# Spin Factor and Spinor Structure of Dirac Propagator in Constant Field

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We use a bosonic path integral representation of Dirac propagator with a spin factor to calculate the propagator in a constant uniform electromagnetic field. Such a way of calculation allows us to get the explicit spinor structure of the propagator in the case under consideration. The representation obtained differs from the Schwinger's one but the equivalence can be checked.

## I. Introduction

Propagators of relativistic particles in an external fields (electromagnetic, non-Abelian or gravitational) contain important information about quantum behavior of these particles. Moreover, if such propagators are known in an arbitrary external field, one can find exact one-particle Green's functions in the corresponding quantum field theory, taking functional integrals over all external fields. Dirac propagator in an external electromagnetic field distinguishes from the one of a scalar particle by a complicated spinor structure. The problem of its path integral representation attracted attention of researchers already for a long time. Feynman, who had written first a path integral for the probability amplitude in nonrelativistic quantum mechanics[1] and then a path integral for the scalar particle propagator[2], had also attempted to derive a representation for Dirac propagator via a bosonic path integral[3]. After Berezin had introduced the integral over grassmannian variables, it turned out to be natural to present Dirac propagator by a path integral over ordinary and grassmannian variables, the latter describe spinning degrees of freedom. Such representations have been discussed in the literature for a long time in different contexts[4-13]. Another attractive problem was to write Dirac propagator via a bosonic path integral only. So, Polyakov[14] assumed that the propagator of free Dirac electron in the three-dimensional Euclidean space-time can be presented by means of a bosonic path integral similar to a scalar particle case modified by the so called spin factor. This idea has been developed in[15] to write the spin factor for Dirac fermions, interacting with a non-Abelian gauge field in $D$-dimensional Euclidean space-time. In those representations the spin factor itself was presented via some additional bosonic path integrals. Surprisingly, it turned out that an explicit form of the spin factor in an arbitrary external field could be found directly[16]. A representation of Dirac propagator in an arbitrary external field via a bosonic and fermionic path integrals[12] was taken and all grassmannian integrations there were done, so that an expression for the spin factor was derived as a given functional of the bosonic trajectory. Having such an expression for the spin factor one can use it to calculate the propagator in some particular cases of external fields. This way of calculation provides automatically the explicit spinor structure of the propagators. In the present paper we are going to use this way of calculations to get the propagator in a constant uniform electromagnetic field. It turns out to be non trivial to compare the representation obtained with the one derived by Schwinger[17].

## II. The spin factor in constant uniform field

The propagator of a spinning particle in an external electromagnetic field $A_\mu(x)$ is the causal Green's function $S^c(x_{out},\, x_{in})$ of Dirac equation in this field,

$$[\gamma^\mu\,(i\partial_\mu - gA_\mu(x)) - m]\,S^c(x,y) = -\delta^4(x-y) \quad , \quad (1)$$

where $x = (x^\mu)$, $[\gamma^\mu\,,\,\gamma^\nu]_+ = 2\eta^{\mu\nu}$, $\eta^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$, $\mu,\nu=\overline{0,3}$.
The propagator can be presented[12] by means of a path integral over ordinary and grassmannian variables. The integration over all the grassmannian variables can be done[16] so that a representation in terms of a bosonic path integral holds:

$$S^c(x_{out}\,,x_{in}) = \frac{\mathrm{i}}{2}\int_0^\infty de_0\int_{x_{in}}^{x_{out}} Dx\, M(e_0)\,\Phi[x,\,e_0]\exp\left\{\mathrm{i}I[x,\,e_0]\right\}, \qquad (2)$$

where the measure $M(e_0)$ has the form

$$M(e_0) = \int Dp\,\exp\left(\frac{\mathrm{i}e_0}{2}\int_0^1 p^2 d\tau\right)\ ; \qquad (3)$$

$I[x,\,e_0]$ is the action of a spinless particle

$$I[x,\,e_0] = -\int_0^1\left[\frac{\dot{x}^2}{2e_0}+\frac{e_0}{2}m^2+g\dot{x}A(x)\right]d\tau\,, \qquad \dot{x}^\mu = \frac{d}{d\tau}x^\mu\,; \qquad (4)$$

and $\Phi[x,\,e_0]$ is the spin factor,

$$\begin{aligned}\Phi[x,\,e_0] = \big[&m + (2e_0)^{-1}\dot{x}^\mu\star K_{\mu\lambda}\left(2\eta^{\lambda\kappa}-ge_0B^{\lambda\kappa}\right)\gamma_\kappa\\ &-\frac{\mathrm{i}g}{4}\left(me_0+\dot{x}^\mu\star K_{\mu\lambda}\gamma^\lambda\right)B_{\kappa\nu}\sigma^{\kappa\nu}+m\frac{g^2e_0^2}{16}B^*_{\alpha\beta}B^{\alpha\beta}\gamma^5\Big]\exp\left\{-\frac{e_0}{2}\int_0^g dg'\mathrm{Tr}\,Q(g')\star\mathcal{F}\right\}\ . \qquad (5)\end{aligned}$$

The following notations are used:

$$\begin{aligned}&\sigma^{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma^\mu\,,\,\gamma^\nu]\,, \qquad \gamma^5=\gamma^0\gamma^1\gamma^2\gamma^3\,,\\ &B_{\mu\nu} = F_{\mu\lambda}\star K^\lambda{}_\nu\,, \qquad B^{*\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}B_{\alpha\beta}\,,\\ &K_{\mu\nu} = \eta_{\mu\nu}+ge_0Q_{\mu\lambda}(g)\star F^\lambda{}_\nu\,,\\ &Q_{\mu\nu}(g) = \frac{1}{2}\varepsilon\star\Lambda^{-1}_{\mu\nu}(g)\star\varepsilon\,, \qquad \Lambda^{-1}_{\mu\lambda}(g)\star\Lambda^{\lambda\nu}(g) = \delta^\nu_\mu\delta(\tau-\tau')\,,\\ &\Lambda_{\mu\nu}(g) = \eta_{\mu\nu}\varepsilon - \frac{ge_0}{2}\varepsilon\star\mathcal{F}_{\mu\nu}\star\varepsilon\,, \qquad (6)\end{aligned}$$

where $\epsilon^{\mu\nu\alpha\beta}$ is Levi-Civita symbol normalized by $\epsilon^{0123}=1$; $\varepsilon$, $\mathcal{F}_{\mu\nu}$, $\Lambda_{\mu\nu}(g)$ and $Q_{\mu\nu}(g)$ are understood as matrices with continuous indices $\tau$, $\tau'$, and integration over $\tau$ is denoted by $\star$, e.g.,

$$\begin{aligned}&\mathcal{F}_{\mu\nu}(\tau,\,\tau') = F_{\mu\nu}\left(x(\tau)\right)\delta(\tau-\tau')\,, \qquad \varepsilon(\tau,\,\tau')=\varepsilon(\tau-\tau')\,,\\ &\varepsilon\star\mathcal{F}_{\mu\nu}\star\varepsilon = \int_0^1 d\tau_1\int_0^1 d\tau_2\,\varepsilon(\tau,\,\tau_1)\mathcal{F}_{\mu\nu}(\tau_1,\,\tau_2)\varepsilon(\tau_2,\,\tau')\,.\end{aligned}$$

Sometimes the Lorentz indices will be also omitted. In this case all the tensors of second rank will be understood as matrices where first indices are contravariant ones of the tensors, indicating the lines, and second indices are covariant ones of the tensors, indicating the columns. We denote by $\mathbf{I}$ the unit $4\times 4$ matrix.

In the case of a constant uniform field $F_{\mu\nu}=const$, which we are going to discuss, $Q$, $K$ and $B$ do not depend on the trajectory $x$, and can be calculated straightforwardly,

$$Q(g)=\frac{1}{2}\left(\mathbf{I}\varepsilon(\tau-\tau')-\tanh\frac{ge_0F}{2}\right)\exp\{e_0gF(\tau-\tau')\}\,,$$
$$K=\left(\mathbf{I}-\tanh\frac{ge_0F}{2}\right)\exp\left(ge_0F\tau\right)\,,\qquad B=\frac{2}{ge_0}\tanh\frac{ge_0F}{2}\,. \tag{7}$$

Using them in (5) and integrating over $\tau$ whenever possible we obtain the spin factor in a constant uniform field,

$$\Phi[x,\,e_0]=\left(\det\cosh\frac{ge_0F}{2}\right)^{1/2}\left\{m\left[1-\frac{\mathrm{i}}{2}\left(\tanh\frac{ge_0F}{2}\right)_{\mu\nu}\sigma^{\mu\nu}\right.\right.$$
$$\left.+\frac{1}{4}\left(\tanh\frac{ge_0F}{2}\right)^{*}_{\mu\nu}\left(\tanh\frac{ge_0F}{2}\right)^{\mu\nu}\gamma^5\right]+\frac{1}{e_0}\left(\int_0^1\dot{x}\exp(ge_0F\tau)d\tau\right)$$
$$\left.\times\left(\mathbf{I}-\tanh\frac{ge_0F}{2}\right)\left[\left(\mathbf{I}-\tanh\frac{ge_0F}{2}\right)\gamma-\frac{\mathrm{i}}{2}\gamma\left(\tanh\frac{ge_0F}{2}\right)_{\mu\nu}\sigma^{\mu\nu}\right]\right\}\,. \tag{8}$$

We can see that in the field under consideration the spin factor is linear in the trajectory $x^\mu(\tau)$. That facilitates the bosonic integration in the expression (2).

## III. The propagator in constant uniform field

In spite of the fact that the spin factor is a gauge invariant object the total propagator is not. It is clear from the expression (2) where one needs to choose a particular gauge for the potentials $A_\mu$. Namely, we are going to use the following potentials

$$A_\mu=-\frac{1}{2}F_{\mu\nu}x^\nu\,, \tag{9}$$

to describe the constant uniform field $F_{\mu\nu}=const$. Thus, one can see that the path integral (2) is quasi-gaussian in the case under consideration. Let us make there the shift $x\rightarrow y+x_{cl}$, with $x_{cl}$ the solution of the classical equations of motion

$$\frac{\delta I}{\delta x}=0\Leftrightarrow\ddot{x}_\mu-ge_0F_{\mu\nu}\dot{x}^\nu=0 \tag{10}$$

subjected to the boundary conditions $x_{cl}(0)=x_{in}\,,\quad x_{cl}(1)=x_{out}$. Then the new trajectories of integration $y$ obey zero boundary conditions, $y(0)=y(1)=0$. Due to the quadratic structure of the action $I[x,e_0]$ and linearity of the spin factor $\Phi[x,e_0]$ one can make the following substitutions in the path integral:

$$I[y+x_{cl},\,e_0]\rightarrow I[y,\,e_0]+I[x_{cl},\,e_0]\,,\quad\Phi[y+x_{cl},\,e_0]\rightarrow\Phi[x_{cl},\,e_0]=\Psi(x_{out},\,x_{in},\,e_0). \tag{11}$$

Doing also a convenient replacement of variables $p\rightarrow\frac{p}{\sqrt{e_0}}$, $y\rightarrow y\sqrt{e_0}$, we get

$$S^c=\frac{\mathrm{i}}{2}\int_0^\infty\frac{de_0}{e_0^2}\,\Psi(x_{out},\,x_{in},\,e_0)\,e^{\mathrm{i}I[x_{cl},\,e_0]}$$
$$\times\int_0^0Dy\int Dp\exp\left\{\frac{\mathrm{i}}{2}\int_0^1\left(p^2-\dot{y}^2-ge_0yF\dot{y}\right)d\tau\right\}\,. \tag{12}$$

One can see that the path integral in (12) is, in fact, the kernel of the Klein-Gordon propagator in the proper-time representation. This path integral can be presented as

$$\int_0^0Dy\int Dp\exp\left\{\frac{\mathrm{i}}{2}\int_0^1\left(p^2-\dot{y}^2-ge_0yF\dot{y}\right)d\tau\right\}=$$

$$\left[\frac{\text{Det}\,(\eta_{\mu\nu}\partial_\tau^2 - ge_0F_{\mu\nu}\partial_\tau)}{\text{Det}\,(\eta_{\mu\nu}\partial_\tau^2)}\right]^{-1/2}\int_0^0 Dy\int Dp\exp\left\{\frac{\mathrm{i}}{2}\int_0^1\left(p^2-\dot{y}^2\right)d\tau\right\}.$$

Cancelling the factor $\text{Det}\,(-\eta_{\mu\nu})$ in the ratio of the determinants, one obtains

$$\frac{\text{Det}\,(\eta_{\mu\nu}\partial_\tau^2 - ge_0F_{\mu\nu}\partial_\tau)}{\text{Det}\,(\eta_{\mu\nu}\partial_\tau^2)} = \frac{\text{Det}\,(-\delta^\mu_\nu\partial_\tau^2 + ge_0F^\mu{}_\nu\partial_\tau)}{\text{Det}\,(-\delta^\mu_\nu\partial_\tau^2)}. \tag{13}$$

One can also do the replacement

$$-\mathbf{I}\partial_\tau^2 + ge_0F\partial_\tau \to -\mathbf{I}\partial_\tau^2 + \frac{g^2e_0^2}{4}F^2, \tag{14}$$

in the r. h. s. of (13) because the spectra of both operators coincide. Indeed,

$$-\mathbf{I}\partial_\tau^2 + ge_0F\partial_\tau = \exp\left(\frac{ge_0}{2}F\tau\right)\left(-\mathbf{I}\partial_\tau^2+\frac{g^2e_0^2}{4}F^2\right)\exp\left(-\frac{ge_0}{2}F\tau\right), \tag{15}$$

and the zero boundary conditions are invariant under the transformation $y \to \exp(\frac{ge_0F\tau}{2})y$. Then, by using (14) and the value of the free path integral [?],

$$\frac{\mathrm{i}}{2}\int_0^0 Dy\int Dp\exp\{\frac{\mathrm{i}}{2}\int d\tau(p^2-\dot{y}^2)\} = \frac{1}{8\pi^2}, \tag{16}$$

related, in fact, to the definition of the measure, we obtain

$$S^c = \frac{1}{8\pi^2}\int_0^\infty\frac{de_0}{e_0^2}\Psi(x_{out},\,x_{in},\,e_0)\,e^{\mathrm{i}I[x_{cl},\,e_0]}\left[\frac{\text{Det}\,(-\mathbf{I}\partial_\tau^2+\frac{g^2e_0^2}{4}F^2)}{\text{Det}\,(-\mathbf{I}\partial_\tau^2)}\right]^{-1/2}. \tag{17}$$

The ratio of the determinants can be now written as

$$\begin{aligned}\frac{\text{Det}\,(-\mathbf{I}\partial_\tau^2+\frac{g^2e_0^2}{4}F^2)}{\text{Det}\,(-\mathbf{I}\partial_\tau^2)} &= \exp\text{Tr}\left[\ln\left(-\mathbf{I}\partial_\tau^2+\frac{g^2e_0^2}{4}F^2\right)-\ln\left(-\mathbf{I}\partial_\tau^2\right)\right]\\ &= \exp\text{Tr}\left[\frac{e_0^2}{2}F^2\int_0^g d\lambda\,\lambda\left(-\mathbf{I}\partial_\tau^2+\frac{\lambda^2e_0^2}{4}F^2\right)^{-1}\right]\\ &= \exp\text{tr}\left[\frac{e_0^2}{2}F^2\int_0^g d\lambda\,\lambda\sum_{n=1}^\infty\left(\pi^2n^2\mathbf{I}+\frac{\lambda^2e_0^2}{4}F^2\right)^{-1}\right].\end{aligned} \tag{18}$$

The trace in the infinite-dimensional space in eq.(18) is taken and the one in the 4-dimensional space remains. Using the formula

$$\sum_{n=1}^\infty\left(\pi^2n^2+\kappa^2\right)^{-1} = \frac{1}{2\kappa}\coth\kappa-\frac{1}{2\kappa^2},$$

which is also valid if $\kappa$ is an arbitrary $4\times4$ matrix, and integrating in (18), we find

$$\frac{\text{Det}\,(-\mathbf{I}\partial_\tau^2+\frac{g^2e_0^2}{4}F^2)}{\text{Det}\,(-\mathbf{I}\partial_\tau^2)} = \det\left(\frac{\sinh\frac{ge_0F}{2}}{\frac{ge_0F}{2}}\right). \tag{19}$$

Thus,

$$S^c = \frac{1}{32\pi^2}\int_0^\infty de_0\left[\det\frac{\sinh\frac{e_0gF}{2}}{gF}\right]^{-1/2}\Psi(x_{out},\,x_{in},\,e_0)e^{\mathrm{i}I[x_{cl},e_0]}, \tag{20}$$

where the function $\Psi(x_{out},\,x_{in},\,e_0)$ is the spin factor on the classical trajectory $x_{cl}$. The latter can be easily found by solving the eq.(10):

$$x_{cl} = \left(\exp(ge_0F)-\mathbf{I}\right)^{-1}\left[\exp(ge_0F\tau)(x_{out}-x_{in})+\exp(ge_0F)x_{in}-x_{out}\right]. \tag{21}$$

Substituting (21) into eqs.(4) and (11), we obtain

$$S^c = \frac{1}{32\pi^2}\int_0^\infty de_0\left[\det\frac{\sinh\frac{ge_0F}{2}}{gF}\right]^{-1/2}\Psi(x_{out},\, x_{in},\, e_0)$$
$$\times\exp\left\{\frac{\mathrm{i}g}{2}x_{out}Fx_{in}-\frac{\mathrm{i}}{2}e_0m^2-\frac{\mathrm{i}g}{4}(x_{out}-x_{in})F\coth(\frac{ge_0F}{2})(x_{out}-x_{in})\right\}, \tag{22}$$

where

$$\Psi(x_{out},x_{in},e_0) = \left[m+\frac{g}{2}(x_{out}-x_{in})F(\coth\frac{ge_0F}{2}-1)\gamma\right]$$
$$\times\ \sqrt{\det\cosh\frac{ge_0F}{2}}\left[1-\frac{i}{2}\left(\tanh\frac{ge_0F}{2}\right)_{\mu\nu}\sigma^{\mu\nu}\right.$$
$$+\ \left.\frac{1}{8}\epsilon^{\alpha\beta\mu\nu}\left(\tanh\frac{ge_0F}{2}\right)_{\alpha\beta}\left(\tanh\frac{ge_0F}{2}\right)_{\mu\nu}\gamma^5\right]. \tag{23}$$

## IV. Comparison with the Schwinger Formula

We are going to compare the representation (22) obtained with the Schwinger formula [?], which he had derived in the same case, using his proper time method. The Schwinger representation for the spinor propagator in a constant uniform field has the form

$$S^c(x_{out},\, x_{in}) = \frac{1}{32\pi^2}\left[\gamma^\mu\left(\mathrm{i}\frac{\partial}{\partial x^\mu_{out}}-gA_\mu(x_{out})\right)+m\right]\int_0^\infty de_0\left[\det\frac{\sinh\frac{ge_0F}{2}}{gF}\right]^{-1/2}$$
$$\times\exp\left\{\frac{\mathrm{i}}{2}\left[gx_{out}Fx_{in}-e_0m^2-(x_{out}-x_{in})\frac{gF}{2}\coth\frac{ge_0F}{2}(x_{out}-x_{in})-\frac{ge_0}{2}F_{\mu\nu}\sigma^{\mu\nu}\right]\right\}. \tag{24}$$

Doing the differentiation with respect to $x^\mu_{out}$, we transform the formula (24) to a form which is convenient to be compared with our representation (22),

$$S^c = \frac{1}{32\pi^2}\int_0^\infty de_0\left[\det\frac{\sinh\frac{ge_0F}{2}}{gF}\right]^{-1/2}\Psi_S(x_{out},x_{in},e_0)\times$$
$$\times\exp\left\{\mathrm{i}\frac{g}{2}x_{out}Fx_{in}-\frac{\mathrm{i}}{2}e_0m^2-\mathrm{i}\frac{g}{4}(x_{out}-x_{in})F\coth\left(\frac{ge_0F}{2}\right)(x_{out}-x_{in})\right\}, \tag{25}$$

where the function $\Psi_S$ is given by

$$\Psi_S(x_{out},x_{in},e_0) = \left[m+\frac{g}{2}(x_{out}-x_{in})F(\coth\frac{ge_0F}{2}-1)\gamma\right]\exp(-\mathrm{i}\frac{e_0g}{4}F_{\mu\nu}\sigma^{\mu\nu}). \tag{26}$$

Thus one needs only to compare the functions $\Psi$ and $\Psi_S$. They coincide since the following formula takes place, where $\omega_{\mu\nu}$ is an arbitrary antisymmetric tensor,

$$\exp\left(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right) = \sqrt{\det\cosh\frac{\omega}{2}}\left[1-\frac{i}{2}\left(\tanh\frac{\omega}{2}\right)_{\mu\nu}\sigma^{\mu\nu}\right.$$
$$+\ \left.\frac{1}{8}\epsilon^{\alpha\beta\mu\nu}\left(\tanh\frac{\omega}{2}\right)_{\alpha\beta}\left(\tanh\frac{\omega}{2}\right)_{\mu\nu}\gamma^5\right]. \tag{27}$$

In fact, the latter formula presents a linear decomposition of a finite Lorentz transformation in the correspondent $\gamma$-matrix structures. We have not found this kind of relation in the literature, and therefore have to comment it. One ought to say that a direct combinatoric proof seems to be rather cumbersome to be presented here. Nevertheless, the validity of (27) can be checked at least in few orders in $\omega$, see Appendix. On the other hand one can consider the general path integral representations derived in[12,16] together with the calculations given in the present paper as a functional integral proof of the formula (27) in general.

## V. Conclusion

Thus, we have got an exact solution for the Dirac propagator in a constant uniform electromagnetic field by means of path integration, using a general representation for the propagator via a bosonic path integral with the spin factor. In fact, there are only few cases where these kind of exact solutions can be found: the constant uniform electromagnetic field considered, electromagnetic plane wave [?, ?], crossed electric and magnetic fields [?, ?], and combination of a constant uniform electromagnetic field with a plane wave field [?, ?, ?]. One can believe that all these cases can be treated similar to the constant uniform field case in terms of the representation used. We find real advanteges in using of the bosonic path integral with the spin factor. First of all, a part of job in such representation is already done, all the grassmannian integrations are fulfiled. Second, the $\gamma$-matrix structure of the final answer appears right away in an explicit form. Besides, such a representation sometimes allows one to get the Dirac propagator without any additional path integrations if the proper-time representation of the corresponding Klein-Gordon one is known.

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## Appendix

Here we are going to prove the validity of the formula (27) in three first orders with respect to $\omega$. Let us denote the left and right sides of (27) by $I_1$ and $I_2$ correspondingly,

$$I_1 = \exp\left(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right), \tag{28}$$

$$\begin{aligned} I_2 &= \left\{\det\ \cosh\frac{\omega}{2}\right\}^{\frac{1}{2}}\left[1-\frac{i}{2}\left(\tanh\frac{\omega}{2}\right)_{\mu\nu}\sigma^{\mu\nu}\right. \\ &\left.+\frac{1}{8}\epsilon^{\alpha\beta\mu\nu}\left(\tanh\frac{\omega}{2}\right)_{\alpha\beta}\left(\tanh\frac{\omega}{2}\right)_{\mu\nu}\gamma^5\right] \end{aligned} \tag{29}$$

Taking into account the identity

$$\left\{\sigma^{\alpha\beta},\sigma^{\mu\nu}\right\} = -2\epsilon^{\alpha\beta\mu\nu}\gamma^5 + 2(\eta^{\alpha\mu}\eta^{\beta\nu}-\eta^{\alpha\nu}\eta^{\beta\mu}), \tag{30}$$

we find

$$(\omega\sigma)^2 = 8\mathcal{F}+\mathcal{G}\gamma^5, \tag{31}$$

where $\mathcal{F}$ and $\mathcal{G}$ are the invariants

$$\mathcal{F} = \frac{1}{4}\omega_{\mu\nu}\omega^{\mu\nu} = -\frac{1}{4}\mathrm{tr}\ \omega^2,\ \ \mathcal{G} = -\frac{1}{4}\omega_{\mu\nu}\omega^{*\mu\nu} = \frac{1}{4}\mathrm{tr}\ \omega\omega^*. \tag{32}$$

Then

$$I_1 = 1-\mathcal{F}4+i\left(-\frac{1}{4}+\frac{\mathcal{F}}{\triangle\forall}\right)\omega\sigma+\frac{i}{48}\mathcal{G}\omega^*\sigma-\frac{\mathcal{G}\gamma^5}{4}+o(g^3), \tag{33}$$

and

$$I_2 = 1-\frac{\mathcal{F}}{\triangle}+i\left(-\frac{1}{4}+\frac{\mathcal{F}}{\infty}\right)\omega\sigma+\frac{i}{48}\omega^3\sigma-\frac{\mathcal{G}\gamma^5}{4}+o(g^3). \tag{34}$$

Using the well-known identity

$$\epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4}\epsilon^{\beta_1\beta_2\beta_3\beta_4} = -\sum_P(-1)^{[P]}\delta_{\alpha_1}^{\beta_{P(1)}}\delta_{\alpha_2}^{\beta_{P(2)}}\delta_{\alpha_1}^{\beta_{P(3)}}\delta_{\alpha_4}^{\beta_{P(4)}}, \tag{35}$$

where $P$ denotes a permutation and $[P]$ is the parity of this permutation, one can check that

$$2\mathcal{F}(\omega\sigma) + (\omega^3\sigma) - \mathcal{G}(\omega^*\sigma). \tag{36}$$

Thus, we see that $I_1$ and $I_2$ coincide up to the terms of the third order in $\omega$.

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