Localization in the Anderson Model with Long Range Hopping

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We give a proof of exponential localization in the Anderson model with long range hopping based on a multiscale analysis.

I. Introduction

We consider the random Hamiltonian

\[ H = \Gamma + V \]  \hspace{1cm} (1.1)

where

1. \( \Gamma \) is a translation invariant self-adjoint operator with exponentially decaying matrix elements, i.e.,
   \[ \Gamma(x,y) = \phi(x - y) \] for some function \( \phi \) on \( \mathbb{Z}^d \) with \( \phi(-x) = \overline{\phi(x)} \) for which there exist \( C < \infty \) and \( \gamma > 0 \) such that
   \[ |\Gamma(x,y)| = |\phi(x - y)| \leq C e^{-\gamma \|x-y\|} \hspace{1cm} (1.2) \]
   for all \( x, y \in \mathbb{Z}^d \).

2. \( V(x), x \in \mathbb{Z}^d \), are independent identically distributed random variables with common probability distribution \( \mu \).

In the usual Anderson model\(^1\) \( \Gamma = -A \), where \( A(x,y) = 1 \) if \( |x - y| = 1 \) and zero otherwise.

In this article we are concerned with localization. We say that the random operator \( H \) exhibits localization in an energy interval \( I \) if \( H \) has only pure point spectrum in \( I \) with probability one. We have exponential localization in \( I \) if we have localization and all the eigenfunctions corresponding to eigenvalues in \( I \) have exponential decay. Localization for \( -A + V \) has been extensively studied\(^{2-20}\).

In this article we extend the von Dreifus-Klein\(^{18}\) proof of localization to random Hamiltonians of the form given in eq. (1.1). (Our methods also extend the original von Dreifus-Spencer\(^{13,14}\) proof of decay of Green's functions). If \( \mu \) satisfy certain regularities conditions, localization for such operators at high disorder or low energy has been proved by Aizenman and Molchanov\(^{18}\). The proof we give here, as other proofs based on a multiscale analysis\(^{6,15}\), do not require regularity of \( \mu \), it only uses certain a priori probabilistic estimates about Green's functions in finite volumes. This has the advantage of allowing the treatment of potentials with singular probability distributions\(^{11,16}\). They can also be used to prove localization inside spectral gaps for small disorder\(^{20}\).

II. Results

We start with notation and definitions.

If \( A \subset \mathbb{Z}^d \), we denote by \( H_A \) the operator \( H \) restricted to \( A \) with zero boundary conditions outside \( A \), i.e.,

\[ H_A(x,y) = \begin{cases} H(x,y) & \text{if } x, y \in A, \\ 0 & \text{otherwise}. \end{cases} \hspace{1cm} (2.1) \]

The corresponding Green's function is \( G_A(z) = (H_A - z)^{-1} \), defined for \( z \notin \sigma(H_A) \). We will write

\[ G_A(z; x, y) = (H_A - z)^{-1}(x, y) \hspace{1cm} \text{for } x, y \in A. \hspace{1cm} (2.2) \]
If \( \Lambda = \mathbb{Z}^d \) we simply write \( G(z; x, y) \). Notice that we omit the dependence of \( H_\Lambda \) and \( G_\Lambda \) on the potential \( V \).

We will use \( \mathbb{P} \) to denote the probability measure in the underlying probability space for the random variables \( V(x), x \in \mathbb{Z}^d \). We will also take \( C = 1 \) in (1.2) without loss of generality.

For \( x \in \mathbb{Z}^d \), \( x = (x_1, \ldots, x_d) \), we set \( \|z\| = \|z\|_\infty \equiv \max\{|x_1|, \ldots, |x_d|\} \). Distances in \( \mathbb{Z}^d \) will always be taken with respect to this norm.

If \( L > 0 \), \( x \in \mathbb{Z}^d \), we will denote by \( \Lambda_L(x) \) the cube centered at \( x \) with sides of length \( L \), i.e.,

\[
\Lambda_L(x) = \{ y \in \mathbb{Z}^d ; \| y - x \| \leq \frac{L}{2} \} \tag{2.3}
\]

We will also use

\[
\hat{\Lambda}_L(x) = \{ y \in \Lambda_L(x) ; \| y - x \| > \frac{L}{2} \} \tag{2.4}
\]

We will say that \( \psi \in \ell^2(\mathbb{Z}^d) \) decays exponentially fast with mass \( m > 0 \) if

\[
\limsup_{\|x\| \to \infty} \frac{\log |\psi(x)|}{\|x\|} < -m \tag{2.5}
\]

The following definition contains the key modification we make in the von Dreifus-Klein proof. We fix \( \beta, 0 < \beta < 1 \).

**DEFINITION** Let \( m > 0 \), \( E \in \mathbb{R} \). A cube \( \Lambda_L(x) \) is \((m, E)\)-regular (for a fixed potential and given \( \beta \)) if

\[
d(E, \sigma(H_{\Lambda_L(x)})) \geq \frac{e^{-L^p}}{2} \tag{2.6}
\]

and

\[
|G_{\Lambda_L(x)}(E; x, y)| \leq e^{-m \|x-y\|} \tag{2.7}
\]

for all \( y \in \hat{\Lambda}_L(x) \). Otherwise we say that \( \Lambda_L(x) \) is \((m, E)\)-singular.

We can now state our main result.

**THEOREM 2.1.** Let \( E_0 \in \mathbb{R} \) and \( L_0 > 0 \). Suppose we have:

(P1)

\[
\mathbb{P}\{ \Lambda_{L_0}(0) \text{ is } (m_0, E_0) \text{- regular} \} \geq 1 - \frac{1}{L_0^{p_0}} \tag{2.8}
\]

for some \( p > 2d \) and \( m_0 < m_0 \leq \frac{\gamma}{2} \).

(P2)

\[
\mathbb{P}\{ d(E, \sigma(H_{\Lambda_L(0)})) < e^{-L^p} \} \leq \frac{1}{L^q} \tag{2.9}
\]

for some \( q \) with \( q > 4p + 6d \), all \( E \) with \( |E - E_0| \leq \eta \) for some \( \eta > 0 \), and all \( L \geq L_0 \).

Then, given \( m \), with \( 0 < m < m_0 \), there exists \( B = B(p, d, \beta, q, \gamma, m_0, m) < \infty \), such that if \( L_0 > B \), we can find \( \delta = \delta(L_0, m_0, m, \beta) > 0 \) so, with probability one, the spectrum of \( H \) in \((E_0 - \delta, E_0 + \delta)\) is pure point and the eigenfunctions corresponding to eigenvalues in \((E_0 - 5, E_0 + 6)\) decay exponentially fast at infinity with mass \( m \).

The validity of (P1) and (P2) are discussed in ref. [15]. Notice that \( B \) and \( \delta \) do not depend on \( E_0 \). Notice also that Theorem 2.1 is still valid if we weaken the requirements on \( p \) and \( m \) to \( p > d \) (as in ref. [15]; notice that if \( p > 2d \) we have \( J = 3 \) in ref. [15]) and \( 0 < m < \gamma \).

As in ref. [15], Theorem 2.1 will follow from Theorems 2.2 and 2.3, which we will now state.

**THEOREM 2.2.** Let \( I \subseteq \mathbb{R} \) be an interval and \( L_0 > 0 \). Suppose we have:

(K1)

\[
\mathbb{P}\{ \text{for any } E \in I \text{ either } \Lambda_{L_0}(x) \text{ or } \Lambda_{L_0}(y) \text{ is } (m_0, E) \text{- regular} \} \geq 1 - \frac{1}{L_0^{p_0}} \tag{2.10}
\]

for some \( p > 2d \), \( m_0 \) with \( 0 < m_0 \leq \frac{\gamma}{2} \), and any \( x, y \in \mathbb{Z}^d \) with \( \|x - y\| > L_0 \).
Then if we fix \( \alpha, 1 < \alpha < \infty \), set \( L_{k+1} = L_k^0 \), \( k = 0, 1, 2, \ldots \), and pick \( m \) with \( 0 < m < m_0 \), we can find \( Q = Q(p, d, \beta, q, \gamma, m_0, \alpha, m) < \infty \), such that if \( L_0 > Q \), we have that for any \( k = 0, 1, 2, \ldots \),

\[
\mathbb{P}\{ \text{for any } E \in I \text{ either } \Delta_{L_k}(x) \text{ or } \Delta_{L_k}(y) \text{ is } (m, E) \text{-regular} \} \geq 1 - \frac{1}{L_k^p}
\]

for any \( x, y \in \mathbb{Z}^d \) with \( ||x - y|| > L_k \).

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**Theorem 2.3.** Let \( I \subset \mathbb{R} \) be an interval, and let \( p, L_0, a, m \) be such that \( p > d \), \( L_0 > 0 \), \( 1 < \alpha < \frac{2p}{d} \), \( 0 < m < \frac{\alpha}{\beta} \). Set \( L_{k+1} = L_k^0 \), \( k = 0, 1, 2, \ldots \). Suppose that we have (2.12) for all \( k = 0, 1, 2, \ldots \) and any \( x, y \in \mathbb{Z}^d \) with \( ||x - y|| > L_k \). Then, with probability one, the spectrum of \( H \) in \( I \) is pure point and the eigenfunctions corresponding to eigenvalues in \( I \) decay exponentially fast at infinity with mass \( m \).

Theorems 2.1-2.3 are essentially the same as in ref. ([15]), except that we have a different definition for when a a cube \( \Lambda_L(x) \) is \((m, E)\)-regular, and we require that \( m_0 < \gamma \) (\( m_0 < \frac{\alpha}{\beta} \) is just to simplify the proofs). The probabilistic part of the proofs are not changed, we modify only the deterministic part of the proofs.

**III. Proof of theorem 2.2**

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**Lemma 3.4.** For \( 1 < \ell < L \) and \( 0 < m \leq \frac{\alpha}{\beta} \), let \( A = \Lambda_L(x_0) \) for some \( x_0 \in \mathbb{Z}^d \), and let \( x, y \in A \) be such that \( \Lambda_L(x) \subset A \) is \((m, E)\)-regular and \( y \notin \Lambda_L(x) \). Then for any \( E \in \mathbb{R} \) we have

\[
|G_A(E; x, y)| \leq e^{-M ||u - z||} |G_A(E; u, y)|
\]

for some \( u \in A \setminus \Lambda_L(x) \), where

\[
M = m - \frac{4}{\ell^p - \beta} - 2 \log \{ (\ell + 1)^d (L + 1)^d \} \ell.
\]

**Proof:** It follows from the resolvent identity that

\[
G_A(E; x, y) = -\sum_{s, t} G_{\Lambda_L(x)}(E; x, s) \Gamma(s, t) G_A(E; t, y),
\]

where we sum over all \( s \in A \setminus \Lambda_L(x) \) and \( t \in A \setminus \Lambda_L(x) \).

Thus there exist \( s' \in \Lambda_L(x) \) and \( t' \in A \setminus \Lambda_L(x) \) such that

\[
|G_A(E; x, y)| \leq (e + 1)^d (L + 1)^d |G_{\Lambda_L(x)}(E; x, s')| e^{-\gamma ||t' - s'||} |G_A(E; t', y)|,
\]

where we used (1.2).

There are two possible situations:

(i) \( s' \notin \Lambda_L(x) \)

In this case it follows from eq. (2.7) that

\[
|G_{\Lambda_L(x)}(E; x, s')| e^{-\gamma ||t' - s'||} \leq e^{-m ||s' - x||} e^{-\gamma ||t' - s'||} \leq e^{-m ||t' - x||},
\]

by the triangular inequality, since \( m < \gamma \).
\( (ii) \ s' \in \Lambda_\ell(x) \setminus \hat{\Lambda}_\ell(x) \)

Now we use (2.6) to get

\[
|G_{\Lambda(x)}(E; x, s')| e^{-\gamma \|t' - s'\|} < 2 e^{\ell \theta} e^{-\gamma \|t' - s'\|} < e^{-\gamma \|t' - s\| + \gamma (\ell/4) + \ell \theta + \log 2} \\
< e^{-\left(\frac{\ell}{2} - \frac{1}{\ell - \beta}\right) \|t' - s\|} \leq e^{-(m - \frac{1}{\ell - \beta}) \|t' - s\|},
\]

where we used the reverse triangular inequality, \( \|s' - x\| \leq (\ell/4), \|t' - x\| > \frac{\ell}{2} \) and \( m \leq \frac{\gamma}{2} \).

(3.1) now follows immediately from (3.4)-(3.6) and \( \|t' - x\| > \frac{\ell}{2} \).

Lemma 3.1 replaces (4.2) in ref. [15]. We now need a definition.

**DEFINITION**  A cube \( \Lambda_\ell(x) \) is non-resonant at the energy \( E \) if \( d(E, \sigma(H_{\Lambda_\ell(x)}) \geq \frac{1}{2} \epsilon^{-\ell \theta} \), i.e., if and only if \( \|G_{\Lambda_\ell(x)}(E)\| \leq 2 e^{\ell \theta} \). In this case we will say that \( \Lambda_\ell(x) \) is \( E - NR \).

The following lemma gives the deterministic part of the induction step in the proof of Theorem 2.2. It replaces Lemma 4.2 in [15].

**LEMMA 3.5.** Let \( L = \ell^\alpha \) with \( 1 < \alpha < 2 \), \( E \in \mathbb{R} \), and \( m_\ell \) with

\[
\left(\frac{36}{\ell \alpha - 1} + \frac{31}{\ell - \beta}\right) \leq m_\ell \leq \frac{\gamma}{2}.
\]

**Proof:** Let \( m = m_\ell \). By (iii) we have at most 3 non-overlapping cubes of side \( \ell \) contained in \( \Lambda_\ell(x) \) that are \( (m, E) \) singular. It follows that we can find \( u_i \in \Lambda_\ell(x), i = 1, \ldots, r \), where \( r \leq 3 \), such that if \( u \in \Lambda_\ell(x) \setminus \bigcup_{i=1}^{t} \Lambda_\ell(u_i) \) with \( d(u, \partial \Lambda_\ell(x)) \geq \frac{\ell}{2} \), then \( \Lambda_\ell(u) \) is \((m, E)-regular\).

A geometric argument now shows that we can find cubes \( \Lambda_{\ell_1} \subset \Lambda_\ell(x) \) with side \( \ell_1 \in \{j\ell, j = 2, 5, 8\}, i = 1, \ldots, t \), such that \( d(\Lambda_{\ell_1}, \Lambda_{\ell_1}) \geq \ell \) if \( i \neq j \),

\[
\Xi \equiv \bigcup_{i=1}^{t} \Lambda_{\ell_1} \supset \bigcup_{i=1}^{r} \Lambda_{\ell_2}(u_i),
\]

and

\[
\sum_{i=1}^{t} \ell_1 \leq 8\ell.
\]

SUBLEMMA 3.6. Suppose \( u \in \mathbb{R} \), with \( d(\Lambda_{\ell_1}(u), \partial \Lambda_\ell(x)) > \ell \), and \( y \in \Lambda_\ell(x) \setminus \Lambda_{\ell_1}(u) \). There exist \( \ell_1 = \ell_1(d, \beta) < \infty \), such that for \( \ell \geq \ell_1 \) we have

\[
|G_{\Lambda_\ell(x)}(E; u, y)| \leq e^{-(m - \frac{1}{e^{2\beta} - 1}) \|t_1 - u\|} \|G_{\Lambda_\ell(x)}(E; t_1, y)|.
\]

for some \( s_1 \in \Lambda_{\ell_1}(u) \) and \( t_1 \in \Lambda_\ell(x) \setminus \Lambda_{\ell_1}(u) \).
Proof: We use the resolvent identity as in (3.3) and (3.4) to get

\[ |G_{\lambda_L}(E; u, y)| \leq (\ell_t + 1)^d (L + 1)^d |G_{\lambda'_t}(E; u, s)| e^{-\gamma \|t - s\|} |G_{\lambda_L}(E; t, y)| , \]

for some \( s \in \lambda'_t(u) \) and \( t \in \lambda_L(x) \setminus \lambda'_t(u) \). Since \( \lambda'_t(u) \) is E-NR by (ii), we have

\[ |G_{\lambda_L}(E; u, y)| \leq (\ell_t + 1)^d (L + 1)^d e^{(s_{\gamma})\|t - s\|} |G_{\lambda_L}(E; t, y)| \]

\[ \leq (8\ell + 1)^d (L + 1)^d e^{(s_{\gamma})\|t - s\|} |G_{\lambda_L}(E; t, y)| . \]  

(3.14)

If \( \|t - s\| \geq \frac{\delta}{2} \), the sublemma follows immediately from (3.14) and (3.7). If not, \( \lambda_L(t) \) must be (m, E)-regular, so we can use Lemma 3.1 to estimate \( |G_{\lambda_L}(E; t, y)| \) in (3.14). We get

\[ |G_{\lambda_L}(E; u, y)| \leq (8\ell + 1)^d (L + 1)^d e^{(s_{\gamma})\|t - s\|} e^{-m'||t' - t||} |G_{\lambda_L}(E; t', y)| \]

\[ \leq e^{-\ell''} e^{(s_{\gamma})\|t - s\|} e^{-m'||t' - t||} |G_{\lambda_L}(E; t', y)| \]

\[ \leq e^{-\ell''} e^{-m'||t' - t||} |G_{\lambda_L}(E; t', y)| \]  

(3.15)

(3.16)

for some \( t' \in \lambda_L(x) \setminus \lambda_L(t) \), where \( m'' \) is given by the right hand side of (3.2) and

\[ m'' = m' - \frac{20}{\ell_t - \beta} > m - \frac{25}{\ell - \beta - \beta'} , \]  

(3.17)

where (3.15) and (3.17) are valid for \( \ell \geq \ell_1 \), for some \( \ell_1 = \ell_1(d, \beta) < \infty \), since \( \|t' - t\| > \frac{\delta}{2} \).

If \( t' \notin \lambda_L(t) \), the sublemma is proved. If not, it follows from (3.16) that

\[ |G_{\lambda_L}(E; u, y)| \leq e^{-\ell''} |G_{\lambda_L}(E; t', y)| \]  

(3.18)

(3.19)

for some \( t'' \in \lambda_L(x) \setminus \lambda_L(t) \). Since we can keep on repeating this argument, we eventually get (3.12), unless \( |G_{\lambda_L}(E; u, y)| = 0 \), in which case there is nothing to prove.

We can now finish the proof of Lemma 3.2. Let \( \ell \geq \ell_1 \); for \( y \notin \lambda_L(x) \) we estimate \( |G_{\lambda_L}(E; x, y)| \) by using either Lemma 3.1 or Sublemma 3.2, as appropriate, starting from the center \( x \) of the box \( \lambda_L(x) \). Setting \( M' = m - \frac{25}{\ell - \beta - \beta'} \), we get, after \( n \) steps,

\[ |G_{\lambda_L}(E; x, y)| \leq e^{-M'|t_1 - t_2| + \cdots + |t_{n - 1} - t_n|} |G_{\lambda_L}(E; u_n, y)| , \]  

(3.20)

with \( u_1,\ldots,u_n \in \lambda_L(x) \). We can assume that each \( \lambda_L \) is visited only once, i. e., \( u_i, u_j \in \Xi \), \( i \neq j \), imply \( \lambda'_L(u_i) \neq \lambda'_L(u_j) \). Thus, using (3.11), we have

\[ \|u_1 - \tilde{z}\| + \|u_2 - \tilde{u}_1\| + \cdots + \|u_n - \tilde{u}_{n-1}\| \geq \|u_1 - x\| + \|u_2 - u_1\| + \cdots + \|u_n - u_{n-1}\| - 8\ell . \]  

(3.21)

(3.22)
The procedure must be stopped the first time (3.21) is violated; in which case we have
\[ \|u_1 - x\| + \|u_2 - u_1\| + \cdots + \|u_n - u_{n-1}\| \geq \|u_n - x\| \geq \|y - x\| - (9\ell + 1). \] (3.23)

It now follows from (3.20), (3.22), (3.23) and (i) that
\[ |G_{\Lambda_t(x)}(E; x, y)| \leq 2e^{L^\beta} e^{-M'(\|x-y\|-(17\ell+1))} \leq e^{-M''\|x-y\|}, \] (3.24)

with
\[ M'' = (m - \frac{25}{\ell^1 - \beta})(1 - \frac{69}{\ell^a - 1}) - \frac{5}{L^1 - \beta} \] (3.25)
since \( \|x - y\| > \frac{L}{\ell} \). It follows that there exists \( Q' \subseteq Q'(d, \gamma, \beta, a) < \infty \), such that if \( \ell \geq Q' \) we have, using (3.7),
\[ M'' \geq m - \left( \frac{69}{\ell^a - 1} + \frac{25}{\ell^1 - \beta} \right) \geq m - \left( \frac{\gamma}{\ell^a - 1} + \frac{36}{\ell^1 - \beta} \right) \] (3.26)

The lemma is proved.

Theorem 2.2 is now proven as in ref. [15]. The probabilistic part of the proof is the same, the deterministic induction step is given by Lemma 3.2.

IV. Proof of theorem 2.3

We recall that \( E \) is a generalized eigenvalue for \( H \) as in (1.1), if there exists a nonzero polynomially bounded function \( \psi \) on \( \mathbb{Z}^d \) such that
\[ \sum_{y \in \mathbb{Z}^d} H(x, y)\psi(y) = E\psi(y) \quad \text{for all} \quad x \in \mathbb{Z}^d. \] (4.1)

In this case \( \psi \) is called a generalized eigenfunction.

We use the following basic result[21,22,23]; notice that in \( \ell^1(\mathbb{Z}^d) \) the proof does not require \( \Gamma = -\Lambda \), it suffices for \( \Gamma \) to be as in (1.1):

With respect to the spectral measure of \( H \), almost every energy is a generalized eigenvalue.

Thus, Theorem 2.3 follows from the following lemma, as in ref. [15].

**LEMMA 4.7.** Under the hypothesis of Theorem 2.3, with probability one the generalized eigenfunctions of

H, corresponding to generalized eigenvalues in \( I \), decay exponentially fast at infinity with mass \( m \).

Lemma 4.1 is proved in the same way as Lemma 3.1 in ref. [15], the necessary modifications will be given below as lemmas. \( H \) will always be as in (1.1) and \( \beta, m, a, L_k \) as in Theorem 2.3. Notice the lemmas are stated for a fixed potential \( V \).

**LEMMA 4.8.** Let \( E \) be a generalized eigenvalue for \( H \), with corresponding generalized eigenfunction \( \psi \). Suppose \( \Lambda_t(x) \) is a \((m, E)\)-regular box, then
\[ |\psi(x)| \leq \sum_{y \in \Lambda_t(x)} e^{-M\|y-x\|} |\psi(y)|, \] (4.2)
with
\[ M = m - \frac{4}{\ell^1 - \beta} - \frac{2\log((\ell + 1)^d)}{\ell}. \] (4.3)

**Proof:** Since \( \Lambda_t(x) \) is a \((m, E)\)-regular, \( E \notin \sigma(H_{\Lambda_t(x)}) \), so it follows from (4.1) that
\[ \psi(x) = - \sum_{y \in \Lambda_t(x), y \notin \Lambda_t(x)} G_{\Lambda_t(x)}(E; u, y) \Gamma(u, y) \psi(y) , \] (4.4)
hence
\[ |\psi(x)| \leq \sum_{y \in \Lambda_t(x), y \notin \Lambda_t(x)} |G_{\Lambda_t(x)}(E; u, y)| e^{-\gamma \|y-u\|} |\psi(y)|. \] (4.5)
The lemma now follows by the same argument as in the proof of Lemma 3.1.

**Lemma 4.9.** Let $E$ be a generalised eigenvalue for $H$, with corresponding generalized eigenfunction $\psi$. Suppose $x_0 \in \mathbb{Z}^d$ is such that $\psi(x_0) \neq 0$. Then there exists $k_1 = k_1(V, E, x_0) < \infty$, such that $\Lambda_{L_k}(x_0)$ is $(m, E)$-singular for all $k \geq k_1$.

**Proof:** Suppose the lemma is false, then there exists a sequence $k_n \to \infty$ such that $\Lambda_{L_{k_n}}(x_0)$ is $(m, E)$-regular for all $n$. But then it follows from Lemma 4.2 that

$$|\psi(x_0)| \leq \lim_{n \to \infty} \sum_{y \in \Lambda_{k_n}(x_0)} e^{-|y-x_0||\psi(y)| = 0},$$

(4.6)

where $\psi$ is polynomially bounded. This is a contradiction.

We set

$$A_{k+1}^{(b)}(x_0) = \Lambda_{2L_{k+1}}(x_0) \setminus \Lambda_{2L_k}(x_0),$$

(4.7)

where $b$ is a positive integer.

**Lemma 4.10.** Let $E$ be a generalised eigenvalue for $H$, with corresponding generalized eigenfunction $\psi$, and $x_0 \in \mathbb{Z}^d$ is such that $\psi(x_0) \neq 0$. Suppose that for all $b$ there exists $\bar{k_b} < \infty$, such that for $k \geq \bar{k_b}$ we have $\Lambda_{L_k}(x)$ $(m, E)$-regular for all $x \in A_{k+1}^{(b)}(x_0)$. Then $\psi$ decays exponentially fast at infinity with mass $m$.

**Proof:** Since $\psi$ is polynomially bounded, there exists $t > 0$ such that

$$|\psi(x)| \leq (1 + ||x - x_0||)^t$$

for all $x \in \mathbb{Z}^d$, (4.8)

if $\psi$ is properly normalized.

Now let $p$: $0 < p < 1$, be given, we pick $b > \frac{1 + p}{1 - p}$, and define

$$\tilde{A}_{k+1}^{(b)}(x_0) = \Lambda_{\tilde{L}_{k+1}}(x_0) = \Lambda_{1 + \frac{b}{1 - p}L_k}(x_0).$$

(4.9)

Then $\tilde{A}_{k+1}^{(b)}(x_0) \subset A_{k+1}^{(b)}(x_0)$, and, if $x \in \tilde{A}_{k+1}^{(b)}(x_0)$, we have

$$d(x, \partial A_{k+1}(x_0)) \geq \rho ||x - x_0||.$$  

(4.10)

Moreover, if $||x - x_0|| > \frac{L_0}{1 - p}$, we have $x \in \tilde{A}_{k+1}^{(b)}(x_0)$ for some $k$.

So let us fix $x \in \tilde{A}_{k+1}^{(b)}(x_0)$, with $k \geq \bar{k_b}$. It follows that $\Lambda_{L_k}(y)$ is $(m, E)$-regular for any $y \in \Lambda_{\rho ||x - x_0||}(x) \subset A_{k+1}(x_0)$. We now apply Lemma 4.2 with $\ell = L_k$; it follows from (4.2) that for any $y \in \Lambda_{\rho ||x - x_0||}(x)$ with $\delta_y = \rho ||x - x_0|| - ||y - x|| > \frac{L_0}{2}$, there exists $u \in \Lambda_{2\delta_y}(y) \setminus \Lambda_{L_k}(y)$, such that

$$|\psi(y)| \leq (2\delta_y + 1)^d e^{-m_1 ||u - y||} |\psi(u)| + \sum_{y \in \Lambda_{2\delta_y}(y)} e^{-m_1 ||u - y||} |\psi(u)|,$$

(4.11)

with $m_1$ given by the right hand side of (4.3). We now use (4.8), so

$$\sum_{y \in \Lambda_{2\delta_y}(y)} e^{-m_1 ||u - y||} |\psi(u)| \leq \sum_{y \in \Lambda_{2\delta_y}(y)} e^{-m_1 ||u - y||} (1 + ||y - x_0||)^t$$

$$\leq \sum_{y \in \Lambda_{2\delta_y}(y)} e^{-m_1 ||u - y||} (1 + bL_{k+1} + ||y - x||)^t$$

$$\leq e^{-m_2 \delta_y},$$

(4.12)
with $m_2 = m - \frac{6}{L_1^2}$, in case $L_k \geq \ell_1$ for some $\ell_1 = \ell_1(d, \beta, a, m, b, t) < \infty$. On the other hand,

$$
(2\delta_y+1)^d e^{-m_1\|u-y\|} \leq (2bL_{k+1}+1)e^{-m_1\|u-y\|} \leq e^{-m_2\|u-y\|}
$$

if $L_k \geq \ell_2$ for some $\ell_2 = \ell_2(d, \beta, a, b) < \infty$.

Thus, given $y \in \Lambda_{\rho\|x-x_0\|}(x)$ with $\delta > \frac{L_k}{2}$, if $L_k \geq \ell_3 = \max(\ell_1, \ell_2)$ there exists $u \in \Lambda_{2\delta_y}(y) \setminus \Lambda_{L_k}(y)$, such that

$$
|\psi(y)| \leq e^{-m_3\|u-y\|}|\psi(u)| + e^{-m_2\delta_y}.
$$

(4.14)

We now start from $x$ and apply (4.14) repeatedly, obtaining, at the $n$th step,

$$
|\psi(x)| \leq e^{-m_2\delta_x} + e^{-m_3\|y_1-x\|}|\psi(y_1)|
$$

\[
\leq e^{-m_2\delta_x} + e^{-m_3\|y_1-x\|} e^{-m_2\delta_{y_1}} + e^{-m_3\|y_2-y_1\|} e^{-m_2\|y_2-y_1\|} |\psi(y_2)|
\]

\[
\leq e^{-m_2\delta_x} + e^{-m_3\|y_1-x\|} + \cdots + e^{-m_3\|y_{n-1}-y_{n-2}\|} + e^{-m_3\|y_{n-1}-y_{n-2}\|} |\psi(y_n)|
\]

(4.15)

for some $y_n \in \Lambda_{\rho\|x-x_0\|}(x)$ with $\|y_n - y_{n-1}\| > \frac{L_k}{2}$, in case the first $n-1$ applications of (4.14) gave us $y_1, \ldots, y_{n-1} \in \Lambda_{\rho\|x-x_0\|}(x)$, $\|y_1-x\|, \ldots, \|y_{n-1}-y_{n-2}\| > \frac{L_k}{2}$, with $\delta_{y_1}, \ldots, \delta_{y_{n-1}} > \frac{L_k}{2}$. We have two cases:

1. We obtain (4.15) with $n$ such that

$$
\frac{2}{L_k} \rho\|x-x_0\| \leq n < \frac{2}{L_k} \rho\|x-x_0\| + 1.
$$

In this case we have

$$
|\psi(x)| \leq n e^{-m_2\rho\|x-x_0\|} + e^{-m_2\|x-x_0\|} |\psi(y_n)|
$$

\[
\leq \left((\frac{2}{L_k} \rho\|x-x_0\| + 1) + |\psi(y_n)|\right) e^{-m_2\rho\|x-x_0\|}
\]

\[
\leq \left(\frac{2bL_{k+1}}{L_k} + 1 + (1 + bL_{k+1})\right) e^{-m_2\rho\|x-x_0\|}
\]

\[
\leq e^{-m_3\rho\|x-x_0\|},
\]

(4.17)

if $L_k \geq \ell_4$ for some $\ell_4 = \ell_4(d, \beta, \alpha, b, m) < \infty$, where

$$
m_3 = m - \frac{6}{L_k^2}.
$$

(4.18)

2. We must stop the procedure with $n < \frac{2}{L_k} \rho\|x-x_0\|$. In this case we must have $\delta_{y_n} \leq \frac{L_k}{2}$, so we must have $\|y_n-x\| \geq \rho\|x-x_0\| - \frac{L_k}{2}$. It now follows from (4.15) that

$$
|\psi(x)| \leq n e^{-m_2\rho\|x-x_0\|} + e^{-m_2\|y_n-x\|} |\psi(y_n)|
$$

\[
\leq n e^{-m_2\rho\|x-x_0\|} + e^{-m_2\left(\rho\|x-x_0\| - \frac{L_k}{2}\right)} |\psi(y_n)|
\]

(4.19)

But since $y_n \in \Lambda_{\rho\|x-x_0\|}(x)$, we know that $\Lambda_{L_k}(y_n)$ is $(m, \rho)$-regular, so it follows from (4.11) and (4.12) with $\frac{L_k}{2}$ substituted for $\delta_y$, that

$$
|\psi(y_n)| \leq e^{-m_2\frac{L_k}{2}}.
$$

(4.20)
Combining (4.19) and (4.20) we get

\[
|\psi(x)| \leq n e^{-m_2 \rho} ||x - x_0|| + e^{-m_2 \rho} ||x - x_0||
\]

\[
\leq \left( \frac{2}{L_k^2} ||x - x_0|| + 1 \right) e^{-m_2 \rho} ||x - x_0||
\]

\[
\leq \left( \frac{2bL_k + 1}{L_k} \right) e^{-m_2 \rho} ||x - x_0||
\]

\[
\leq e^{-m_3 \rho} ||x - x_0||
\]

(4.21)

if \( L_k \geq L_4 \).

It follows from (4.17), (4.21) and (4.18) that, given \( \rho', 0 < \rho' < 1 \), we can find \( \hat{k} < \infty \) such that, if \( k \geq \hat{k} \) we have

\[
|\psi(x)| \leq e^{-\rho' \rho ||x - x_0||}
\]

(4.22)

in case \( ||x - x_0|| \geq \frac{L_k}{1 - \rho} \).

We can conclude that \( \psi \) decays exponentially, and

\[
\limsup_{||x|| \to \infty} \frac{\log |\psi(x)|}{||x||} \leq -\rho' \rho m
\]

(4.23)

for any \( \rho \) and \( \rho' \in (O, 1) \).

Lemma 4.4 is proved.

Acknowledgments

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References