First-Order Conservation Laws in a Chiral Medium II: A Covariant Approach

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The first-order conservation laws associated with the symmetric zilch Z and its antisymmetric companion Y previously derived in a chiral medium at rest are extended to an arbitrary frame.

I. Introduction

Lipkin[1] discovered nine new conservation laws for the electromagnetic field in vacuum, calling "zilch" the corresponding conserved quantities. He showed the existence of a third-rank tensor density $Z_{\mu\nu\alpha}$ (see Eqs.(3.1) and (3.3)) which is a $\mu\nu$-symmetric combination of bilinear functions of the electromagnetic field $F_{\alpha\beta}$ and containing a first-order derivative of it, that is of the form $F\partial F$, which is conserved in the sense that $\partial^\alpha Z_{\mu\nu\alpha} = 0$. Therefore, the space integral $Z_{\mu\nu} = \int Z_{\mu\nu o} d^3x$ is a constant of motion, which he called the "zilch" of the electromagnetic field. This is a symmetric tensor with nine independent components since it is a traceless tensor $Z_\mu{}^\mu = 0$. Later[2] we have added one more set of six conservation laws of the type discovered by Lipkin represented by an antisymmetric combination $Y_{\mu\nu\alpha}$ (see Eq.(3.2)) of the field quantities and its derivatives. The corresponding constants of motion $Y_{\mu\nu} = \int Y_{\mu\nu o} d^3x = -Y_{\nu\mu}$ has been called the companions of zilch. As all these laws involve a first-order derivative of the field quantities they have been referred to as first-order conservation laws. Next[3], Bailyn and the author have extended these vacuum conservation laws to the case of a field in a normal medium. In a previous paper[4] (here after called I) we have extended the study to chiral media, which are commonly known as optically active media in the optical regime. The term chirality refers to the lack of congruence between an object and its mirror image, either by rotation and or translation as manifested, for instance, in the hands of a human being. A chiral medium is a medium caracterized by a left-handedness or right-handedness in its microstructure. They can, for instance, rotate polarized light to the left or to the right depending on its handedness. The chiral first-order conservation laws were derived for a chiral medium at rest. Here we wish to extend these rest frame laws to an arbitrary frame, generalizing in this way the covariant approach already studied for a normal medium[3].

The starting point for theoretical work on the interaction of the electromagnetic field with a chiral medium is the formulation of proper constitutive relations for the medium. This is discussed in Sec.II together with the presentation of the electromagnetic field equations in the moving frame. In Sec.III we discuss the conservation laws.

II. The field equations

We shall begin by writing the electromagnetic field equations in the arbitrary frame starting from those in the medium rest frame. We shall be concerned with a chiral medium characterized by the Drude- Born- Fedorov constitutive relations[5]

$$\begin{aligned} \mathbf{D}^{(o)} &= \varepsilon \mathbf{E}^{(o)} + \varepsilon\beta \nabla \times \mathbf{E}^{(o)}, \\ \mathbf{B}^{(o)} &= \mu \mathbf{H}^{(o)} + \mu\beta \nabla \times \mathbf{H}^{(o)} \quad , \end{aligned} \tag{2.1}$$

in the rest frame S^o of the medium. As in I we restrict ourselves to homogeneous and nondispersive media, where ε, μ and β are constants. The pseudoscalar β measures the degree of chirality. As in I, Maxwell's equations in this frame in the absence of charges ($\nabla.\mathbf{D}^{(o)} = 0, \nabla.\mathbf{B}^{(o)} = 0$, $\nabla \times \mathbf{H}^{(o)} = \mathbf{D}^{(o)}_{,o}$ and $\nabla \times \mathbf{E}^{(o)} = -\mathbf{B}^{(o)}_{,o}$, where $a_{,o} = \partial a/\partial t$) can be rewritten in terms of the fields $\mathbf{E}^{(o)}$ and $\mathbf{B}^{(o)}$ as

$$\nabla.\mathbf{E}^{(o)} = 0 \quad , \quad \nabla \times \mathbf{B}^{(o)} = \mu \mathbf{J}^{(o)} + \varepsilon\mu \mathbf{E}^{(o)}_{,o} \tag{2.2}$$

$$\nabla.\mathbf{B}^{(o)} = 0 \quad , \quad \nabla \times \mathbf{E}^{(o)} = -\mathbf{B}^{(o)}_{,o} \quad , \tag{2.3}$$

where

$$\mathbf{J}^{(o)} = \varepsilon\beta \nabla \times \mathbf{K}^{(o)} \tag{2.4}$$

with

$$\begin{aligned} \mathbf{K}^{(o)} &= 2\mathbf{E}^{(o)}_{,o} + \beta \nabla \times \mathbf{E}^{(o)}_{,o} \\ &= 2\mathbf{E}^{(o)}_{,o} - \beta \mathbf{B}^{(o)}_{,oo} \quad . \end{aligned} \tag{2.5}$$

We also have

$$\mathbf{J}^{(o)} = \varepsilon\beta(-2\mathbf{B}^{(o)}_{,oo} - \beta\nabla^2 \mathbf{E}^{(o)}_{,o}) \quad . \tag{2.6}$$

Equations (2.2) and (2.3) are formally similar to Maxwell's equations in a normal medium with current $\mathbf{J}^{(o)}$ and charge density $\rho^{(o)} = 0$, obeying from (2.4) the equation of continuity $\nabla.\mathbf{J}^{(o)} + \rho^{(o)}_{,o} = 0$. We shall call $\mathbf{J}^{(o)}$ the rest frame chiral current. In an arbitrary frame S Eqs. (2.2) and (2.3) can be written ($\partial a/\partial x^\alpha = a_{,\alpha}$).

$$F^{\alpha\lambda}{}_{,\alpha} - (\varepsilon\mu - 1)F^{\lambda\alpha}{}_{,\gamma} u_\alpha u^\gamma = \mu J^\lambda \quad , \tag{2.7}$$

and

$$F_{\alpha\beta,\gamma} + F_{\gamma\alpha,\beta} + F_{\beta\gamma,\alpha} = 0 \quad , \tag{2.8}$$

respectively. Here u_α is the uniform velocity of the medium, $F_{\alpha\beta}$ is the electromagnetic field tensor with components $F_{oi} = E_i$ and $F_{ij} = -\varepsilon_{ijk}B_k$. We also introduce the dual of the field tensor

$$G^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} \quad , \tag{2.9}$$

with components $G_{oi} = B_i$ and $G_{ij} = +\varepsilon_{ijk}E_k$, with which the chiral current can be written

$$J^\lambda = \varepsilon\beta\varepsilon^{\alpha\lambda\rho\sigma} u_\alpha K_{\rho,\sigma} \quad , \tag{2.10}$$

with $\varepsilon^{0123} = +1$ and

$$K^\rho = 2\dot{F}^\rho - \beta \ddot{G}^\rho \quad , \tag{2.11}$$

where we used the indication

$$F^\rho = F^{\rho\alpha} u_\alpha \quad ; \quad G^\rho = G^{\rho\alpha} u_\alpha \tag{2.12}$$

and the dot operation is defined by

$$\dot{a} = u^\alpha a_{,\alpha} \quad , \tag{2.13}$$

which reduces to the ordinary time derivative in the medium rest frame. In this frame F^i reduces to the components of the electric field (and F^o vanishes) and G^i to those of the magnetic field (and G^o vanishes). Therefore, J^o and K^o will vanish and the space parts of Eqs. (2.10) and (2.11) will reduce to (2.4) and (2.5), respectively. Equations (2.7) and (2.8) reduce to (2.2) and (2.3), respectively. We shall need also the current in the form

$$J^\alpha = \varepsilon\beta(-2\ddot{G}^\alpha + \beta h^{\mu\nu} \dot{F}^\alpha{}_{,\mu\nu}) \tag{2.14}$$

where $h^{\mu\nu}$ is related to the metric tensor $g^{\mu\nu}$ by

$$h^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu . \tag{2.15}$$

In the medium rest frame the space part of (2.14) reduces to (2.6) and, of course, J^o vanishes. Notice that from (2.10) the equation of continuity is obeyed

$$J^\lambda{}_{,\lambda} = 0 \quad . \tag{2.16}$$

From Eq. (2.7) we have, upon contraction with u_λ and noting that $u_\lambda J^\lambda = 0$ from (2.10),

$$F^\alpha{}_{,\alpha} = F^{\alpha\lambda}{}_{,\alpha} u_\lambda = 0 \quad . \tag{2.17}$$

This is nothing but the first equation in (2.2) in the medium rest frame.

We shall need the field equations expressed in terms of the dual field. The inverse of (2.9) is

$$F^{\alpha\lambda} = -\frac{1}{2}\varepsilon^{\alpha\lambda\beta\delta} G_{\beta\delta} \quad . \tag{2.18}$$

If we substitute this in the first term of Eq. (2.7) and contract with $\varepsilon_{\rho\lambda\sigma\gamma}$ we obtain

$$G_{\sigma\gamma,\rho} + G_{\rho\sigma,\gamma} + G_{\gamma\rho,\sigma} - (\varepsilon\mu - 1)\varepsilon_{\rho\lambda\sigma\gamma} \dot{F}^\lambda = \mu\varepsilon_{\rho\lambda\sigma\gamma} J^\lambda \quad , \tag{2.19}$$

where we have used the identity

$$\varepsilon^{\lambda\alpha\beta\mu}\varepsilon_{\lambda\rho\sigma\gamma} = -\delta^\alpha_\rho \delta^\beta_{[\sigma}\delta^\mu_{\gamma]} + \delta^\alpha_\sigma \delta^\beta_{[\rho}\delta^\mu_{\gamma]} - \delta^\alpha_\gamma \delta^\beta_{[\rho}\delta^\mu_{\sigma]} \quad , \tag{2.20}$$

with $[\sigma\gamma] = \sigma\gamma - \gamma\sigma$. We could write (2.19) entirely in terms of G (since $-\varepsilon_{\rho\lambda\sigma\gamma}F^{\lambda\alpha}u_\alpha = G_{\sigma\gamma}u_\rho + G_{\rho\sigma}u_\gamma + G_{\gamma\rho}u_\sigma$) but we will find convenient not to do so. The homogeneous equation (2.8) becomes

$$G^{\gamma\sigma}{}_{,\gamma} = 0 \quad , \tag{2.21}$$

which immediatly follows from the contraction of (2.8) with $\varepsilon^{\gamma\sigma\alpha\beta}$. Finally, we shall need the wave equation for the field tensor. If we differentiate Eq. (2.8) with respect to γ and use (2.7) we obtain

$$F_{\alpha\beta,\gamma}{}^{\gamma} + (\varepsilon\mu - 1)\ddot{F}_{\alpha\beta} = -\mu(J_{\alpha,\beta} - J_{\beta,\alpha}). \quad (2.22)$$

Here use has been made of the relation

$$\dot{F}_{\alpha\beta} = u^{\gamma} F_{\alpha\beta,\gamma} = F_{\alpha,\beta} - F_{\beta,\alpha} \quad , \quad (2.23)$$

which follows from (2.8) upon contraction with u^{γ}. In the medium rest frame Eq. (2.22) reduces to Eqs. (I-8) and (I-9).

III. The conservation laws

The symmetric zilch pseudotensor Z [1] and its antisymmetric companion Y [2] can be written as

$$Z^{\lambda\nu\rho} = X^{\lambda\nu\rho} + X^{\nu\lambda\rho} \quad , \quad (3.1)$$

$$Y^{\lambda\nu\rho} = X^{\lambda\nu\rho} - X^{\nu\lambda\rho} \quad , \quad (3.2)$$

where X is given by

$$X^{\lambda\nu}{}_{\rho} = G^{\lambda\alpha} F_{\alpha}{}^{\nu}{}_{,\rho} - \frac{1}{4} g^{\lambda\nu} G^{\beta\alpha} F_{\alpha\beta,\rho}. \quad (3.3)$$

Notice that this pseudotensor is traceless in the first two indices, $X^{\lambda}{}_{\lambda\rho} = 0$. Z is symmetric and traceless in the first two indices and Y is antisymmetric. If we differentiate Eq. (3.3) with respect to ρ we obtain

$$X^{\lambda\nu\rho}{}_{,\rho} = G^{\lambda\alpha} F_{\alpha}{}^{\nu}{}_{,\rho}{}^{\rho} - \frac{1}{4} g^{\lambda\nu} G^{\beta\alpha} F_{\alpha\beta,\rho}{}^{\rho}, \quad (3.4)$$

where we have used the identity[3]

$$G^{\lambda\alpha}{}_{,\rho} F_{\alpha}{}^{\nu}{}_{,}{}^{\rho} = \frac{1}{4} g^{\lambda\nu} G^{\beta\alpha}{}_{,\rho} F_{\alpha\beta,}{}^{\rho}, \quad (3.5)$$

that can be verified directly. Using the wave equation (2.22) in (3.4) and making a dot-differentiation by parts in the term proportional to $(\varepsilon\mu - 1)$ we obtain

$$X^{\lambda\nu\rho}{}_{,\rho} + (\varepsilon\mu - 1)\dot{X}^{\lambda\nu\sigma} u_{\sigma} = \mu \left[G^{\lambda\alpha} (J^{\nu}{}_{,\alpha} - J_{\alpha,}{}^{\nu}) - \frac{1}{2} g^{\lambda\nu} G^{\beta\alpha} J_{\beta,\alpha} \right] \quad , \quad (3.6)$$

where we have used a second identity[3],

$$\dot{G}^{\lambda\alpha} \dot{F}_{\alpha}{}^{\nu} = \frac{1}{4} g^{\lambda\nu} \dot{G}^{\beta\alpha} \dot{F}_{\alpha\beta} \quad , \quad (3.7)$$

which can also be verified directly. We shall need later the more general identity

$$G^{\lambda\alpha}{}_{,\rho} F_{\alpha}{}^{\gamma}{}_{,\sigma} + (\rho, \sigma) = \frac{1}{2} g^{\lambda\gamma} G^{\beta\alpha}{}_{,\rho} F_{\alpha\beta,\sigma} \quad , \quad (3.8)$$

where (ρ, σ) stands for the previous term with ρ and σ interchanged. Of course, (3.5) follows from (3.8) by contraction of ρ and σ and (3.7) follows by contraction with $u^{\rho} u^{\sigma}$.

Notice also that contraction with $u_{\lambda} u_{\gamma}$ leads to

$$G^{\alpha}{}_{,\rho} F_{\alpha,\rho} + (\rho, \sigma) = \frac{1}{2} G^{\alpha\beta}{}_{,\rho} F_{\alpha\beta,\sigma} \quad . \quad (3.9)$$

Introducing

$$\overline{X}^{\lambda\nu\rho} = X^{\lambda\nu\rho} + (\varepsilon\mu - 1) X^{\lambda\nu\sigma} u_{\sigma} u^{\rho} \quad , \quad (3.10)$$

we can write Eq.(3.6) as

$$\overline{X}^{\lambda\nu\rho}{}_{,\rho} = \mu T^{\lambda\nu} \quad , \quad (3.11)$$

where $T^{\lambda\nu}$ is the pseudotensor on the right-hand side of that equation. Making a convenient ν-differentiation by parts and using (2.21), $T^{\lambda\nu}$ can be put into the form

$$T^{\lambda\nu} = \left(G^{\lambda\alpha} J^{\nu} - g^{\nu\alpha} G^{\lambda\beta} J_{\beta} - \frac{1}{2} g^{\lambda\nu} G^{\beta\alpha} J_{\beta} \right)_{,\alpha} + G^{\lambda}{}_{\alpha,}{}^{\nu} J^{\alpha} \quad . \quad (3.12)$$

Our goal is to write $T^{\lambda\nu}$ as a divergence and, therefore, the task now is to write the last term of (3.12) as a divergence. Using (2.9) and (2.10) together with (2.20), the last term of (3.12) can be written as

$$G_{\lambda\alpha,\nu}J^{\alpha}=\varepsilon\beta[A_{\lambda\nu}-u_{\lambda}B_{\nu}-(F_{\sigma,\nu}K_{\lambda})_{,}{}^{\sigma}] \tag{3.13}$$

where

$$A_{\lambda\nu}=F_{\sigma,\nu}K^{\sigma}{}_{,\lambda}\ , \tag{3.14}$$

$$B_{\nu}=F_{\sigma\lambda,\nu}K^{\sigma}{}_{,}{}^{\gamma}\ , \tag{3.15}$$

and we have used (2.17) for the last term of (3.13). Before we go on we call attention to the fact that there are many ways one can proceed to write $T_{\lambda\nu}$ as a divergence. We could, for instance, use directly the expression (2.14) for J^{α} with or without the ν-differentiation by parts. Also the transformations we shall perform for $A_{\lambda\nu}$ and B_{ν} are not unique, with other possibilities sometimes even shorter than those we have followed. However, those that we have chosen lead to a quicker road when we want to recuperate the results for the rest frame[4].
Using (2.11) in (3.14) and performing a few differentiation by parts we obtain

$$\begin{aligned}A_{\lambda\nu} &= (F_{\sigma}\dot{F}^{\sigma}{}_{,\lambda})_{,\nu}+(F_{\sigma,\nu}\dot{F}^{\sigma})_{,\lambda}-(F^{\sigma}F_{\sigma,\lambda\nu})_{,}{}^{\rho}u_{\rho}\\ &\quad-\beta\left[(F_{\sigma,\nu}\ddot{G}^{\sigma})_{,\lambda}-F_{\sigma,\lambda\nu}\ddot{G}^{\sigma}\right]\ .\end{aligned} \tag{3.16}$$

We show in the appendix that the last tensor can be written as a divergence with the result

$$\begin{aligned}F_{\sigma,\lambda\nu}\ddot{G}^{\sigma} &= (F_{\lambda,\nu}\ddot{G}^{\sigma})_{,\sigma}+u_{\lambda}(\dot{G}_{\sigma\beta,\nu}\dot{F}^{\sigma})_{,}{}^{\beta}+(\dot{G}_{\beta,\nu}\dot{F}_{\lambda})_{,}{}^{\beta}\\ &\quad-\frac{1}{2}\left[(\dot{G}_{\beta,\nu}\dot{F}^{\beta})_{,\lambda}-(\dot{G}_{\beta,\lambda}\dot{F}^{\beta})_{,\nu}-(G^{\alpha\beta}{}_{,\lambda}\dot{F}_{\nu\alpha})_{,\beta}\right]\ .\end{aligned} \tag{3.17}$$

Taking this result back to (3.16) we can then write

$$A_{\lambda\nu}=A_{\lambda\nu\rho,}{}^{\rho} \tag{3.18}$$

where

$$\begin{aligned}A_{\lambda\nu\rho} &= F_{\sigma}\dot{F}^{\sigma}{}_{,\lambda}g_{\nu\rho}+F_{\sigma,\nu}\dot{F}^{\sigma}g_{\lambda\rho}-F^{\sigma}F_{\sigma,\lambda\nu}u_{\rho}\\ &\quad-\beta[F_{\sigma,\nu}\ddot{G}^{\sigma}g_{\lambda\rho}-F_{\lambda,\nu}\ddot{G}_{\rho}-u_{\lambda}\dot{G}_{\sigma\rho,\nu}\dot{F}^{\sigma}-\dot{G}_{\rho,\nu}\dot{F}_{\lambda}+\frac{1}{2}\dot{G}_{\beta,\nu}\dot{F}^{\beta}g_{\lambda\rho}\\ &\quad-\frac{1}{2}\dot{G}_{\beta,\lambda}\dot{F}^{\beta}g_{\nu\rho}-\frac{1}{2}\dot{G}^{\alpha}{}_{\rho,\lambda}\dot{F}_{\nu\alpha}]\ .\end{aligned} \tag{3.19}$$

Instead of using (3.15) for B_{ν} it is convenient to use the relation that follows from (3.13) itself upon contraction with u^{λ}. As

$$u^{\lambda}K_{\lambda}=0\ , \tag{3.20}$$

from (2.11) and (2.12), and as

$$u^{\lambda}K^{\sigma}{}_{,\lambda}=\dot{K}^{\sigma}\ , \tag{3.21}$$

as indicated in (2.13), we obtain, from (3.13) and (3.14),

$$B_{\nu}=\frac{1}{\varepsilon\beta}G_{\rho\gamma,\nu}J^{\rho}u^{\gamma}+F_{\sigma,\nu}\dot{K}^{\sigma}\ . \tag{3.22}$$

Next, with (2.19) we have

$$G_{\rho\gamma,\nu}J^{\rho}u^{\gamma}=\dot{G}_{\rho\nu}J^{\rho}+(G_{\nu}J^{\rho})_{,\rho}+(\varepsilon\mu-1)\varepsilon_{\rho\gamma\sigma\nu}\dot{F}^{\gamma}u^{\sigma}J^{\rho}\ , \tag{3.23}$$

where we have used (2.16) for the second term on the right-hand side. The first and the last can also be written as divergences. We give the details in the appendix and quote here only the final results. We have

$$\dot{G}_{\rho\nu}J^{\rho} = \varepsilon\beta C_{\nu\rho,}{}^{\rho} \tag{3.24}$$

where

$$\begin{aligned} C_{\nu\rho} &= \frac{1}{2}u_\nu u_\rho \dot{F}_{\alpha\gamma}\dot{F}^{\alpha\gamma} + 2\dot{F}_\nu\dot{F}_\rho - \dot{F}_\gamma\dot{F}^\gamma g_{\nu\rho} \\ &\quad +\beta h^{\sigma\gamma}(\dot{G}^{\alpha}{}_{\nu}\dot{F}_{\alpha,\sigma}g_{\gamma\rho} + \frac{1}{2}u_\nu G_{\alpha\rho,\gamma}F_{\sigma}{}^{\alpha}) \end{aligned} \tag{3.25}$$

and

$$\varepsilon_{\rho\gamma\sigma\nu}\dot{F}^{\gamma}u^{\sigma}J^{\rho} = \varepsilon\beta D_{\nu\rho,}{}^{\rho} \tag{3.26}$$

where

$$\begin{aligned} D_{\nu\rho} &= K_\nu\dot{F}_\rho - h_{\nu\rho}\dot{F}_\gamma\dot{F}^\gamma + \beta\Big[\ddot{G}^\gamma\dot{F}_\gamma g_{\nu\rho} \\ &\quad -\frac{1}{2}u_\nu\ddot{G}_{\alpha\rho}\dot{F}^\alpha - \ddot{G}_\rho\dot{F}_\nu - u_\nu(u_\rho\ddot{G}^\gamma\dot{F}_\gamma - \frac{1}{2}\ddot{G}_{\alpha\rho}\dot{F}^\alpha)\Big] \quad . \end{aligned} \tag{3.27}$$

With (3.24) and (3.26), Eq.(3.23) can then be written

$$G_{\lambda\gamma,\nu}J^{\lambda}u^{\gamma} = (\varepsilon\beta C_{\nu\rho} + G_\nu J_\rho + (\varepsilon\mu - 1)\varepsilon\beta D_{\nu\rho})_{,}{}^{\rho} \quad . \tag{3.28}$$

Finally, we also show in the appendix that the last term of (3.22) can be put into the form

$$F_{\sigma,\nu}\dot{K}^{\sigma} = H_{\nu\rho,}{}^{\rho} \tag{3.29}$$

where

$$\begin{aligned} H_{\nu\rho} &= 2F_{\sigma,\nu}\dot{F}^{\sigma}u_\rho - \dot{F}_\sigma\dot{F}^\sigma g_{\nu\rho} \\ &\quad -\beta\left[F_{\sigma,\nu}\ddot{G}^{\sigma}u^{\rho} - \frac{1}{2}u^{\nu}\ddot{G}_{\alpha\rho}\dot{F}^{\alpha} - \dot{F}_\nu\ddot{G}_\rho\right] \quad . \end{aligned} \tag{3.30}$$

From (3.28) and (3.29) we see that B_ν in (3.22) is a divergence,

$$B_\nu = B_{\nu\rho,}{}^{\rho} \tag{3.31}$$

where

$$B_{\nu\rho} = C_{\nu\rho} + \frac{1}{\varepsilon\beta}G_\nu J_\rho + (\varepsilon\mu - 1)D_{\nu\rho} + H_{\nu\rho} \quad . \tag{3.32}$$

Substituting (3.31) and (3.18) in (3.13) we conclude that $T^{\lambda\nu}$ in (3.12) is in fact a divergence,

$$T^{\lambda\nu} = T^{\lambda\nu\rho}{}_{,\rho} \tag{3.33}$$

where

$$\begin{aligned} T^{\lambda\nu\rho} &= G^{\lambda\rho}J^{\nu} - G^{\lambda\beta}J_\beta g^{\nu\rho} - \frac{1}{2}G^{\beta\rho}J_\beta g^{\lambda\nu} \\ &\quad +\varepsilon\beta[A^{\lambda\nu\rho} - u^{\lambda}B^{\nu\rho} - F^{\rho}{}_{,}{}^{\nu}K^{\lambda}] \end{aligned} \tag{3.34}$$

$A^{\lambda\nu\rho}$ and $B^{\nu\rho}$ are given in Eqs.(3.19) and (3.32) together with (3.25), (3.27) and (3.30). From (3.33) and (3.11) it then follows that

$$I^{\lambda\nu\rho}{}_{,\rho} = 0 \tag{3.35}$$

where

$$I^{\lambda\nu\rho} = \overline{X}^{\lambda\nu\rho} - \mu T^{\lambda\nu\rho} \quad . \tag{3.36}$$

From (3.35) we conclude that $I^{\lambda\nu o}$ is to be interpreted as the density of a conserved quantity and $I^{\lambda\nu i}$ as expressing its flux. The conserved quantity, when the flux across a closed surface S involving a volume V vanishes, is

$$I^{\lambda\nu} = \int I^{\lambda\nu o}d^3x \tag{3.37}$$

where the density is

$$I^{\lambda\nu o} = \overline{X}^{\lambda\nu o} - \mu T^{\lambda\nu o} \quad . \tag{3.38}$$

As $I^{\lambda\nu\rho}$ is traceless in the first two indices, $I^{\lambda\nu}$ is traceless $I^{\lambda} = 0$. This leaves us with fifteen independent first-order conservation laws as anticipated in the non-covariant calculations done in the medium rest frame[4]. If we add and subtract $I^{\nu\lambda}$ to $I^{\lambda\nu}$ we shall have the conservation laws associated to the symmetric zilch and its antisymmetric companion, which are then the irreducible parts of $I^{\lambda\nu}$. The divergence in (3.35) splits into

$$I^{(\lambda\nu)\rho}{}_{,\rho} = 0 \qquad\qquad I^{[\lambda\nu]\rho}{}_{,\rho} = 0 \tag{3.39}$$

where, from (3.36), (3.1), (3.2) and (3.10),

$$I^{(\lambda\nu)\rho} = \overline{Z}^{\lambda\nu\rho} - \mu T^{(\lambda\nu)\rho} \quad , \tag{3.40}$$

$$I^{[\lambda\nu]\rho} = \overline{Y}^{\lambda\nu\rho} - \mu T^{[\lambda\nu]\rho} \quad , \tag{3.41}$$

with $\overline{Z}$ and $\overline{Y}$ defined as in (3.10) for $\overline{X}$. The symmetric constants of motion are

$$I^{(\lambda\nu)} = \int I^{(\lambda\nu)o} d^3x \tag{3.42}$$

with nine independent components, and the antisymmetric are

$$I^{[\lambda\nu]} = \int I^{[\lambda\nu]o} d^3x \tag{3.43}$$

with six independent components.

In the medium rest frame S^o we recuperate the results obtained before[4].. In fact, let us calculate (3.38) in S^o. From (3.10) and (3.3) we obtain for $\lambda = i$, $\nu = 0$ then $\lambda = 0$, $\nu = i$ and then $\lambda = i$, $\nu = j$,

$$\overline{X}^{ioo}_{(o)} = \varepsilon\mu(\mathbf{E}^{(o)} \times \mathbf{E}^{(o)}{}_{,o})_i \quad , \tag{3.44}$$

$$\overline{X}^{oio}_{(o)} = \varepsilon\mu(\mathbf{B}^{(o)} \times \mathbf{B}^{(o)}{}_{,o})_i = -\varepsilon\mu\left(\mathbf{B}^{(o)} \times (\nabla \times \mathbf{E}^{(o)})\right)_i \quad , \tag{3.45}$$

$$\overline{X}^{ijo}_{(o)} = \varepsilon\mu\left(-B_i^{(o)}\dot{E}_j^{(o)} + E_j^{(o)}\dot{B}_i^{(o)} + \frac{1}{2}\delta_{ij}(\mathbf{B}^{(o)}.\dot{\mathbf{E}}^{(o)} - \dot{\mathbf{B}}^{(o)}.\mathbf{E}^{(o)})\right) \quad . \tag{3.46}$$

Next we calculate $T^{\lambda\nu o}_{(o)}$. From (3.34) we get

$$T^{ioo}_{(o)} = -(\mathbf{E}^{(o)} \times \mathbf{J}^{(o)})_i \quad , \tag{3.47}$$

$$T^{oio}_{(o)} = \varepsilon\beta\left[\left((E_k\dot{E}_k)_{,i} - \frac{\beta}{2}(\dot{B}^{(o)}_{k,i}\dot{E}^{(o)}_k + \ddot{B}^{(o)}_a\varepsilon_{iaj}\dot{B}^{(o)}_j\right)\right] \quad , \tag{3.48}$$

and

$$\begin{aligned} T^{ijo}_{(o)} \quad = \quad & B_i^{(o)}J_j^{(o)} - \frac{1}{2}\mathbf{B}^{(o)}.\mathbf{J}^{(o)}\delta_{ij} \\ & -\varepsilon\beta\left[E_k^{(o)}E^{(o)}_{k,ij} + \frac{\beta}{2}\dot{B}^{(o)}_{k,i}\varepsilon_{jkm}\dot{B}^{(o)}_m\right] \quad . \end{aligned} \tag{3.49}$$

Taking (3.44) and (3.47) in (3.38) we obtain from (3.37), and the second Maxwell equation (2.2)

$$I^{io}_{(o)} = \int \mathbf{E}^{(o)} \times (\nabla \times \mathbf{B}^{(o)}) d^3x \quad . \tag{3.50}$$

This is the constant of motion in Eq.(I-16), with its integrand displayed in (I-14). Now we substitute (3.45) and (3.48) in (3.38) and calculate (3.37). Let us show that all of (3.48) can actually be written as a 3-divergence and, therefore, will not contribute to (3.37). Using Maxwell's equations we get

$$T^{oio}_{(o)} = \varepsilon\beta\left(-(E_k\dot{E}_k)_{,i} + \frac{1}{2}\beta\left[(\dot{\mathbf{B}}^{(o)}.\dot{\mathbf{E}}^{(o)})_{,i} - (\dot{B}^{(o)}_k\dot{E}_i{}^{(o)})_{,k}\right]\right) \quad , \tag{3.51}$$

whose volume integral will then vanishes. Therefore, only (3.45) can contribute to (3.37) and we obtain,

$$I^{oi}_{(o)} = -\varepsilon\mu \int \mathbf{B}^{(o)} \times (\nabla \times \mathbf{E}^{(o)}) d^3x \quad , \tag{3.52}$$

which agrees with Eq.(I-22).

For the last case we use in the last tensor of (3.49) $\dot{B}^{(o)}_{k,i} = \dot{B}^{(o)}_{i,k} + \varepsilon_{ikj}(\nabla \times \dot{\mathbf{B}}^{(o)})_j$ and perform an integration by parts to obtain

$$\begin{aligned} \dot{B}^{(o)}_{k,i}\varepsilon_{jkm}\dot{B}^{(o)}_m &= (\dot{B}_i{}^{(o)}\varepsilon_{jkm}\dot{B}_m{}^{(o)})_{,k} - 2\dot{B}_i{}^{(o)}(\nabla \times \dot{\mathbf{B}}^{(o)})_j \\ &+ \delta_{ij}(\nabla \times \dot{\mathbf{B}}^{(o)}).\dot{\mathbf{B}}^{(o)} \quad . \end{aligned} \tag{3.53}$$

The 3-divergence in this expression cannot contribute to the volume integral and we obtain, from (3.46),

$$\begin{aligned} I^{ij}_{(o)} &= \int \Big[- B^{(o)}_i(\nabla \times \mathbf{B}^{(o)})_j - \varepsilon\mu E_j{}^{(o)}(\nabla \times \mathbf{E}^{(o)})_i \\ &+ \frac{1}{2}\delta_{ij}\big(\mathbf{B}^{(o)}.(\nabla \times \mathbf{B}^{(o)}) + \varepsilon\mu \mathbf{E}^{(o)}.(\nabla \times \mathbf{E}^{(o)})\big) \\ &- \varepsilon\mu\beta\big(E_k{}^{(o)}E^{(o)}_{k,ij} + B\dot{B}_i{}^{(o)}\nabla^2 E_j{}^{(o)} \\ &- \frac{1}{2}\beta\delta_{ij}\dot{B}_k{}^{(o)}\nabla^2 E^{(o)}_k\big)\Big] \quad . \end{aligned} \tag{3.54}$$

This agrees with the result that we have in (I-32) when we use in that paper the relation

$$(\dot{\mathbf{B}}^{(o)})^2 = -E^{(o)}_i\nabla^2 E^{(o)}_i + (E^{(o)}_j E^{(o)}_{j,i})_{,i} - (E^{(o)}_j E^{(o)}_{i,j})_{,i} \quad , \tag{3.55}$$

that follows from Maxwell's equations.

Appendix

1. Using (2.23), the last term in (3.16) can be written

$$F_{\sigma,\lambda\nu}\ddot{G}^{\sigma} = \dot{F}_{\sigma\lambda,\nu}\ddot{G}^{\sigma} + (F_{\lambda,\nu}\ddot{G}^{\sigma})_{,\sigma} \quad . \tag{A1}$$

With (2.9) and (2.18) the first contribution can be written

$$\dot{F}_{\sigma\lambda,\nu}\ddot{G}^{\sigma} = \frac{1}{2}u_{\lambda}\dot{G}_{\rho\beta,\nu}\ddot{F}^{\rho\beta} + \dot{G}^{\beta}{}_{,\nu}\ddot{F}_{\lambda\beta} \quad . \tag{A2}$$

Using again (2.23) and performing a few differentiation by parts we get

$$\dot{F}_{\sigma\lambda,\nu}\ddot{G}^{\sigma} = u_{\lambda}(\dot{G}_{\rho\beta,\nu}\dot{F}^{\rho})_{,}{}^{\beta} + (\dot{G}_{\beta,\nu}\dot{F}_{\lambda})_{,}{}^{\beta} - (\dot{G}_{\beta,\nu}\dot{F}^{\beta})_{,\lambda} + \dot{G}_{\beta,\nu\lambda}\dot{F}^{\beta} \quad . \tag{A3}$$

For the last term we write, with the help of (3.9) for dotted quantities,

$$\dot{G}_{\beta,\nu\lambda}\dot{F}^{\beta} = \frac{1}{2}\left[(\dot{G}_{\beta,\nu}\dot{F}^{\beta})_{,\lambda} + (\dot{G}_{\beta,\lambda}\dot{F}^{\beta})_{,\nu} - \frac{1}{2}\dot{G}^{\alpha\beta}{}_{,\lambda}\dot{F}_{\alpha\beta,\nu}\right] \quad . \tag{A5}$$

Finally, for the last term of this expression we use (2.8) to write it as

$$-\frac{1}{2}\dot{G}^{\alpha\beta}{}_{,\lambda}\dot{F}_{\alpha\beta,\nu} = (\dot{G}^{\alpha\beta}{}_{,\lambda}\dot{F}_{\nu\alpha})_{,\beta} \quad . \tag{A6}$$

Substituting this in the preceeding equation and going from there to (A3) we can see that (A1) leads to (3.17).

2. Using (2.14) for the first term on the right-hand side of (3.23) we can write

$$\dot{G}_{\rho\nu}J^{\rho} = \varepsilon\beta(-2C_{1\nu} + \beta C_{2\nu}) \tag{A6}$$

where, after using (2.9), (2.18) and (2.23),

$$C_{1\nu} = -\frac{1}{2}\left[u_\nu \dot{F}_{\alpha\gamma}\ddot{F}^{\alpha\gamma} + 2(\dot{F}_\nu \dot{F}_\gamma)_,{}^{\gamma} - 2\dot{F}_\gamma \dot{F}^{\gamma}{}_{,\nu}\right] \quad . \tag{A7}$$

and

$$C_{2\nu} = h^{\sigma\rho}\left[(\dot{G}_{\alpha\nu}\dot{F}^{\alpha}{}_{,\sigma})_{,\rho} - \dot{G}_{\alpha\nu,\rho}\dot{F}^{\alpha}{}_{,\sigma}\right] \quad . \tag{A8}$$

From (3.8) the last term of (A8) gives, after using Eqs.(2.8) and (2.21),

$$h^{\sigma\rho}\dot{G}_{\alpha\nu,\rho}\dot{F}^{\alpha}{}_{,\sigma} = -\frac{1}{2}h^{\sigma\rho}u_\nu(G^{\alpha\beta}{}_{,\rho}F_{\sigma\alpha})_{,\beta} \quad . \tag{A9}$$

Taking this in (A8) we see that (A6) leads to (3.24).

3. Consider now the last term of (3.23). From (2.10) we obtain

$$\varepsilon_{\rho\gamma\sigma\nu}\dot{F}^{\gamma}u^{\sigma}J^{\rho} = \varepsilon\beta\left[(K_\nu \dot{F}^{\gamma})_{,\gamma} - K_{\gamma,\nu}\dot{F}^{\gamma} + u_\nu \dot{K}_\gamma \dot{F}^{\gamma}\right] \quad . \tag{A10}$$

Next we use (2.11) to write the middle term as

$$K_{\gamma,\nu}\dot{F}^{\gamma} = (\dot{F}_\gamma \dot{F}^{\gamma})_{,\nu} - \beta(\ddot{G}_\gamma \dot{F}^{\gamma})_{,\nu} + \beta\ddot{G}_\gamma \dot{F}^{\gamma}{}_{,\nu} \quad . \tag{A11}$$

For the last term of this expression we obtain, with (2.23) and (3.7),

$$\ddot{G}_\gamma \dot{F}^{\gamma}{}_{,\nu} = \frac{1}{2}u_\nu(\ddot{G}^{\alpha\beta}\dot{F}_\alpha)_{,\beta} + (\ddot{G}_\gamma \dot{F}_\nu)_,{}^{\gamma} \quad . \tag{A12}$$

For the last term in the right-hand side of (A10) we use (2.11), perform a dot-differentiation by parts in the term proportional to β, use (3.9) contracted write $u^\rho u^\sigma$ and use (2.23) followed by (2.21). The final result is

$$\dot{K}_\gamma \dot{F}^{\gamma} = (\dot{F}_\gamma \dot{F}^{\gamma})_{,\rho}u^{\rho} - \beta\left[(\ddot{G}_\gamma \dot{F}^{\gamma})_{,\rho}u^{\rho} - \frac{1}{2}(\ddot{G}^{\alpha\gamma}\dot{F}_\alpha)_{,\gamma}\right] \quad . \tag{A13}$$

Taking (A13) and (A12) in (A10) we obtain (3.26).

4. Finally, we consider the last term of (3.22). Using again (2.11), making a dot-differentiation by parts and using (2.23) followed by (3.7) contracted with u_λ, we obtain

$$F_{\sigma,\nu}\dot{K}^{\sigma} = 2(F_{\sigma,\nu}\dot{F}^{\sigma})_{,\rho}u_\rho - (\dot{F}_\sigma \dot{F}^{\sigma})_{,\nu} - \beta\left[(F_{\sigma,\nu}\ddot{G}^{\sigma})_{,\rho}u^{\rho} - \frac{1}{2}u_\nu(\ddot{G}^{\alpha\beta}\dot{F}_\alpha)_{,\beta} - (\dot{F}_\nu \ddot{G}^{\sigma})_{,\sigma}\right] \quad . \tag{A14}$$

This result leads to (3.29).

References

1. D. M. Lipkin, J. Math. Phys. **5**, 696 (1964).
2. S. Ragusa, Nuovo Cimento B, **101**, 121 (1988).
3. S. Ragusa and M. Bailyn, Nuovo Cimento B, **107**, 17 (1992).
4. S. Ragusa, J. Phys. A: Math. Gen. **27**, 2887 (1994).
   Equation (n) of this paper will be indicated by Eq.(I-n). X in this paper is what we call $I_{(o)}$ in the present one.
5. F. I. Fedorov, Opt. Spectrosc. (USSR) **6**, 49 (1959).