Newton's Gravity and the Theories with an Arbitrary RØ2 Coupling

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It is shown that Einstein's gravitational theory with an arbitrary $R\phi^2$ coupling agrees with Newton's law of universal gravitation. It is also shown that under suitable conditions the same is true for higher order derivative gravity plus a scalar field arbitrarily coupled to the curvature scalar.

Few theories can compare in the accuracy of their predictions with Newton's theory of universal gravitation. The discovery of Neptune, and the rediscovery of Ceres, are among the spectacular sucesses that testify to the accuracy of the theory. Unfortunately Newton's theory is not perfect: the predicted motions of the perihelia for the inner planets deviate somewhat from the observed values. Although Newton's theory is not perfect, it is an excellent approximation in the limiting case of motion at low velocity in a weak gravitational field. As a consequence, any relativistic theory of gravitation ought to agree with Newton's theory in this limiting case.

On the other hand, the reasons for adding to Einstein's action for gravitation a nonminimal functional of a scalar field are manifold[1]. It is widely believed, moreover, that this nonminimal functional may be written in the following form[2]

$$S_S = \int d^4x\sqrt{-g}\frac{1}{2}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2 - V(\phi) + \lambda R\phi^2\right] \quad (1)$$

where λ is a dimensionless parameter, and $V(\phi)$ is the interaction potential, i.e. a polynomial over the field ϕ (higher than second degree). Here the Ricci tensor is defined by $R_{\mu\nu} = -\partial_\alpha\Gamma^\alpha{}_{\mu\nu} + ...$; and the metric convention is $g_{\mu\nu} = diag(1,-1,-1,-1)$. Accordingly, let us focus our attention on the gravitational theories described by the action functional

$$S[g,\phi] = S_G + S_S \ , \quad (2.a)$$

where

$$S_G \equiv \int d^4x\sqrt{-g}\frac{R}{2A} \ , \quad (2.b)$$

with $A = 8\pi G$ in natural units. On physical grounds, two fundamental questions may be raised as far as the preceding theories are concerned: Are those gravitational nonminimally coupled theories compatible with the wealc equivalence principle? If this is so, do they agree with Newton's gravity in the limiting case of motion at low velocity in a wealc gravitational field?

The first question was recently answered by Accioly et al.[3] They found that gravitational nonminimally coupled theories, in general, do not violate the weak equivalence principle. In the following we shall show that the answer to the second question is affirmative as well.

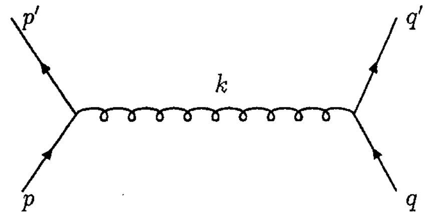

Figure 1: Lowest-order graviton exchange force.

As it is well known, a covariant photon exchange between elastically scattered particles leads naturally to the concept of a force or, equivalently, a static electromagnetic potential. The same is true for the single-graviton exchange between two massive particles as shown in Fig.1 in connection with the static Newtonian force $F = -Gm_1m_2/r^2$. Let us then find the invariant amplitude for the Feynman diagram corresponding to Fig.1 in the framework of the theories defined by the action (1), wherein we shall assume that $V(\phi) = 0$ for the sake of simplicity. Of course the massive external particles in Fig.1 have zero spin. To avoid the clumsiness involved in the evaluation of invariant amplitudes within the context of linearized gravity we restrict our semi-classical computations to conformally flat spaces. For these spacetimes the metric can be written as

$$g_{\mu\nu}(x) = \Omega^2(x)\eta_{\mu\nu} \ , \tag{3}$$

where $\Omega(x)$ is a position-dependent function and $\eta_{\mu\nu}$ is the flat space Lorentz metric.

Combining Eq. (2) with Eq. (3) yields[4]

$$S = \int d^4x \left[\frac{\Omega\Box\Omega}{-2B^2} + \frac{1}{2}\left(\Omega^2\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \Omega^4\, m^2\, \phi^2 + 6\lambda\phi^2\Omega\Box\Omega\right)\right] \ , \tag{4}$$

where $B^2 \equiv -\frac{A}{6}$ and $\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$.

In the weak field approximation Ω^2 can be written as

$$\Omega^2 = \exp\left[2Bf(x)\right] \ , \tag{5}$$

where $f(x)$ is a position-dependent function which for physical reasons we assume to have the following properties

$$|2Bf| \ll 1$$
$$\lim_{x\to\pm\infty} f(x) \to 0 \quad \forall\, x^\mu$$
$$\lim_{x\to\pm\infty} \partial_\mu f(x) \to 0 \quad \forall\, x^\mu \ .$$

Substituting Eq. (5) into Eq. (4) we obtain

$$\begin{aligned} S &= \int d^4x\, \frac{f\Box f}{-2} + \int d^4x \frac{1}{2}\left(\partial_\mu\phi\, \partial^\mu\phi - m^2\phi^2\right) \\ &+ \int d^4x B\left[f\left(\partial_\mu\phi\partial^\mu\phi - 2m^2\phi^2\right) + 3\lambda\phi^2\Box f\right] \ , \end{aligned} \tag{6}$$

where all indices in Eq. (6) are raised(lowered) by $\eta^{\mu\nu}$ ($\eta_{\mu\nu}$).

Hence, the "graviton propagator" and the vertex function associated with the coupling $\phi(p)-\phi(p')-f(k)$

are given respectively by

$$D(k) = \frac{i}{k^2}$$

$$V(p,p') = -2iB\left[p\cdot p' + 2m^2 + 3\lambda\,(p+p')^2\right]$$

where all momenta are supposed to be incoming as far as the above vertex is concerned.

Introducing the Mandelstam variable $t = k^2$, the invariant amplitude for the process depicted in Fig.1 becomes

$$\mathcal{M} = -4iB^2\left[\frac{m_1^2 m_2^2}{t} + (m_1^2+m_2^2)\left(\frac{1}{2}+3\lambda\right) + t\left(\frac{1}{2}+3\lambda\right)^2\right] \tag{7}$$

The gravitational potential is given by

$$V(\vec{r}) = \frac{1}{4m_1m_2}\frac{1}{(2\pi)^3}\int d^3\vec{k}\;F(NR)\,e^{i\vec{k}\cdot\vec{r}}\;, \tag{8}$$

where $F(NR)$ is the nonrelativistic limit of $i\mathcal{M}$. It can be obtained from Eq. (7), and the potential may be expressed as

$$V(\vec{r}) = -\frac{B^2 m_1 m_2}{4\pi r} + \frac{B^2(m_1^2+m_2^2)}{m_1m_2}\left(3\lambda+\frac{1}{2}\right)\delta^3(\vec{r}) + \frac{B^2\left(\frac{1}{2}+3\lambda\right)^2}{m_1m_2}\nabla^2\,\delta^3(\vec{r})\;. \tag{9}$$

Therefore the agreement with the Newtonian theory is verified if $B^2 \equiv 4\pi G$, which directly implies that Einstein's constant in conformally flat spaces is given by $A = -24\pi G$. It follows then that Einstein-nonminimally coupled scalar field theory is compatible with Newton's law of universal gravitation. The second and third terms in Eq. (9) are semi-classical corrections to the Newton force.

It can be shown using much algebra that the preceding result remains valid for arbitrary Riemannian spaces.

It is regrettable, however, that Einstein's theory gives no room for any quantum theory which is free of contradictions. Yet, over the last twenty years higher-derivative gravity has been considered as the first and best candidate for the quantum description, although it is not unitary within the usual pertubation scheme. Incidentally, it is believed that the problem of non-unitarity can be overcome by using a non-perturbative approach[5]. Consequently, it would be interesting to investigate whether or not R^2 - gravity plus a scalar field arbitrarily coupled to the curvature scalar agrees with Newton's theory in the limiting case previously mentioned. The action for higher-derivative gravity is generally written in the form

$$S_G = \int d^4x\sqrt{-g}\left[\frac{R}{2A} - \frac{\alpha R^2}{2} - \frac{\beta}{2}R_{\mu\nu}R^{\mu\nu}\right]\;, \tag{10}$$

where both α and β are dimensionless parameters. In the weak field approximation this action assumes the form

$$S_G = \int d^4x\frac{1}{-2}f\left[\Box + 12B^2(3\alpha+\beta)\Box\Box\right]f\;, \tag{11}$$

from which the "graviton propagator" is easily obtained as

$$D(k) = i\left[\frac{1}{k^2} - \frac{1}{k^2-M^2}\right]\;, \tag{12}$$

$$M^2 \equiv \frac{1}{12B^2(3\alpha+\beta)} \tag{13}$$

This result agrees with the one found by Brunini and Gomes[6]. The above propagator has a good $\left(\sim\frac{1}{(k^2)^2}\right)$

ultraviolet behaviour at the expense of a negative metric massive ghost. Of course to avoid the ghost becomes tachyonic one must impose $(3\alpha + \beta) > 0$.

Repeating the rasoning that led previously to Eq. (7) we now find that

$$\mathcal{M} = -4iB^2 \left[\frac{1}{t} - \frac{1}{t-M^2}\right] \left[t^2 \left(\frac{1}{2}+3\lambda\right)^2 + (m_1^2+m_2^2)\left(\frac{1}{2}+3\lambda\right) t + m_1^2\, m_2^2\right] \tag{14}$$

The gravitational potential is then given by

$$V(\vec{r}) = Gm_1m_2 \left[-\frac{1}{r} + \frac{K\, e^{-Mr}}{r}\right] - \frac{4\pi G\left[M\left(\frac{1}{2}+3\lambda\right)\right]^2}{m_1m_2}\, \delta^3(\vec{r}) \tag{15}$$

where

$$K \equiv 1 + \frac{M^2\left(\frac{1}{2}+3\lambda\right)\left[m_1^2+m_2^2+M^2\left(\frac{1}{2}+3\lambda\right)\right]}{m_1^2m_2^2}$$

So Eq.(15) may be approximate to approach the newtonian limit $1/r$ as closely as we wish, by ensuring that M is large enough; of course we are assuming that the parameter M is positive (absence of tachyonic ghosts). It can be shown that this result remains valid for arbitrary Riemannian spaces.

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Embedding of the Riemann's Space

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We study the Gonner identity for the Riemann's 4-spaces of class one, and emphasize to the singular case $b^{ae}G_{ae} = 0$, where G„ is the Einstein's tensor and bae is the second fundamental form of R4 with respect to $E5$ We obtain bij for the special case (p = O) in Rq embedded in $E5$.

I. Introduction

A space-time accepts embedding into E_5 if and only if there exists the second fundamental form tensor $b_{ac} = b_{ca}$ satisfying the Gauss-Codazzi equations:

$$R_{ijkc} = \varepsilon(b_{ik}b_{jc} - b_{ic}b_{jk}) \qquad (1.a) \ ,$$

and

$$b_{ij;k} = b_{ik;j} \ , \qquad (1.b)$$

where $\varepsilon = \pm 1$, R_{abcd} is the R_4 curvature tensor and k stands for the covariant derivative. From Eq. (1.a) we can prove[1,2] that

$$pb_{ij} = \frac{K_2}{48}g_{ij} - \frac{1}{2}R_{imnj}G^{mn} \ , \qquad (2.a)$$

where K_2 is the Lanczos invariant[3,4] ($^*R^{*abcd}$ is tlie double dual of the Riemann's tensor defined by $^*R^*_{abcd} = \frac{1}{4}\eta_{abqr}R^{qrij}\eta_{ijcd}$):

$$K_2 \equiv^* R^{*abcd}R_{abcd} \ . \qquad (2.b)$$

Also

$$p \equiv \frac{\varepsilon}{3}b^{ac}G_{ac} \ , \qquad (2.c)$$

and $G_{ac} = R_{ac} - g_{ac}\frac{R}{2}$ is the Einstein's tensor. The importance of Eq. (2.a) relies on the fact that it allows us to find b_{ij} when p # 0. It is routine to verify that the quantities b_{ij} satisfies Eqs. (1.a) and (1.b). However, if p = 0 the problem remains open.

In the section II we will indicate two cases with p = 0: empty space-time and R_4 of Einstein-Maxwell of class one. In section III we will determine b_{ij} when p = 0 for the special case with $det(G_{ac}) \neq 0$ and $b^r{}_r$ # 0.

Notation and terminology are as in ref [5].

II. Space-times with p = 0

We will indicate at least two situations where Eq. (2.c) reduces to zero.

a) R_4 empty

From Eq. (2.c) it is clear that $G_{ab} = 0$ implies $p = 0$. It is well known that the empty 4-space is ruled out, because the impossibility of its ernbedding into E_5[6,7].

b) The R_4 Einstein-Maxwell

In this case the gravitation is produced by an electromagnetic field, which is given by $F_{ij} = -F_{ji}$. *Collinson* proved[8], that:
"only a R_4 of type N in the Petrov classification with F_{ij} null (its invariant are zero) can be embedded into E_5."

(3.a)

This necessary condition avoids, for instance, that the Reissner-Nordstrom solution could be of class one.

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By studying the statement (3.a) and the corresponding Gauss-Codazzi equation we find that:

$$p = 0, \quad b = trace(b^i{}_j) = 0, \quad det(G_{ar}) = 0 \ , \qquad (3.b)$$

In spite of this, Collinson[8] was able to get the embedding. It should be pointed out that in this way R_4 losses intrinsic rigidity, i.e. p = 0 does not necessary avoids embedding, but instead the condition p = 0 might change the intrinsic rigidity of the 4-space. It is not known if there are other situations with p = 0. This can be seen if we construct metrics such that:

$$K_2 g_{ij} = 24 R_{imnj} G^{mn} \ . \qquad (4)$$

(see Eq. (2.a)).

III. Determination of b_{ij} for a special case with $p = 0$.

Here we analyze a specific situation where in principle one gets b_{ac} for p = 0.

In fact, by substituting Eqs. (1.a) in (4) we obtain:

$$\frac{K_2}{24} g_{ij} = \varepsilon b_i{}^n b_j{}^m G_{nm} \ . \qquad (5.a)$$

By introducing the condition

$$det(G_{ar}) \neq 0 \ , \qquad (5.b)$$

we are able to write for the square of the second fundamental form:

$$b_{ic} b^c{}_j = e \frac{K_2}{24} G^{-1}_{ij} \ . \qquad (5.c)$$

Before going further, notice that Eq. (5.b) doesn't hold neither for an Einstein-Maxwell metric (see Eq. (3.b)), nor in Fermi's[9] 3-space of R_4.

Eq. (1.a) gives:

$$b b_{ij} = b_{ic} b^c{}_j - \varepsilon R_{ij} \ , \qquad (6.a)$$

where $R_{ac} \square R^i{}_{aci}$ is the Ricci's tensor. Combining Eq. (6.a) and Eq. (5.c) gives

$$b b_{ij} = \varepsilon \left(\frac{K_2}{24} G^{-1}_{ij} - R_{ij} \right) \ , \qquad (6.b)$$

and thus ($b \equiv b^r{}_r$) :

$$b^2 = \varepsilon \left(\frac{K_2}{24} G^{-1a}_{a} - R \right) \geq 0 \ . \qquad (6.c)$$

From Eq. (6.c) we can easily determine E when $b \neq 0$. If $b \neq 0$, then Eq. (6.b) gives b_{ij} as a function of the internal space time geometry. Summarizing:

" A R_4 of class one with

$$p = 0, \quad det(G_{ar}) \neq 0 \quad \text{and} \quad b \neq 0 \qquad (7)$$

is intrinsically rigid"

From Eqs. (6.b) and (6.c), we can find b_{ij} for the special case when R_4 is embedded into E_5 under the conditions (7).

Our research will continue trying to find a space-time of class one satisfying (7)

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