{ab} = 0$, where T_{ab} is defined by equation (19).

Taking the Lie derivative of equation (19) with respect to k^c, we have

$$\mathcal{L}_k T_{ab} = \phi^2[k_a(k^c k_{b;c} + k_c k^c_{;d}) + k_b(k^d k_{a;d} + k_d k^d_{;a})] \; . \tag{20}$$

Also, since a null electromagnetic field with equation (19) is often known to be a pure radiation field, we therefore from equation (20), have the following

Theorem 5. Pure radiation fields admit TRC if and only if the tangent vector k_a defines a null geodesic.

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