

Compressible Strips, Chiral Luttinger Liquids, and All That Jazz

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When the quantum Hall effect occurs in a two-dimensional electron gas, all low-energy elementary excitations are localized near the system edge. The edge acts in many ways like a one-dimensional ring of electrons, except that a finite current flows around the ring in equilibrium. This article is a brief and informal review of some of the physics of quantum Hall system edges. We discuss the implications of macroscopic *compressible and incompressible strip* models for microscopic chiral Luttinger liquid models and make an important distinction between the origin of non-Fermi-liquid behavior in fractional quantum Hall edges and in usual one-dimensional electron gas systems.

I. Introduction

The quantum Hall effect^[1] is an anomaly which occurs in the transport properties of two-dimensional electron system in the regime of strong perpendicular magnetic fields. At certain magnetic fields it is found that the voltage drop in the system in the direction of current flow, responsible for dissipation in the system, vanishes at low temperatures. Our understanding of this transport anomaly is not absolutely complete. Nevertheless, there is fairly broad agreement that edge state^[2,3] theories, in which the nonequilibrium transport currents are carried near the edge of the system, capture the essence of the phenomenon. We present a version of the edge state picture which accommodates both integer and fractional quantum Hall effects in Section II of this article. It follows from the edge state picture that when the quantum Hall effect occurs, the ground state of the two-dimensional electron system *must* possess a set of low-energy excitations which are localized near the edge of the system. In this article we discuss a sequence of pictures of edge excitations in the quantum Hall effect. The order in which we present the pictures is loosely one of increasing sophistication and power, although there are important lessons at each step and the more powerful descriptions are not in every sense the ones with greater generality. In Section III we discuss edge states for a system of non-interacting

electrons in which the particles are confined to a finite area by an external potential. In Section IV we grapple with the Coulomb interaction between electrons. The long-range of this interaction can have overwhelming importance in determining the edge electronic structure when the edge charge density profile has a width which is long compared to microscopic lengths, the so-called ‘smooth edge’ regime. The discussion of Section IV uses both Thomas-Fermi and Hartree-Fock approximations to treat electron-electron interactions. A many-particle description beyond the Hartree-Fock approximation is necessary to describe edge states in the case of the fractional quantum Hall effect. In Section V we discuss the ground state and excited states of edges in the fractional case using a language of many-body wavefunctions. This approach shows that the edges are very similar in integer and fractional cases but fails to reveal some important differences which appear in quantities like the tunneling density-of-states at the edge of the system. We address these issues by using Wen’s^[4,5] chiral Luttinger liquid theory to discuss integer and fractional edges in Section VI. In Section VII we present some concluding remarks.

II. Incompressibility and Edge States

The thermodynamic compressibility of a system of interacting particles is proportional to the derivative

of the chemical potential with respect to density. It can happen that at zero temperature the chemical potential has a discontinuity at a density n^* : the energy to add a particle to the system (μ^+) differs, at this density, from the energy to remove a particle from the system (μ^-). The system is then said to be incompressible. In an incompressible system a finite energy is required to create independent positive and negative charges which are capable of carrying current through the bulk. The number of these free charges present in the system will have an activated temperature dependence and will vanish for $T \rightarrow 0$. Incompressible systems are usually insulating. Paradoxically, as we explain below, incompressibility is precisely the condition required for the quantum Hall effect to occur. The twist is that in the case of the quantum Hall effect, the density n^* at which the incompressibility occurs must depend on magnetic field. In my view *incompressibility at a magnetic-field-dependent density* is the key to the quantum Hall effect. This property requires the existence of the gapless edge excitations which are the subject of this article.

To make this more concrete, consider a large but finite two-dimensional electron gas at zero temperature, as illustrated in Fig. 1. We consider the case in which the chemical potential lies in the ‘charge gap’; $\mu \in (\mu^-, \mu^+)$. We want to consider the change in the equilibrium local currents, present in the system because of the breaking of time-reversal-invariance by the magnetic field, when we make an infinitesimal change in the chemical potential, $\delta\mu$. Because μ lies in the charge gap the change in the local current density anywhere in the bulk of the system must be zero. The current density can change, if it does anywhere, only at the edge of the system. It follows from charge conservation that, if there is a change in the current flowing along the edge of the system, it must be the same at any point along the edge. We can relate this change in current to the change in the orbital magnetization:

$$\delta I = \frac{c}{A} \delta M. \quad (1)$$

Eq. (1) is just the equation for the magnetic moment of a current loop. However,

$$\delta M = \left. \frac{\partial M}{\partial \mu} \right|_B \delta \mu = \left. \frac{\partial N}{\partial B} \right|_\mu \delta \mu. \quad (2)$$

The second equality in Eq. (2) follows from a Maxwell relation. Combining Eq. (1) and Eq. (2) we obtain the

following result for the rate at which the equilibrium edge current changes with chemical potential when the chemical potential lies in a charge gap:

$$\frac{\delta I}{\delta \mu} = c \left. \frac{\partial n}{\partial B} \right|_\mu. \quad (3)$$

The fact that $\delta I / \delta \mu \neq 0$ implies that whenever the charge gap occurs at a density which depends on magnetic field, there *must* be gapless excitations at the edge of the system. From a more microscopic point of view, Eq. (3) arises because the edge currents are related to the way in which the spectrum evolves with changes in the the vector potential and hence in the magnetic field^[2].

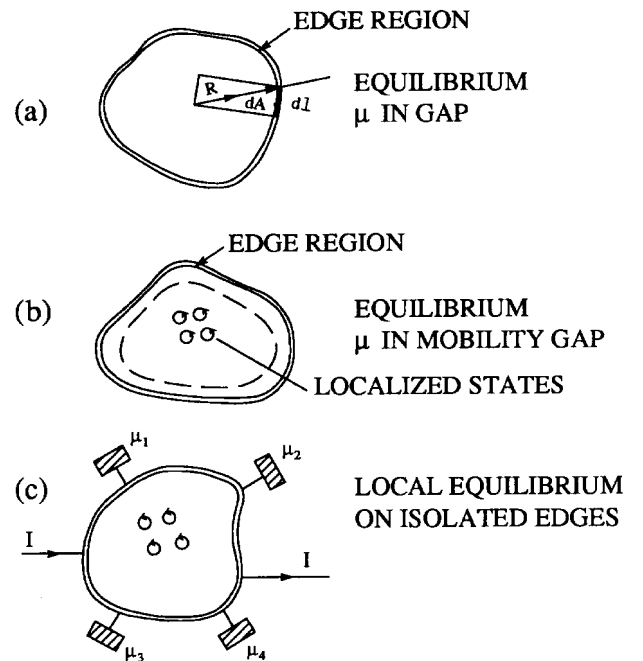


Figure 1. A large but finite two-dimensional electron gas. In panel (a) the chemical potential lies in a gap and the only low-energy excitations are localized at the edge of the system. In panel (b) the chemical potential lies in a mobility gap so that there are low-energy excitations in the bulk but they are localized away from the edge. In panel (c) a net current is carried from source to drain by having local equilibria at different chemical potentials on upper and lower edges.

This property of the edge states is expected to persist even if the chemical potential lies only in a mobility gap and not in a true gap, as illustrated schematically in Fig. 1. A net current can be carried from source to drain across the system by changing the local chemical potentials only at the edges and having different chemical potentials along the two edges connecting source

and drain. When bulk states are localized, the two edges and the bulk are effectively decoupled from each other. Eq. (3) then also applies to transport currents, relating the current carried from source to drain to the chemical potential difference between the two edges, equal to eV_H where V_H is the Hall voltage. There is no voltage drop along an edge since each edge is in local equilibrium and hence no dissipation inside the sample. Eq. (3) was proposed as an explanation for the quantum Hall effect by Pavel Štředa and is commonly known as the Štředa formula^[6].

In using this picture to explain transport experiments in bulk systems it is necessary to claim that the transport current will be carried entirely at the edge of the system even when bulk states occur at the Fermi level, as long as these states are localized. There are difficulties with this argument as a complete explanation for all transport phenomena associated with the quantum Hall effect, but that is another story and we will not pursue it here. In our view, however, there is no difficulty with the conclusion relevant to the present paper; gapless edge excitations satisfying Eq. (3) must be present whenever the quantum Hall effect occurs.

III. Non-Interacting Electron Picture

Throughout this article we will consider a disk geometry where electrons are confined to a finite area centered on the origin by a circularly symmetric confining potential, $V_{\text{conf}}(r)$. We have in mind the situation where $V_{\text{conf}}(r)$ rises from zero to a large value near $r = R$, where R is loosely speaking the radius of the disk in which the electron system is confined. We choose this geometry, for which the electron system has a single edge, since we limit our attention here to the properties of an isolated quantum Hall edges and will not discuss the physics of interaction or scattering between edges^[7]. In this geometry it is convenient to choose the symmetric gauge, $\vec{A} = B(-y, x, 0)/2$. For $V_{\text{conf}}(r) \equiv 0$, the single-electron kinetic energy eigenfunctions and eigenenergies in this gauge have definite angular momentum and are known analytically.^[3] The spectrum consists of a set of degenerate Landau levels. The kinetic energy in the n -th Landau level is $\hbar\omega_c(n + 1/2)$ where $\omega_c = eB/mc$ is the cyclotron frequency; states occur in the n -th Landau level with^[8] angular momentum $m = -n, -n + 1, \dots$. In each Landau level states with larger angular momentum are lo-

calized further from the origin. For example, the symmetric gauge wavefunctions in the lowest Landau level are:

$$\phi_m(z) = \left(\frac{1}{2\pi\ell^2 2^m m!}\right)^{1/2} z^m \exp(-|z|^2/4\ell^2) \quad (4)$$

where $2\pi\ell^2 B = hc/e$ is the magnetic flux quantum and ℓ is the magnetic length which serves as the fundamental microscopic length scale in the strong magnetic field regime. It is easy to verify that, for large m , ϕ_m is localized near a circle with radius $R_m = \sqrt{2(m+1)}\ell$. (Note that for large m the separation between adjacent values of R_m is $\ell^2/R_m \ll \ell$.) In the strong magnetic field limit, which we will adopt consistently in this article, the confinement potential does not mix different Landau levels. Since there is only one state with each angular momentum in each Landau level, that means that the only effect of the confinement potential is to increase the energy of the symmetric gauge eigenstates when R_m becomes larger than $\sim R$. The typical situation is illustrated schematically in Fig. 2. Here the $n = 0$ and $n = 1$ Landau levels are occupied in the bulk and the chemical potential μ lies in the gap $\Delta = \hbar\omega_c$ between the highest energy occupied Landau level ($E = 3\hbar\omega_c/2$) and the lowest energy unoccupied Landau level ($E = 5\hbar\omega_c/2$). The states we are interested in in this article are the ground state and the low energy excited states obtained by making one or more particle-hole excitations at the edge. We will actually discuss only the simplest situation where a single Landau level crosses the chemical potential at the edge of the system and the analogous *single branch* situations in the case of the fractional quantum Hall effect.^[9] We will also neglect the spin degree of freedom of the electrons throughout this article.

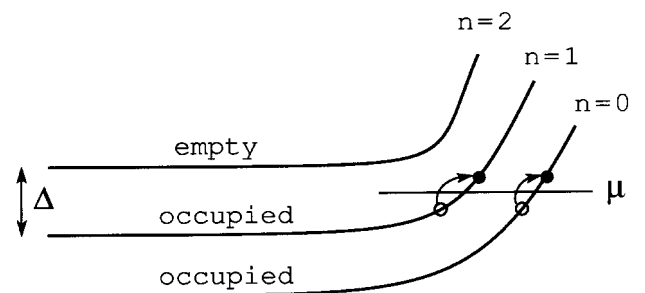


Figure 2. Schematic spectrum for non-interacting electrons confined to a circular disk in a strong magnetic field. In the limit of large disks the dependence of the energy on m can usually be considered to be continuous. The situation depicted has Landau level filling factor $\nu = 2$ in the bulk of the system.

An important property of the ground state of the non-interacting electron system in the case of interest, is that it remains an exact eigenstate of the system (but not necessarily the ground state!) when interactions are present. That is because the total angular momentum K for this state is

$$M_0 = \sum_{m=0}^{N-1} m = N(N-1)/2 \quad (5)$$

and all other states in the Hilbert space (truncated to the lowest Landau level) have larger angular momentum.^[10] For large disks and total angular momentum near M_0 the excitation energy of a non-interacting electron state will be

$$\Delta E = \gamma M \quad (6)$$

where $M \equiv K - M_0$ is the excess angular momentum and γ is the energy separation between single-particle states with adjacent angular momenta and energies near the Fermi energy. γ is related to the electric field, E_{edge} from the confining potential at the edge of the disk:

$$\gamma = e E_{\text{edge}} \frac{dR_m}{dm} = e E_{\text{edge}} \ell^2 / R \quad (7)$$

This expression for γ can be understood in a more appealing way. In a strong magnetic field charged particles execute rapid cyclotron orbits centered on a point which slowly drifts in the direction perpendicular to both the magnetic field and the local electric field. For an electron at the edge of the disk the velocity of this ‘E cross B’ drift is $v_{\text{edge}} = c E_{\text{edge}} / B$. The energy level separation can therefore be written in the form

$$\gamma = \hbar v_{\text{edge}} / R_{\text{edge}} = \hbar / T \quad (8)$$

where T is the period of the slow drift motion of edge electrons around the disk, in agreement with expectations based on semiclassical quantization.

Since the excitation energy depends only on the angular momentum increase compared to the ground state it is useful to classify states by M . It is easy to count the number of distinct many-body states with a given value of M as illustrated in Fig. 3. For $M = 1$ only one many-particle state is permitted by the Pauli exclusion principle; it is obtained by promoting the ground state electron with $m = N - 1$ to $m = N$. For $M = 2$, particle hole excitations are possible from $m = N - 1$ to $m = N + 1$ and from $m = N - 2$ to $m = N$. In general

M many-particle states with excess angular momentum M can be created by making a single-particle hole excitation of the ground state. For $M \geq 4$ additional states can be created by making multiple particle-hole excitations. The first of these is a state with two particle-hole excitations which occurs at $M = 4$ and is illustrated in Fig. 3.

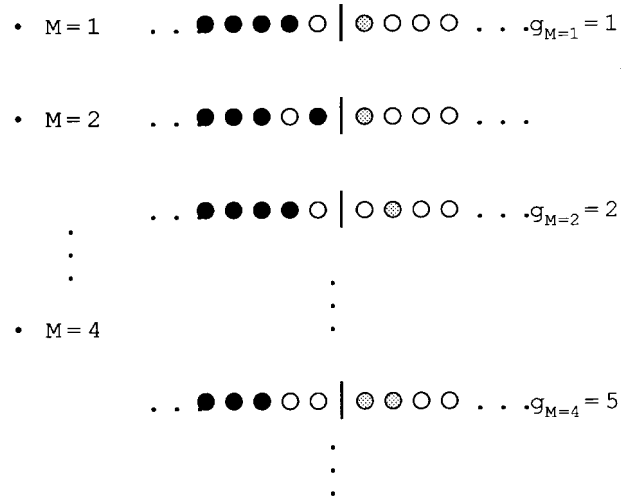


Figure 3. Non-interacting many electron eigenstates for small excess angular momentum M specified by occupation numbers for the single-particle states with energies near the chemical potential μ . The vertical bars separate single-particle states with $\epsilon_m < \mu$ from those with $\epsilon_m > \mu$. A solid circle indicates that $n_m = 1$ in both the ground state and in the particular excited state; a shaded circle indicates that $n_m = 1$ in the particular excited state but not in the ground state; an empty circle indicates that $n_m = 0$.

IV. Compressible and Incompressible Strips: The Thomas-Fermi Picture

It is very instructive to apply a Thomas-Fermi approximation to the edge of an electron disk in the quantum Hall regime^[11]. A modern framework for discussing the Thomas-Fermi approximation is provided by density-functional theory, which has been generalized in recent years to accommodate magnetic fields.^[12] The Thomas-Fermi approximation is intended to be applicable to the case where the charge density changes very slowly on atomic length scales. In this regime we can hope that the local-density-approximation, where the energy density at \vec{r} is assumed to be equal to the energy density of a uniform system with density $n = n(\vec{r})$, is valid. The main lessons to be learned from the Thomas-Fermi approximation for quantum Hall edges are connected with the long-range of the

Coulomb interaction and our discussion here is always for the case of these physically realistic interactions. When the Thomas-Fermi approximation is applied for systems of charged particles, the electrostatic interac-

tion energy, for which a local density approximation is obviously inappropriate, must be treated separately. The Thomas-Fermi energy functional is therefore

$$E[n] = E_H[n] + \int d^2\vec{r} n(\vec{r}) \epsilon(n(\vec{r})) + \int d^2\vec{r} n(\vec{r}) V_{\text{ext}}(\vec{r}), \quad (9)$$

where $E_H[n]$ is the electrostatic (Hartree) energy, V_{ext} is the external potential, and $\epsilon(n)$ is the energy per-particle of a uniform density system (in a magnetic field in our case) which is placed in an electrically neutralizing positively charged background to eliminate the electrostatic energy. Minimizing this energy functional with respect to $n(\vec{r})$ at fixed total particle number gives the Thomas-Fermi approximation expression for the density profile,

$$V_{\text{ext}}(\vec{r}) + V_H(\vec{r}) + \mu_{2D}(n(\vec{r})) = \mu. \quad (10)$$

Here $V_H(\vec{r})$ is the (non-local) electrostatic potential, μ is the chemical potential, and

$$\mu_{2D}(n) = \frac{dn\epsilon(n)}{dn} \quad (11)$$

is the density-dependent chemical potential of a uniform density electron system in a neutralizing background.

Eq. (11) can and has been used to discuss the density profile at the edge in the case of both integer^[11] and fractional^[13] quantum Hall effects. We will discuss here the Thomas-Fermi version of the random-phase-approximation, where we include only the electrostatic and kinetic energy contributions to the energy and fractional quantum Hall effect features are not captured. In this approximation (neglecting spin for simplicity)

$$\mu_{2D}(n) = \hbar\omega_c([\nu] + 1/2) \quad (12)$$

where $[\nu]$ is the integer part of the Landau level filling factor $\nu = 2\pi\ell^2 N/A$. Let's assume for the moment that the non-interacting electron ground state remains the ground state in the presence of interactions. Looking on the macroscopic length scale appropriate

to the Thomas-Fermi approximation, the charge density of this state is constant: $n = (2\pi\ell^2)^{-1}$ for $R < R_N$ and $n = 0$ for $R > R_N$. The electrostatic potential produced by this charge density is

$$V_H(r) = \frac{e^2}{\epsilon\ell} \sqrt{2NF} \left(\frac{1}{2}, \frac{-1}{2}, 1; \frac{r^2}{R_N^2} \right). \quad (13)$$

where $F(a, b, c; x)$ is the confluent hypergeometric function.

The important point for us is that the scale of this potential is larger than the microscopic interaction energy scale e^2/ℓ by a factor proportional to \sqrt{N} . Since e^2/ℓ and $\hbar\omega_c$ are of the same order it follows that except for the case of small N 'quantum dot' systems,^[14-17] the maximum density droplet wavefunction cannot be the ground state *unless* the sum of an external potential and the Hartree electrostatic potential is close to constant, *i.e.* unless the external potential is similar to that from a neutralizing positive background which is co-planar with the electrons. When this is not the case, the charge density profile will be close to that which would be obtained in a purely electrostatic theory by solving the equation $V_{\text{ext}}(r) + V_H(r) = \mu$. When $\mu_{2D}(n)$ is included, the solution of the Thomas-Fermi equation for the charge-density will consist of relatively broad regions where $V_{\text{ext}}(r) + V_H(r) = \mu - \hbar\omega_c(N + 1/2)$ for some fixed integer N and the charge density varies, and relatively narrow regions separating adjacent values of N where the charge density is fixed.

In this picture of quantum Hall system edges, regions where the electron density varies are referred to as compressible strips and regions where the electron density is fixed are referred to as incompressible strips. Evidently, low-energy excitations can be created in the compressible strips by varying the charge density along

the edge or by altering slightly the charge density profile across the edge. To relate the Thomas-Fermi theory results to microscopic theory, we must, at a minimum, use a Hartree or Hartree-Fock approximation. The results of such a theory are illustrated schematically in Fig. 4. In this illustration we have in mind a ‘smooth edge’ where the external potential is created by charges on gates or surface states some distance from the two-dimensional electron system. The extreme limit of the ‘smooth edge’ model systems is one where the external potential is taken to be parabolic. For this parabolic confinement model, which is often applicable to quantum dot systems, the electrostatic approximation to the Thomas-Fermi equations can be solved analytically:^[18,11]

$$n(r) = n_0 \sqrt{1 - (r/R)^2}, \quad (14)$$

where n_0 and R , the radius of the electron disk, are constants fixed by the total electron number and the curvature of the parabolic potential. When $\mu_{2D}(n)$ is included in the Thomas-Fermi equations, the density will distort slightly and develop the compressible and incompressible strips discussed above and illustrated in Fig. 4. For smooth edges, electrostatic considerations will be so dominant that nearly the same density profile will be produced by any microscopic theory. The simplest microscopic theory that includes interactions is a Hartree^[19] or Hartree-Fock theory in which the energy of each single particle state is modified by electrostatic (or Hartree) and, in the Hartree-Fock case, exchange corrections. In such a theory the regions in space where the local density does not correspond to an integral Landau level filling factor, *i.e.* the *compressible strips* must have fractional occupation numbers for the Hartree-Fock single-particle states. It follows that the total variation of the Hartree-Fock energy across the angular momenta in each compressible strip must be smaller than the thermal energy $k_B T$. One point of view on this result is to regard the system as locally metallic in the compressible strips so that the confinement potential is strongly screened.^[20] Of course, the Hartree-Fock approximation is not justified in this regime but it does provide one valid lesson. Throughout the compressible region, single-particle states with nearby angular momenta will have a finite probability of being occupied or unoccupied as the result of

thermal or, in a more general theory, quantum fluctuations. The sharp Fermi edge between occupied and unoccupied states that we would have for non-interacting electrons is lost. It seems clear that the spectrum of low-energy excitations can be exceedingly complicated in the smooth edge regime.

Thomas-Fermi

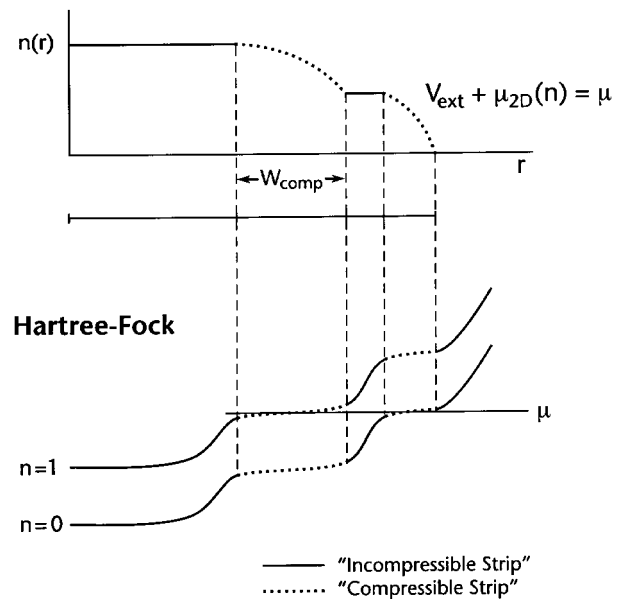


Figure 4. Schematic illustration of the Thomas-Fermi theory and Hartree-Fock theory pictures of the edge of a quantum Hall system with Landau level filling factor $\nu = 2$ in the bulk. The Thomas-Fermi theory gives an approximation for the charge density profile across the edge. Regions where the charge density varies, indicated by dashed lines, correspond to constant values for $V_{\text{ext}} + V_H$ and are known as compressible strips. The edge excitations occur in the compressible strips. Hartree-Fock theory produces approximate values for the quasiparticle energies as a function of angular momentum in the edge region. When the electron density at the edge varies gradually on an atomic length scale the Hartree-Fock eigenvalues will have a weak dependence on angular momentum where they cross the Fermi level.

In the following sections we will implicitly assume that the external confining potential permits the electron density at the edge to fall off on a microscopic length scale, *i.e.* that we are in the ‘abrupt edge’ regime. We will also implicitly assume that the interactions between the particles are short-ranged, appealing if pressed to the ubiquitous presence of nearby gates which dress all electrons with image charges. You have been fairly warned that these assumptions can be dangerous especially when electrostatic imperatives

force an electron-density that changes slowly on microscopic length scales. The task of determining the excitation energy (temperature), if any, below which the ‘abrupt edge’ models we will now discuss apply for ‘smooth edge’ systems remains an important challenge. In quantum dot systems the transition between ‘abrupt edge’ and ‘smooth edge’ regimes is initiated by *edge reconstructions*.^[16] Similar ‘reconstructions’ may occur at the edges of bulk systems and when they occur they will complicate the excitation spectrum and all physical properties.

V. Many-Body Wavefunction Picture

In this section we discuss the edge excitation spectrum of interacting electrons using a language of many-particle wavefunctions. For the case of the integer quantum Hall effect we will essentially recover the picture of the excitation spectrum obtained previously for non-interacting electrons by counting occupation numbers. We could have used the Hartree-Fock approximation and occupation number counting to generalize these results to interacting electrons. However, the Hartree-Fock approximation is completely at sea when it comes to the fractional case. Discussions of the fractional edge using an independent electron language are can be comforting but are, in my view, misleading. Nevertheless, we will see that there is a one-to-one correspondence between the edge excitation spectrum for non-interacting electrons at integer filling factors and the fractional edge excitation spectrum.

Many-electron wavefunctions where all electrons are confined to the lowest Landau level must be sums of products of one-particle wavefunctions from the lowest Landau level. From Eq. (4) it follows that any N electron wavefunction has the form

$$\Psi[z] = P(z_1, \dots, z_N) \prod_{\ell} \exp(-|z_{\ell}|^2/4), \quad (15)$$

where we have adopted ℓ as the unit of length and $P(z_1, \dots, z_N)$ is a polynomial in the two-dimensional complex coordinates. This property^[21] of the wavefunctions will be exploited in this section. The first important observation is that since $\Psi[z]$ is a wavefunction for many identical fermions it must change sign when any two particles are interchanged, and therefore must vanish as any two particles positions approach each other. Since $P(z_1, \dots, z_N)$ is a polynomial in each complex coordinate it follows^[22] that

$$P(z_1, \dots, z_N) = \prod_{i < j} (z_i - z_j) Q[z] \quad (16)$$

where $Q[z]$ is any polynomial which is symmetric under particle interchange. It is important to note that the total angular momentum of all the particles (K) is just the degree of the polynomial $P[z]$, *i.e.* the sum of the powers to which the individual particle complex coordinates are raised. Since the total angular momentum is a good quantum number the polynomial part of any many-electron eigenstate will be a homogeneous degree polynomial, *i.e.* all terms in the many-particle polynomial must have the same degree. Additionally, the total angular momenta corresponding to a polynomial which, as in Eq. (16), is the product of two polynomials is the sum of the angular momenta associated with those polynomials.

From the discussion of Section III it is clear that for non-interacting electrons in any monotonically increasing confinement potential, the lowest energy state will be the state with the minimum total angular momentum. In Eq. (16) that corresponds to choosing $Q[z]$ to have degree zero, *i.e.* to $Q[z] \propto 1$. It is easy to verify that the wavefunction when $Q[z]$ is a constant is in fact the Slater determinant formed by occupying the single-particle states with $m = 0, \dots, N - 1$. For interacting electrons this state will remain the ground state provided the confinement potential is strong enough to overcome the repulsive interactions between electrons which favor states with larger total angular momentum. When this is the ground state, low-energy excited states with excess angular momentum $M = K - N(N - 1)/2$ are linear combinations of the states constructed by choosing all possible symmetric polynomials^[23] of degree M for $Q[z]$.

We’ll discuss the enumeration of these polynomials in a moment but pause now to explain how this analysis may be generalized to the case of the fractional quantum Hall effect. We limit our attention here^[24] to the simplest fractional quantum Hall effects which occur at Landau level filling factors $\nu = 1/m$ for any odd integer m ; in some senses the $m = 1$ case can be regarded as a special case of the fractional quantum Hall effect. The physics of the chemical potential jump which occurs at these filling factors was explained in the pioneering paper of Laughlin^[3,25]. For $\nu < 1/3$, for example, it is possible to find states in the Hilbert space in which pairs of electrons are never found in a state with relative angular momentum equal to one. This is the two-body state in which two electrons are

closest together. All many-particle states which avoid placing pairs in this state will have low energy. If this condition^[26] is satisfied,

$$P(z_1, \dots, z_N) \equiv \prod_{i < j} (z_i - z_j)^3 Q[z] \quad (17)$$

for any symmetric polynomial $Q[z]$. If the system has an abrupt edge the ground state will have $Q[z] \equiv 1$ just as in the non-interacting case. The edge excitations correspond to the same set of symmetric polynomials as in the $\nu = 1$ case. In the case of a model system with a short-range interaction and a parabolic confinement potential, it is easy to place^[17] the argument we have sketched above on firm ground. It is known from numerical studies that the bulk chemical potential discontinuity survives when the model interaction is replaced by the realistic Coulomb interaction. As long as the external potential is such that we are in the ‘abrupt edge’ regime, the edge states in the realistic case will be in one-to-one correspondence with those of Eq. (17).

The wavefunction

$$\Psi[z] \equiv Q[z] \prod_l \exp(-|z_l|^2/4) \quad (18)$$

is a wavefunction for N bosons in a strong magnetic field. Thus the enumeration of the edge excitations in terms of symmetric polynomials discussed above is equivalent to enumerating all many boson wavefunctions with a given value of the total angular momentum. The boson angular momentum

$$M = \sum_{m=0}^{\infty} m n_m \quad (19)$$

where n_m are the boson occupation numbers, is equivalent (for $\nu = 1/m$) to the excess angular momentum

$M = K - mN(N - 1)/2$ of the fermion wavefunctions. In the state with $M = 0$, $n_0 = N$, all other boson occupation numbers are zero, and $Q[z]$ is a constant. In the boson language the ground state is a Bose condensate. The lone state with $M = 1$ has $n_1 = 1, n_0 = N - 1$; the symmetric polynomial for this boson wavefunction is

$$Q[z] = z_1 + z_2 + \dots + z_N. \quad (20)$$

For the integer $\nu = 1$ case, it can be shown explicitly that the corresponding many-fermion state is the $M = 1$ state with a single particle-hole excitation at the edge, discussed in Section III. The set of excitations at general values of M can be described equally well in either fermion or boson languages. Some of the states which occur at small values of M are listed in Table I.

In the parabolic confinement case the total energy depends only on the excess total angular momentum: $\delta E = \gamma M$. The number of many-boson states with total angular momentum M , $g(M)$ can be calculated by considering a system of non-interacting bosons with single-particle energy γm so that $E = \sum_m (\gamma m) \cdot n_m = \gamma M$. The partition function is

$$Z = \sum_M g(M) e^{-\gamma M/k_B T} = \sum_M x^M g(M) \quad (21)$$

where $x = e^{-\gamma/k_B T}$. For $N \rightarrow \infty$, the $m = 0$ state acts like a reservoir with chemical potential $\mu = 0$ so that the partition function calculation can be done in the grand canonical ensemble. The degeneracies $g(M)$ can be read off the power series expansion of the partition function:

$$\begin{aligned} Z &= \prod_{k=1}^{\infty} \frac{1}{1 - x^k} = (1 - x)^{-1} (1 - x^2)^{-1} (1 - x^3)^{-1} \dots \\ &= (1 + x + x^2 + x^3 + \dots)(1 + x^2 + x^4 + \dots)(1 + x^3 + x^6 + \dots) \dots \\ &= (1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + 30x^9 + 42x^{10} + \dots) \end{aligned} \quad (22)$$

For^[17] large M $g(M) \sim e^{\sqrt{\frac{2}{3}} \pi \cdot M^{1/2}}$. The function $g(M)$ is well known to number theorists from the the-

ory of partitions^[27] in which it is known as the partition function, not to be confused with the physics partition

Table I. Quantum occupation numbers in boson and fermion descriptions for edge excitations with small excess angular momentum M . g_M is the number of states with excess angular momentum M . The fermion occupation numbers are relative to the maximum density droplet state. Only non-zero values are listed for both fermion and boson descriptions. $L = N - 1$ is the highest angular momentum which is occupied in the maximum density droplet state.

M	g_M	Fermion Description	Boson Description
1	1	$n_{L+1} = 1, n_L = -1$	$n_1 = 1$
2	2	$n_{L+2} = 1, n_L = -1; n_{L+1} = 1, n_{L-1} = -1$	$n_2 = 1; n_1 = 2$
3	3	$n_{L+3} = 1, n_L = -1; n_{L+2} = 1, n_{L-1} = -1$ $n_{L+1} = 1, n_{L-2} = -1$	$n_3 = 1; n_2 = 1, n_1 = 1;$ $n_1 = 3$
4	5	$n_{L+4} = 1, n_L = -1; n_{L+3} = 1, n_{L-1} = -1$ $n_{L+2} = 1, n_{L-2} = -1; n_{L+1} = 1, n_{L-3} = -1$ $n_{L+2} = 1, n_{L+1} = 1, n_L = -1, n_{L-1} = -1$	$n_4 = 1; n_3 = 1, n_1 = 1; n_2 = 2$ $n_2 = 1, n_1 = 2; n_1 = 4$

function above! For parabolic confinement potentials and short-ranged repulsive interactions, the degeneracy of the edge excitations at a given excess angular momentum is exact in both integer and fractional $\nu = 1/m$ cases. For general confinement potentials and general electron-electron interactions these degeneracies will be lifted. However, there is reason to expect that in the thermodynamic limit excitations with $M \ll N^{1/2}$ will be nearly degenerate. One way to see this is to use the chiral Luttinger liquid picture of quantum Hall edges which we discuss in the following section. This approach will allow us to do more than enumerate excitations of the system and, in particular will enable us to discuss the density-of-states for tunneling into the edge of a quantum Hall system.

VI. Chiral Luttinger Liquid Picture

The chiral Luttinger liquid picture^[5] of quantum Hall systems is an adaptation of the Luttinger liquid theory of one-dimensional electron systems. We start this section with a brief outline of the portion of that theory that we require. Readers in search of greater depth should look elsewhere.^[28] As in higher dimensions, low excitation energies states in a one-dimensional fermion system will involve only single-particle states near the Fermi wavevector. Since the differences in wavevector among the relevant states at a given Fermi edge are small, the excitations produced by rearranging them occur on length scales which are long compared to microscopic lengths. It is therefore reasonable to argue that the energy density in the system at any point in space should depend only on the local density of left-moving ($k < 0$) and right-moving ($k > 0$) electrons, $n_L(x)$ and $n_R(x)$:

$$E[n_L, n_R] = E_0 + \int dx \left[\frac{\alpha_{LL}}{2} \delta n_L^2(x) + \frac{\alpha_{RR}}{2} \delta n_R^2(x) + \alpha_{LR} \delta n_L(x) \delta n_R(x) \right]. \quad (23)$$

It is, perhaps, not completely obvious that the density provides a complete parameterization of the low-energy excitations, and indeed in the fractional Hall case there are situations where the analog of Eq. (23) is incorrect. Here α_{LL} , α_{LR} and α_{RR} are determined by the second derivatives of the energy per unit length with respect to n_L and n_R for a uniform system and can be determined in principle by a microscopic calcu-

lation. $\delta n_L(x)$ and $\delta n_R(x)$ are differences of the density from the ground state density. Note that we have as a convenience chosen the chemical potential to be zero in dropping a term proportional to $\int dx (\delta n_L(x) + \delta n_R(x))$. We start by considering the case where $\alpha_{LR} = 0$ so that the left-moving electrons and right moving electrons are decoupled. Focus for this case on the energy of the right moving electrons. We Fourier expand the density and

note that

$$\int dx \delta n_R^2(x) = \frac{1}{L} \sum_{q \neq 0} n_{-qR} n_{qR} \quad (24)$$

so that the energy can be written in the form

$$E_R = E_0 + \frac{\alpha_{LL}}{2L} \sum_{q \neq 0} n_{-qR} n_{qR}. \quad (25)$$

The energy above can be used as an effective Hamiltonian for low-energy long-wavelength excitations. The simplification at the heart of the Luttinger liquid theory is the observation that when the Hilbert space is truncated to include only low-energy, long-wavelength excitations (in particular when the number of left-moving and right-moving electrons is fixed) Fourier components of the charge density do not commute. For example consider the second quantization expression for n_{qR} in terms of creation and annihilation operators with $k > 0$:

$$n_{qR} = \sum_{k > 0} c_{k+q}^\dagger c_k. \quad (26)$$

An example of the dependence of the effect of products of these operators on the order in which they act is more instructive than the actual algebraic calculation of the commutators. Note for example that

$$n_{-qR} |\Psi_0\rangle = 0 \quad (27)$$

where $q > 0$ and $|\Psi_0\rangle$ is the state with all right-going electron states with $k < k_F$ occupied and all right-going states with $k > k_F$ empty. (The alert reader will have noticed that this state of ‘right-going’ electrons corresponds precisely to the ‘maximum density droplet’ states which occur in the quantum Hall effect.) n_{-qR} annihilates this state because there are no right-electron states with a smaller total momentum than $|\Psi_0\rangle$. On the other hand for $q = M2\pi/L$, $n_{qR}|\Psi_0\rangle$ yields a sum of M terms in which single-particle hole excitations have been formed in $|\Psi_0\rangle$. For example, if we represent occupied states by solid circles and unoccupied states by open circles, as in Fig. 3, for $M = 2$ we have

$$\begin{aligned} n_{qR} |\Psi_0\rangle &= |\dots \bullet \circ \bullet | \bullet \circ \circ \dots\rangle \\ &+ |\dots \bullet \bullet \circ | \circ \bullet \circ \dots\rangle. \end{aligned} \quad (28)$$

Each of the M terms produced by $n_{qR}|\Psi_0\rangle$ is mapped back to $|\Psi_0\rangle$ by n_{-qR} . Therefore $n_{qR}n_{-qR}|\Psi_0\rangle = 0$

whereas $n_{-qR}n_{qR}|\Psi_0\rangle = M|\Psi_0\rangle$. The general form of the commutation relation is readily established by a little careful algebra:^[28]

$$[n_{-q'R}, n_{qR}] = \frac{qL}{2\pi} \delta_{q,q'}. \quad (29)$$

This holds as long as we truncate the Hilbert space to states with a fixed number of right-going electrons and assume that states far from the Fermi edge are always occupied.

We can define creation and annihilation operators for density wave excitations of right-going electrons. For $q > 0$

$$a_q = \sqrt{\frac{2\pi}{qL}} n_{-qR} \quad (30)$$

$$a_q^\dagger = \sqrt{\frac{2\pi}{qL}} n_{qR} \quad (31)$$

With these definitions Eq. (29) yields

$$[a_{q'}, a_q^\dagger] = \delta_{q,q'} \quad (32)$$

so that the density waves satisfy bosonic commutation relations. Also note that

$$[\hat{M}, a_q] = -\frac{qL}{2\pi} a_q \quad (33)$$

$$[\hat{M}, a_q^\dagger] = \frac{qL}{2\pi} a_q^\dagger \quad (34)$$

where \hat{M} is the total angular momentum operator. The contribution to the Hamiltonian from right-going electrons is therefore

$$H_R = \sum_{q > 0} \hbar v q a_q^\dagger a_q \quad (35)$$

where

$$v = \frac{\alpha_{RR}}{2\pi\hbar} = \frac{1}{2\pi L\hbar} \frac{d^2 E_0}{dn_R^2} = \frac{1}{2\pi\hbar} \frac{d\mu_R}{dn_R} \quad (36)$$

At low-energies the system is equivalent to a system of one-dimensional phonons traveling to the right with velocity v . In the limit of non-interacting electrons

$$v = \frac{\hbar k_F}{m^*} \equiv v_F \quad (37)$$

as expected.

Without interactions between left and right-moving electrons a Luttinger liquid is quite trivial. In particular the ground state ($|\Psi_0\rangle$) is a single-Slater determinant with a sharp Fermi edge. For one-dimensional

electron gas systems the interesting physics^[28] occurs only when left and right-moving electrons are allowed to interact. Most notably, arbitrarily weak interactions destroy the sharp Fermi edge which is the hallmark of Fermi liquids and which survives interactions in higher dimensions. In the case of quantum Hall edges, however, the above restriction to electrons moving in only one direction is not a temporary pedagogical device. The model with only right moving electrons discussed above can be taken over *mutatis mutandis* as a model of the edge excitations for an electron system with $\nu = 1$. The role played by the one-dimensional electron density is taken over by the integral of the two-dimensional electron density along a line perpendicular to the edge. Results discussed in earlier sections can be discussed instead in the language of chiral Luttinger liquids. The set of boson states are the states of the chiral phonon system which has modes with only one sign of momentum and velocity.

For $\nu = 1$ the analysis applies whether or not the electrons interact. We now turn our attention to a discussion of the fractional case. Do all steps of the above discussion generalize? We can argue that if we are interested only in low-energy long-wavelength excitations, the energy can be expressed in the form

$$E = E_0 + \frac{\alpha}{2L} \sum_{q \neq 0} n_{-q} n_q. \quad (38)$$

As we comment later, this expression can fail at the edge of fractional quantum Hall systems although it is appropriate for $\nu = 1/m$. What about the commutator? There is an important difference in the line of argument in this case, since single-particle states far from the edge of the system are not certain to be occupied. Instead the average occupation number is $\nu = 1/m$ and there are large quantum fluctuations in the local configuration of the system even in the interior. However, we know^[24] from the discussion in terms of many-body wavefunctions in the previous section that the low-energy excitations at $\nu = 1/m$ can be described as the excitations of a boson system, exactly like those at $\nu = 1$, which suggests that something like Eq. (29) must still be satisfied when the Hilbert space is projected to low energies. If we replace the commutator

by its expectation value in the ground state we obtain

$$[n_{-q'}, n_q] = \nu \cdot \frac{qL}{2\pi} \delta_{q,q'} \quad (39)$$

which differs from Eq. (29) only through the factor ν . It seems clear for the case of $\nu = 1/m$ this replacement can be justified on the grounds that the interior is essentially frozen (but in this case *not* simply by the Pauli exclusion principle) at excitation energies smaller than the gap for bulk excitations. What we need to show is that Eq. (39) applies as an operator identity in the entire low-energy portion of the Hilbert space. Below, however, we follow a different line of argument.

Appealing to the microscopic analysis in terms of many-body wavefunctions we know that the excitation spectrum for $\nu = 1/m$ is equivalent to that of a system of bosons. We conjecture that the commutator $[n_{-q'}, n_q] = \alpha q \delta_{q,q'}$. To determine the constant of proportionality we will require that the rate of change of the equilibrium edge current with chemical potential be $e\nu/h$. From the edge state picture of the quantum Hall effect discussed in Section II, it is clear that this is equivalent to requiring the Hall conductivity to be quantized at $\nu e^2/h$. Since our theory will yield a set of phonon modes which travel with a common velocity v it is clear that the change in equilibrium edge current is related to the change in equilibrium density by

$$\delta I = e\nu \delta n. \quad (40)$$

When the chemical potential for the single edge system is shifted slightly from its reference value (which we chose to be zero) the grand potential is given by

$$E[n] = E_0 + \mu \delta n + \alpha \frac{(\delta n)^2}{2} \quad (41)$$

Minimizing with respect to δn we find that

$$\delta n = \frac{\delta \mu}{\alpha} \quad (42)$$

so that

$$\frac{\delta I}{\delta \mu} = \frac{e\nu}{\alpha} \quad (43)$$

In order for this to be consistent with the quantum Hall effect ($\delta I = (e\nu/h)\delta\mu$) our theory must yield a edge phonon velocity given by

$$v = \frac{\alpha}{h} \cdot \nu. \quad (44)$$

The extra factor of ν appearing in this equation compared to Eq. (36) requires the same factor of ν to appear in Eq. (39). We discuss below the qualitative changes in the physics^[4,5] of fractional edge states which are implied by this outwardly innocent numerical factor.

It is worth remarking that the line of argument leading to this specific chiral Luttinger liquid theory of the fractional quantum Hall effect is not completely rigorous. In fact we know that this simplest possible theory with a single branch of chiral bosons does not apply for all filling factors^[24,4,29], even though (nearly) all steps in the argument are superficially completely general. The reader is encouraged to think seriously about what could go wrong with our arguments. Certainly the possibility of adiabatically connecting all low-energy states with corresponding states of the non-interacting electron system, available for one-dimensional electron gases and for quantum Hall systems at integer filling factors but not at fractional filling factors, adds confidence when it is available. In our view, the microscopic many-particle wavefunction approach which establishes a one-to-one mapping between integer and fractional edge excitations (for $\nu = 1/m!$) is an important part of the theoretical underpinning of the Luttinger liquid model of fractional Hall edges. Once we know that the edge excitations map to those of a chiral boson gas and that the fractional quantum Hall effect occurs, it appears that no freedom is left in the construction of a low-energy long-wavelength effective theory. The reader is reminded however, of the smooth edge regime, where in our view both the many-particle wavefunction mapping and the chiral Luttinger liquid theory of the edge are likely to fail.

An important aspect of Luttinger liquid theory is the expression for electron field operators in terms of bosons.^[28] This relationship is established by requiring the exact identity

$$[\rho(x), \hat{\psi}^\dagger(x')] = \delta(x - x') \hat{\psi}^\dagger(x') \quad (45)$$

to be reproduced by the effective low-energy theory. This equation simply requires the electron charge density to increase by the required amount when an elec-

tron is added to the system. The electron creation operator should also be consistent with Fermi statistics for the electrons:

$$\{\psi^\dagger(x), \psi^\dagger(x')\} = 0. \quad (46)$$

In order to satisfy Eq. (??), the field operator must be given by

$$\hat{\psi}^\dagger(x) = ce^{i\nu^{-1}\phi(x)} \quad (47)$$

where $d\phi(x)/dx = n(x)$ and c is a constant which cannot be determined by the theory. The factor of ν^{-1} in the argument of the exponential of Eq. (47) is required because of the factor of ν in the commutator of density Fourier components which in turn was required to make the theory consistent with the fractional quantum Hall effect. When the exponential is expanded the $k - th$ order terms generate states with total boson occupation number k and are multiplied in the fractional case by the factor ν^{-k} ; multi-phonon terms are increased in importance. It is worth remarking^[4] that the anti-commutation relation between fermion creation operators in the effective theory is satisfied only when ν^{-1} is an odd integer. This provides an indication, independent of microscopic considerations, that the simplest single-branch chiral boson effective Hamiltonian can be correct only when $\nu = 1/m$ for odd m . Wen^[5] has surveyed, using this criterion, the multi-branch generalization of the simplest effective Hamiltonian theory which are possible at any given rational filling factor. His conclusions are consistent with arguments^[24] based on the microscopic theory of the fractional quantum Hall effect.

Eq. (47) has been carefully checked numerically^[30] and appears to be correct. The ν^{-1} factor leads to predictions of qualitative changes in a number of properties of fractional edges. The quantity which is most directly altered is the tunneling density-of-states. Consider, for example, the state created when an electron, localized on a magnetic length scale, is added to the ground state at the edge of a N -electron system with $\nu = 1/m$:

$$\begin{aligned}\hat{\psi}^\dagger(0)|\Psi_0\rangle &\sim \exp\left(-\sum_{n>0}\frac{a_n^\dagger}{\sqrt{n\nu}}\right)|\psi_0\rangle \\ &= 1 + \frac{1 \text{ phonon term}}{\nu^{1/2}} + \frac{2 \text{ phonon terms}}{\nu} + \dots\end{aligned}\quad (48)$$

The tunneling density states is given by a sum over the ground and excited states of the $N + 1$ particle system:

$$A(\epsilon) = \sum_n \delta(E_n - E_0 - \epsilon) |\langle \Psi_n | \hat{\psi}^\dagger(0) | \Psi_0 \rangle|^2 \quad (49)$$

Because of the increased weighting of multiphonon states, which become more numerous at energies farther from the chemical potential, the spectral function is larger at larger $\epsilon - \mu$ in the fractional case. An explicit calculation^[4,5] yields a spectral function which grows like $(\epsilon - \mu)^{\nu^{-1}-1}$. It is intuitively clear that the spectral function should be small at low-energies in the fractional case since the added electron will not share the very specific correlations common to all the low-energy states. It is amazing that by simply requiring the low-energy theory to be consistent with the fractional quantum Hall effect we get a very specific prediction for the way in which this qualitative notion is manifested in the tunneling density of states.

VII. Endnote

There is, as always, a lot more which could be said. However other duties, including even ones for which I am paid, are insisting that I must stop here. These notes have focused on the microscopic origins, and also possibly (in the ‘smooth edge’ case) the limitations of Luttinger liquid theories for quantum Hall edges. A different and at least equally interesting article could be written on applications of Luttinger liquid models in the fractional quantum Hall regime. Among these it appears that those which describe^[7] tunneling^[31] between quantum Hall edge systems due to disorder are the most promising for experimental tests. Indeed, initial experiments^[32] appear to confirm theoretical predictions. It is likely that more experimental work will

be added to the literature soon. The important question of the role of long-range interactions in Luttinger liquid theories, which we have for the most part dodged here, is also beginning to be addressed.^[33]

These informal notes are intended to be widely accessible. I hope that they will be of some use to people who are expert on either theoretical or experimental aspects of the quantum Hall effect but have not been following theories of quantum Hall edges. Likewise I hope that they can be useful to experts on the one-dimensional electron gas who have not been following the quantum Hall effect. Comments, critical or complimentary, are welcome. The ideas here have been shaped by discussions with members of the condensed matter theory group at Indiana University, especially S.M. Girvin, R. Haussmann, S. Mitra, K. Moon, J.J. Palacios, D. Pfannkuche, E. Sorensen, K. Tevosyan, K. Yang, and U. Zülicke. Discussions with L. Brey, M. Fisher, M. Johnson, C. Kane, L. Martin, J. Oaknin, C. Tejedor, S.R.-E. Yang and X.-G. Wen are also gratefully acknowledged. The responsibility for surviving misapprehensions rests with me. This work was supported by the National Science Foundation under grant DMR-9416906.

References

1. K. v. Klitzing, G. Dorda, and M. Pepper, *Phys. Rev. Lett.* **45**, 494 (1980); D.C. Tsui, H.L. Störmer, and A.C. Gossard, *Phys. Rev. Lett.* **48**, 1761 (1986).
2. R.B. Laughlin, *Phys. Rev. B* **23**, 5632 (1981); B.I. Halperin, *Phys. Rev. B* **25**, 2185 (1982); A.H. MacDonald and P. Streda, *Phys. Rev. B*

- 29, 1616 (1984); M. Buttiker, Phys. Rev. B **38**, 9375 (1988).
3. See for example, A.H. MacDonald, in *Les Houches, Session LXI, 1994, Physique Quantique Mesoscopique*, edited by E. Akkermans, G. Montambaux, and J.L. Pichard (Elsevier, Amsterdam, 1995).
 4. X.G. Wen, Phys. Rev. B **41**, 12838 (1990); D.H. Lee and X.G. Wen, Phys. Rev. Lett. **66**, 1765 (1991); X.G. Wen, Phys. Rev. B **44**, 5708 (1991).
 5. For reviews see X.G. Wen, Int. J. Mod. Phys. **B6**, 1711 (1992); X.G. Wen, to appear in *Advances in Physics* (1995).
 6. P. Středa, J. Phys. C, **15**, L717 (1982); L. Smrčka and P. Středa, J. Phys. C, **10**, 2153 (1977).
 7. Lack of time and space limits the scope of this article to systems without internal boundaries which have a single outer edge. There is, appropriately, considerable interest in annular and equivalent geometries which have inner and outer edges carrying currents in opposite directions. These geometries allow the problem of scattering between edges by disorder, related to transport experiments, to be considered. For work on this problem see C.L. Kane and M.P.A. Fisher, Phys. Rev. Lett. **68**, 1220 (1992); Phys. Rev. B **46**, 15233 (1992); M. Ogata and H. Fukuyama, Phys. Rev. Lett. **47**, 4631 (1993); K.A. Matveev, D. Yue, and L.I. Glazman, Phys. Rev. Lett. **49**, 1966 (1994); K.A. Matveev, D. Yue, and L.I. Glazman, Phys. Rev. B **49**, 1966 (1994).
 8. For negatively charged particles the angular momenta in the lowest Landau level are negative for a magnetic field in the \hat{z} direction. We choose a magnetic field in the $-\hat{z}$ direction to have the convenience of positive angular momenta in the lowest Landau level.
 9. For recent work on 'multi-branch' systems see C.L. Kane and Matthew P.A. Fisher, Phys. Rev. B **51**, 13449 (1995) and work cited therein.
 10. For an elementary discussion of some properties of this *maximum density droplet* state see A.H. MacDonald, S.R. Eric Yang, and M.D. Johnson, *Aust. J. Phys.* **46**, 345 (1993).
 11. C.W.L. Beenakker, Phys. Rev. Lett. **64**, 216 (1990); D.B. Chklovskii, B.I. Shklovskii, and L.I. Glazman, Phys. Rev. B **46**, 4026 (1992); *ibid* **46**, 15606(E) (1992); P.L. McEuen, E.B. Foxman, Jari Kinaret, U. Meirav, M.A. Kastner, Ned S. Wingreen, and S.J. Wind, Phys. Rev. B **45**, 11419 (1992); D.B. Chklovskii, K.A. Matveev, and B.I. Shklovskii, Phys. Rev. B **47**, 12605 (1993); M.M. Fogler, E.I. Levin, and B.I. Shklovskii, Phys. Rev. B **49**, 13767 (1994); Karheinz Lier and Rolf R. Gerhardtts, Phys. Rev. B **50**, 7757 (1994).
 12. See for example M. Ferconi and G. Vignale, Phys. Rev. B **50**, 14722 (1994).
 13. M. Ferconi, M.R. Geller, and G. Vignale, preprint (1995).
 14. For reviews on quantum dots see U. Merkt, *Advances in Solid State Physics*, **30**, 77 (1990); Tapash Chakraborty, *Comments on Condensed Matter Physics* **16**, 35 (1992); M.A. Kastner, *Rev. Mod. Phys.* **64**, 849 (1992); N.F. Johnson, *J. Phys. C* **7**, 965 (1995). Systems containing of order 100 or fewer electrons are generally referred to as quantum dots.
 15. For experimental studies of quantum dots in strong magnetic fields see P.L. McEuen, E.B. Foxman, U. Meirav, M.A. Kastner, Y. Meir, Ned S. Wingreen, and S.J. Wind, Phys. Rev. Lett. **66**, 1926 (1991); ; E.B. Foxman, P. L. McEuen, U. Meirav, Ned S. Wingreen, Yigal Meir, Paul A. Belk, N. R. Belk, and M.A. Kastner, and S.J. Wind, Phys. Rev. B **47**, 10020 (1993); R.C. Ashoori, H.L. Störmer, J.S. Weiner, L.N. Pfeiffer, S.J. Pearton, K.W. Baldwin, and K.W. West, Phys. Rev. Lett. **68**, 3088 (1992); R.C. Ashoori, H.L. Störmer, J.S. Weiner, L.N. Pfeiffer, K.W. Baldwin, and K.W. West, Phys. Rev. Lett. **71**, 613 (1993); O. Klein, C. de Chamon, D. Tang, D.M. Abusch-Magder, X.-G. Wen, and M.A. Kastner, Phys. Rev. Lett. **74**, 785 (1995). T. Schmidt *et al.*, Phys. Rev. B **51**, 5570 (1995).
 16. For theoretical studies of quantum dot systems in strong magnetic fields see J. J. Palacios and P. Hawrylak, Phys. Rev. B **51**, 1769 (1995); J. J. Palacios *et al.*, Phys. Rev. B **50**, 5760 (1994);

- J. H. Oaknin *et al.*, Phys. Rev. Lett. **74**, 5120 (1995); C. de Chamon and X.-G. Wen, Phys. Rev. B **49**, 8227 (1994); J. Dempsey, B.Y. Gelfand, and B.I. Halperin, Phys. Rev. Lett. **70**, 3639 (1993); S.-R. Eric Yang, A.H. MacDonald, and M.D. Johnson, Phys. Rev. Lett. **71**, 3194, (1993).
17. A.H. MacDonald and M.D. Johnson, Phys. Rev. Lett. **70**, 3107.
 18. S.S. Nazin and V.B. Shikin, Zh. Eksp. Fiz. **85**, 530 (1983) [Sov. Phys. JETP **58**, 210 (1983)]; V.B. Shikin, T. Demel' and D. Heitmann, Zh. Eksp. Teor. Fiz. **96**, 1406 (1989) [Sov. Phys. JETP **69**, 797 (1989)].
 19. L. Brey, J.J. Palacios, and C. Tejedor, Phys. Rev. B **47**, 13884 (1993).
 20. U. Wulf, V. Gudmundsson, and R.R. Gerhardtts, Phys. Rev. B **38**, 4218 (1988); A. Manolescu, Phys. Rev. B **45**, 11829 (1992).
 21. The Hilbert space of the lowest Landau level is the Hilbert space of analytic functions: S.M. Girvin and T. Jach, Phys. Rev. B **29**, 5617 (1984).
 22. A.H. MacDonald and D.B. Murray, Phys. Rev. B **32** 2291 (1985).
 23. Michael Stone, Phys. Rev. B **42**, 8399 (1990); Michael Stone, Ann. Phys. (NY) **207**, 38 (1991); Michael Stone H.W. Wyld, and R.L. Schult, Phys. Rev. B **45**, 14156 (1992).
 24. For a more general discussion see A.H. MacDonald, Phys. Rev. Lett. **64**, 220 (1990).
 25. R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
 26. The antisymmetry requirement generally requires $P[z]$ must be an odd function of $z_i - z_j$ for any i and j so that $P[z]$ contains only odd terms in a Taylor series expansion of its dependence on this difference coordinate. The coefficient of the $(z_i - z_j)^1$ term must vanish if the pair never has relative angular momentum equal to one so that the leading term is $(z_i - z_j)^3$. It then follows from analyticity that each difference coordinate to the third power is a factor of $P[z]$.
 27. See for example, George E. Andrews, *The Theory of Partitions* (Addison-Wesley, Reading, 1976). A famous theorem by Hardy and Ramanujan in the theory of partitions gives an exact formula expressing $g(M)$ in terms of a finite sum.
 28. See for example, J. Sólym, Adv. Phys. **28**, 201 (1979); F.D.M. Haldane, J. Phys. C **14**, 2585 (1981); G.D. Mahan, *Many-Particle Physics*, (Plenum, New York, 1990) Chapter 4; J. Voit, preprint (1995). [To appear in Reports on Progress in Physics.] K. Schönhammer and V. Meden, submitted to Am.J. Phys. (1995).
 29. M.D. Johnson and A.H. MacDonald, Phys. Rev. Lett. **67**, 2060 (1991).
 30. J.J. Palacios and A.H. MacDonald, preprint (1995) and work cited therein.
 31. K. Moon, H. Yi, C.L. Kane, S.M. Girvin, M.P.A. Fisher, Phys. Rev. Lett. **71**, 4381 (1993); P. Fendley, A.W.W. Ludwig, and H. Saleur, Phys. Rev. Lett. **74**, 3005 (1995).
 32. F.P. Milliken, C.P. Umbach, and R.A. Webb, Solid State Comm. (in press) (1995).
 33. L.I. Glazman, I.M. Ruzin, and B.I. Shklovskii, Phys. Rev. B **45**, 8454 (1992); H.J. Schulz, Phys. Rev. Lett. **71**, 1864 (1993); Yuval Oreg and Alexander M. Finkel'stein, Phys. Rev. Lett. **74**, 3668 (1995); K. Moon and S.M. Girvin, preprint (1995); U. Zülicke and A.H. MacDonald, in preparation (1995).