Loop Variables and Holonomy Transformations for a Class of Space-Times

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We show that the loop variables for static spherically symmetric space-times are elements of the Lorentz group $SO(3,1)$, or more generally, they are elements of the covering group of the Lorentz group in order to include fermions. The analogous results concerning the cylindrically symmetric space-time are given. In this case we particularize our results to the $(2+1)$-dimensional space-time showing that the loop variables are elements of $SO(2,1)$ or its covering group. Some examples and applications are discussed.

I. Introduction

In the loop space formalism for gauge theories the fields depend on paths rather than on space-time points, and a gauge field is described by associating with each path in space-time an element of the corresponding gauge group. The fundamental quantity that arises from this path-dependent approach, the non-integrable phase factor (or loop variable, in our terminology) represents the electromagnetic field or a general gauge field more adequately than the field strength or the integral of the vector potential. In the electromagnetic case, for example, as observed by Wu and Yang, in a situation where global aspects are taken into consideration the field strength underdescribes the theory and the integral of the vector potential for every loop overdescribes it. The exact description is given by the factor $\exp(\int_c A_\mu dx^\mu)$.

The extension of the loop space formalism to the theory of gravity was first considered by Mandelstam who established several equations involving the loop variables, and also by Yang, Menskii and Voronov and Makeenko. Recently, Bollini et al. computed the loop variables for the gravitational field corresponding to the Kerr metric.

Einstein gravity in $(2+1)$-dimensional space-time has recently developed into an area of active research. One reason for this interest is that there are systems whose symmetry properties reduce the effective number of dimensions. In gravity this occurs for the space-time created by an infinite cosmic string, which we shall consider here. On the other hand, this interest has been stimulated by the peculiar and non-generic properties of this field theory.

Space-time is flat outside matter in three-dimensional gravity as well as outside a cosmic string and hence there can exist no static interaction between sources. The effects of the sources show up in global aspects of the geometry and we find topology assuming the role played by curvature in the $(3+1)$-dimensional theory. Although the local curvature of source free regions in $(2+1)$-dimensional gravity is unaffected by any matter in the space-time, it is important to understand that matter can still produce nontrivial global effects. In order to study these effects we shall use the only possible observables in this theory which must come from non-local variables such as the loop variables matrices.

The loop variables in the theory of gravity are matrices representing parallel transport along contours in a space-time with a given affine connection. They are connected with the holonomy transformations which contain important topological information. These mathematical objects contain information, for example, about how vectors change when parallel transported around a closed curve. They also can be thought of as measuring the failure of a single coordinate patch to extend all the way around a closed curve.

Suppose that we have a vector $v^a$ at a point $p$ of a closed curve $C$ in a space-time. Then, one can produce a vector $\tilde{v}^a$ at $p$ which, in general, will be different from $v^a$, by parallel transporting $v^a$ around $C$. In this case, we associate with the point $p$ and the curve $C$ a linear map $U^a_\beta$ such that for any vector $\gamma^a$ at $p$, the vector $\tilde{v}^a$ at $p$ results from parallel transporting $v^a$ around $C$ and is given by $\tilde{v}^a = U^a_\beta v^\beta$. The linear map $U^a_\beta$ is called the holonomy transformation associated with the point $p$ and the curve $C$. If we choose a tetrad frame and a parameter $\lambda \in [0,1]$ for the curve $C$ such that $C(0) = C(1) = p$, then in parallel transporting a vector $v^a$ from $C(\lambda)$ to $C(X + d\lambda)$, the vector components change by $\delta v^a = M^a_\beta [x(\lambda)] v^\beta \lambda$, where $M^a_\beta$ is a linear
map which depends on the tetrad, the affine connection of the space-time and the value of $A$. Then, it follows that the holonomy transformation $U_{\beta}^a$ is given by the ordered matrix product of the N linear maps as

$$U_{\beta}^a = \lim_{N \to \infty} \prod_{i=1}^{N} \left\{ \delta_{\beta}^{\gamma} + \frac{1}{N} M_{\beta}^\gamma [x(\lambda)]_{\lambda=i/N} \right\} . \quad (1.1)$$

One often writes the expression in Eq. (1.1) as

$$U(C) = P \exp \left( \int_{C} M \right) . \quad (1.2)$$

where $P$ means ordered product along a curve $C$. Equation (1.2) should be understood as simply an abbreviation for the expression in Eq. (1). Note that if $M_{\beta}^\gamma$ is independent of $A$, then it follows from Eq. (1) that $U_{\beta}^a$ is given by $U_{\beta}^a = [\exp(M)]_{\beta}^a$. Under a change of coordinates $x \to x^i = L_x U_{\beta}^a$ transforms as $L(U_{\beta}^a) L^{-1}$.

In this paper we shall use the notation

$$\omega = P \exp \left( \int_{A}^{B} \frac{dx^\mu}{\gamma} \right), \quad (1.3)$$

where $\Gamma^\mu$ is the tetradic connection and $A, B$ are the initial and final points of the path. Then, associated with every path $C$ from point $A$ to point $B$, we have a loop variable given by Eq. (1.3) which is a function of the path $C$ as a geometrical object.

The aim of this paper is to study the theory of gravity using loop variables on the basis of a metric formalism. In Section II we compute the loop variables for a static spherically symmetric space-time and the results are applied to the black hole-string metric for an uncharged non-rotating hole. Section III contains similar results concerning the cylindrically symmetric space-time in $(2+1)$ and $(3+1)$ dimensions and a brief discussion on the gravitational analogue of the Aharonov-Bohm effect and on the study of space-time configuration from the global point of view. Finally, in Section IV, we add some concluding remarks.

II. Loop variables in a spherically symmetric space-time

The space-time metric which represents a static spherically symmetric solution of the Einstein's field equations can be written as

$$ds^2 = e^{2\Phi(r)} dt^2 - e^{2\Lambda(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 , \quad (11.1)$$

where $\Phi(r)$ and $\Lambda(r)$ are functions of $r$ only, $t$ is the time-like coordinate ($-\infty < t < \infty$) and $\phi$ and $\theta$ are spherical coordinates.

We wish to incorporate a string defect in this metric because we are interested in the effect of the string in this background space-time. We can easily introduce a conical singularity describing a straight cosmic string assuming that a string is a defect in space-time and is to be introduced by removing a sector of angle, say $8\pi \mu$ ($\mu$ is the linear mass density of the string) and identifying the sides of the sector, that is, identifying $(\phi, r + 2\pi(1 - 4\mu))$ rather than with $(\phi + 2\pi, r)$, so making the periodicity arbitrary. Thus, the $(r, \phi)$ plane is topologically equivalent to a cone of angle $\sin^{-1}(1 - 4\mu)$.

The static spherically symmetric metric with a string passing through is simply given by

$$ds^2 = e^{2\Phi(r)} dt^2 - e^{2\Lambda(r)} dr^2 - r^2 d\theta^2 - r^2 (1 - 4\mu)^2 \sin^2 \theta d\phi^2 , \quad (11.2)$$

where $0 \leq \phi \leq 2\pi$.

In order to compute the loop variables we have to write an explicit expression for the tetradic connection $\Gamma^a_{\mu}$.

Let us introduce a set of four vectors $e^a_{(\mu)} (a = 0, 1, 2, 3$ is a tetradic index) which are orthonormal at each point with respect to the metric with Minkowski signature, that is, $g_{\mu\nu} e^a_{(\mu)} e^b_{(\nu)} = \eta_{ab} = \text{diag}(+1, -1, -1, -1)$. We assume that the $e^a_{(\mu)}$'s are matrix invertible, that is, that there exists an inverse frame $e^a_{(\nu)}$ given by $e^a_{(\nu)} e^b_{(\mu)} = \delta^a_{\nu} \delta^b_{\mu}$ and $e^a_{(\nu)} e^b_{(\mu)} = \delta^a_{\nu} \delta^b_{\mu}$.

Define the one-forms $\omega^a (a = 1, 2, 3, 4)$ as

$$\omega^0 = e^{\Phi(r)} dt , \quad \omega^1 = e^{\Lambda(r)} dr , \quad \omega^2 = r d\theta , \quad \omega^3 = (1 - 4\mu) r \sin \theta d\phi . \quad (11.3)$$

Then, in a coordinate system $(x^0 = t, x^1 = r, x^2 = \theta$ and $x^3 = \phi)$ the tetrad frame defined by $\omega^a = e^a_{(\mu)} dx^\mu$ is given by

$$e^a_{(\mu)} = \left( \begin{array}{cccc} e^{\Lambda(r)} & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & (1 - 4\mu) r \sin \theta & 0 \\ 0 & 0 & 0 & e^{\Phi(r)} \end{array} \right).$$

Using the Cartan's structure equations $d\omega^a = e^a_{(\mu)} e^\mu_{(\nu)} \Lambda_{\nu}^b dx^b$, we get the following expressions for the tetradic connections $\Gamma^a_{\mu b} (a, b$ are tetradic indices)

$$\Gamma^0_{10} = \Gamma^0_{20} = -e^{-\Lambda(r)} \frac{d}{dr}(e^{\Phi(r)}),$$
$$\Gamma^0_{21} = \Gamma^0_{31} = -e^{-\Lambda(r)} \sin \theta e^{-\Phi(r)} ,$$
$$\Gamma^3_{13} = \Gamma^3_{23} = -e^{-\Lambda(r)} \cos \theta.$$

(11.4)
First of all we shall consider general curves in the $xy$ plane (and planes parallel to it), at fixed times. In this case we have

$$\Gamma_\mu dx^\mu = \Gamma_\phi d\phi, \quad (II.5)$$

and

$$\Gamma_\phi = -i(1 - 4\mu) \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sin \theta e^{-\Lambda(r)} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cos \theta$$

$$= -i(1 - 4\mu)(\sin \theta e^{-\Lambda(r)} J_{13} + \cos \theta J_{23}). \quad (II.6)$$

In Eq.(II.6), $J_{13}$ and $J_{23}$ are, respectively, the generators of rotations about the $y$- and $x$-axis in $\mathbb{R}^3$. Therefore, for a general curve in the $xy$-plane, the loop variable is given by

$$U_{\phi_2\phi_1}(C) = \exp[-i(\phi_2 - \phi_1)(1 - 4\mu)] \left( \sin \theta e^{-\Lambda(r)} J_{13} + \cos \theta J_{23} \right). \quad (II.7)$$

When the curve is closed we get from Eq.(II.7) the following expression for the holonomy transformation

$$U_{2\pi,0}(C) = \exp[-2\pi i(1 - 4\mu)(\sin \theta e^{-\Lambda(r)} J_{13} + \cos \theta J_{23})] \quad (II.8)$$

Now, consider a curve $r(\lambda)$, with $\theta(\lambda)$ coiisided in a meridian plane. In this case we have

$$\Gamma_\lambda d\lambda = \left( \Gamma_\phi \frac{d\theta}{d\lambda} + \Gamma_\gamma \frac{dr}{d\lambda} \right) d\lambda. \quad (II.9)$$

From Eqs.(II.4) we see that $\Gamma_\gamma = 0$ and $\Gamma_\phi = i e^{-\Lambda(r)} J_{12}$, independent of $\theta$. Then the loop variables for a general curve in the meridian plane is given by

$$U_{\theta_2\theta_1}(C) = \exp[i e^{-\Lambda(r)}(\theta_2 - \theta_1)]J_{12}, \quad (II.10)$$

where

$$J_{12} = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is the generator of rotations about the local $z$-axis in $\mathbb{R}^3$.

From Eq.(II.10) we get for a closed curve

$$U_{2\pi,0}(C) = \exp[-2\pi i(1 - e^{-\Lambda})J_{12}] \quad (II.11)$$

$$= \begin{pmatrix} \cos[2\pi(1 - e^{-\Lambda})] & \sin[2\pi(1 - e^{-\Lambda})] & 0 & 0 \\ -\sin[2\pi(1 - e^{-\Lambda})] & \cos[2\pi(1 - e^{-\Lambda})] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (II.12)$$

Eq.(II.11) represents a rotation about the $Oz$ axis through an angle $2\pi(1 - e^{-\Lambda})$. Finally consider a translation in time. In this case $\Gamma_\mu dx^\mu = \Gamma_t dt$ where

$$\Gamma_t = -i \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} e^{-\Lambda(r)} \frac{d}{dr} (e^{\Phi(r)})$$

$$= -ie^{-\Lambda(r)} \frac{d}{dr} (e^{\Phi(r)}) J_{01}. \quad (II.13)$$

$J_{01}$ being the generator of a boost in the $Oz$-direction. Using Eq.(II.12) we get for a time translation between $t_1$ and $t_2$, the following expression for the loop variable

$$U_{t_2t_1}(C) = \exp \left[ -ie^{-\Lambda(r)} \frac{d}{dr} (e^{\Phi(r)}) (t_2 - t_1) J_{01} \right]$$

$$= \begin{pmatrix} \cos y & 0 & 0 & \sinh y \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh y & 0 & 0 & \cosh y \end{pmatrix}, \quad (II.14)$$

where $y = e^{-\Lambda(r)} \frac{d}{dr} (e^{\Phi(r)})$ is the boost parameter. Eq.(II.13) represents a boost in the $(x,t)$ direction.

Using the above results we can write a general expression for $U(C)$. In the general case $U(C)$ reads
where \( J_{ab} \) are the generators of the Lorentz group \( SO(3,1) \) and \( \Gamma^a_\mu \) are the appropriate tetrad connections. From the above result we conclude that the holonomy for \((3 + 1)\)-dimensional static space-time spherically symmetric, is the homomorphism that maps the homotopy class of all the curves to the rotations and Loosts in \( SO(3,1) \). As ordinary vectors live in tangent space to the manifold and for static space-times there is no slip of spinning and consequently no translations, the transformations that act on this space are the Lorentz ones and therefore the parallel transport matrices (loop variables) must be elements of the Lorentz group. In general, the \( J_{ab} \)'s generate the representation of the Lorentz group which acts on the transported quantity which can be a vector or a spinor. In the spinor case, instead of the group \( SO(3,1) \) we have a covering group of this one. Therefore, when we have fermions, the loop variables are elements of the covering group of the Lorentz group.

From the previous results we see that the wedge removal affects the loop variable in the \( xy \) plane only, so that, a vector parallel transported along a curve in the \( xy \) plane will detect the presence of the stria.

As an example consider the black hole-string metric for an uncharged non-rotating hole which is given by

\[
ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + (1 + 4\mu) \sin^2 \theta d\varphi^2),
\]

where \( 0 \leq \varphi \leq 2\pi \).

In view of the previous results we should expect the presence of the stria through the role to modify the holonomy transformation for general curves in the \( xy \) plane, which is given, in this case, by

\[
U_{2\pi,0}(C) = \exp \left\{ +2\pi i \left(1 - 4\mu\right) \left(1 - \frac{2M}{r}\right)^{1/2} J_{13} \right\}.
\]

Consider a path formed by two beams which is assumed to circle the \( z \) axis and to be for fixed \( \theta \), say \( \pi/2 \). Then, the relevant phase is

\[
U_{2\pi,0}(C) := \exp \left\{ +2\pi i \left[ 1 - (1 - 4\mu) \left(1 - \frac{2M}{r}\right)^{1/2} \right] J_{13} \right\},
\]

where we have introduced a factor \( \exp(2\pi i J_{13}) \) in order to take into account the rotation of the local tetrad frame with respect to a tetrad of fixed orientation and which is equal to the \( 4 \times 4 \) identity matrix.

Eq. (11.18) give us the phase acquired by a vector when parallel transported around the source for \( 0 = \pi/2 \), which is associated with the non-triviality of the holonomy transformation for all values of \( r \neq 2M \). Note that the wedge removal produced by the presence of a stria appears in the phase through the parameter \( \mu \).

We can obtain a similar result iii the spinor case. However, in order to incorporate fermions we have to use the spinorial representation of the Lorentz group. So, we change \( J_{ab} \) by \( \Sigma_{13} = \frac{1}{2} [\gamma_1, \gamma_3] \), where \( \gamma_1 \) and \( \gamma_3 \) are Dirac matrices in the standard representation. Then, for a path in the \( xy \) plane, for \( \theta = \pi/2 \), we get

\[
U_{2\pi,0}(C) = \exp \left\{ +2\pi i \left[ 1 - (1 - 4\mu) \left(1 - \frac{2M}{r}\right)^{1/2} \right] \Sigma_{13} \right\}.
\]

From Eq. (11.19) we see that the holonomy transformation for the black hole-string metric and for Schwarzschild metric (\( \mu = 0 \)) also is trivial only for \( r = 2M \), where the metric is infinite. For physical sources however, this singularity occurs inside the source where the exterior solution does not apply. Thus, the phase can never reach the trivial value, or in other words, the operability of this value is limited by physical considerations.

### III. Loop variables in a cylindrically symmetric space-time and applications

The most general static cylindrically symmetric metric may be expressed in the form

\[
ds^2 = e^{2\nu} dt^2 - e^{2\lambda} (dp^2 + dz^2) - e^{2\psi} d\varphi^2,
\]

where \( \lambda \) is the time-like coordinate \( - \infty < \lambda < \infty \), \( p, \varphi \) and \( z \) are ordinary cylindrical coordinates with \( 0 \leq \rho < \infty, 0 \leq \varphi \leq 2\pi \) and \( -\infty < z < \infty \) and \( \rho, \varphi \) and \( z \) are functions of \( \lambda \).

Proceeding as in Sectioii we define the forms

\[
\begin{align*}
\omega^0 &= e^{\nu} dt, \\
\omega^1 &= e^{\lambda} \cos \varphi dp - e^{\psi} \sin \varphi d\varphi, \\
\omega^2 &= e^{\lambda} \sin \varphi dp + e^{\psi} \sin \varphi d\varphi, \\
\omega^3 &= e^{\lambda} dz.
\end{align*}
\]

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Then, in a coordinate system ($x^1 = p, x^2 = \varphi, x^3 = z$ and $x^0 = t$), the tetrad frame defined by $e^{(a)}_\mu dx^\mu$ is given by

$$
e_1^{(1)} = e^\varphi \cos \varphi, \quad e_1^{(2)} = -e^\varphi \sin \varphi, \quad e_1^{(3)} = e^\lambda, \quad e_1^{(0)} = +e^\nu.
$$

(III.3)

Proceeding analogously to the spherically symmetric case we can show that the loop variables are given by Eq.(II.14), with the tetradic connections given by

$$
\Gamma^1_{i1} = \Gamma^1_{t1} = -e^{-\lambda} \frac{d}{dp}(e^\varphi) \cos \varphi,
\Gamma^2_{t2} = -e^{-\lambda} \frac{d}{dp}(e^\varphi) \sin \varphi,
\Gamma^1_{\varphi \varphi} = -1 + e^{-\lambda} \frac{d}{dp}(e^\varphi),
\Gamma^1_{z3} = -e^{-\lambda} \frac{d}{dp}(e^\lambda) \cos \varphi,
\Gamma^3_{z2} = -e^{-\lambda} \frac{d}{dp}(e^\nu) \sin \varphi.
$$

(III.4)

Consider now the $(2 + 1)$-dimensional case. Then Eqs.(III.4) reduces to

$$
\Gamma^1_{i0} = \Gamma^1_{t1} = -e^{-\lambda} \frac{d}{dp}(e^\varphi) \cos \varphi,
\Gamma^2_{t2} = -e^{-\lambda} \frac{d}{dp}(e^\varphi) \sin \varphi,
\Gamma^1_{\varphi \varphi} = -1 + e^{-\lambda} \frac{d}{dp}(e^\varphi).
$$

(III.5)

Using these connections and considering general curves in the $xy$-plane, translation in time and radial segments it is easy to show that the loop variables are given by Eq.(II.14) where now the $J_{ab}$’s are generators of the group $SO(2,1)$ or in general, $J_{ab}$’s are generators of the covering group of the group $SO(2,1)$.

Now let us define the deficit angle and establish its connection with the holonomy transformation. The deficit angle is one number and the holonomy transformation is a set of linear maps (one for each point and closed curve). One must then obtain from the linear map a single number, the deficit angle which is a property of axially symmetric, asymptotically conical space-times (at infinity, these space-times are asymptotically a cone rather than a plane). To obtain the single linear map we consider a point $p$ on the curve $C$. Since the space-time is axially symmetric, it does not matter which point we choose. Then $U^a_\beta$, as defined previously, is the holonomy transformation associated with the point $p$ and a curve $C$, where $C$ is an integral curve of the axial Killing field in the asymptotic region. With $U^a_\beta$, the deficit angle $\chi$ can be defined by

$$
\cos \chi = U^a_\beta \tilde{A}_a \tilde{A}_\beta,
$$

(III.6)

where $\tilde{A}_a$ is the unit vector in the direction of the axial Killing field. Using tetradic indices we can write

$$
\cos \chi = \tilde{A}^a \eta A_a,
$$

(III.7)

where $\tilde{A}^a = U^a_\beta A^\beta$.

As $\tilde{A}^a$ is a unit vector, the elements of $U$ are the components of the parallel translated vector. From this and Eq. (III.7), it follows that, the corresponding diagonal element of $U$ is the cosine of the angle between the vectors. Then, we can write in this case $\cos \chi_a = U^a_\alpha$, where $a$ is a tetradic index.

Considering $a = 1$, we have

$$
\cos \chi_1 = \cos \left[ 2\pi \left( 1 - e^{-\lambda} \frac{d}{dp}(e^\varphi) \right) \right],
$$

(III.8)

$$
|\chi_1| = 2\pi \left( 1 - e^{-\lambda} \frac{d}{dp}(e^\varphi) \right) + 2\pi n.
$$

As $e^{-\lambda} \rightarrow 0$, we must have $\chi_1 \rightarrow 0$, and we choose $n = 0$ so that

$$
|\chi_1| = 2\pi \left( 1 - e^{-\lambda} \frac{d}{dp}(e^\varphi) \right).
$$

(III.9)

Eq.(III.9) corresponds to the general formula for the angular deficit for a class of static cylindrically symmetric space-times metric given by Eq.(III.1).

Now, let us apply our results to the change in a vector as well as in a spinor when parallel transported along a closed curve in the space-time of a static cylindrically symmetric cosmic string. As we know, the space-time corresponding to this solution has the geometry of a cone $\mathbb{R}^2$. The curvature vanishes everywhere except in the vertex. Then, if a vector (or a spinor) is carried around a closed curve encircling the vertex, after the transport is completed, the vector (or the spinor) changes due to the global effect of the enclosed curvature.

For the cosmic string solution, the metric is a particular case of the one given by Eq.(III.1) with $e^\varphi = e^\lambda = 1$ and $e^\nu = (1 - 4\mu)\rho$, where $\mu$ is the linear mass density of the string and we have considered Newton’s constant $G = 1$. Explicitly, the line element of the space-time described by an infinite, straight and static cylindrically symmetric cosmic string, lying along the $z$-axis, is given by

$$
ds^2 = dt^2 - dp^2 - (1 - 4\mu)\rho^2 \varphi^2 - dz^2,
$$

(III.10)

with the deficit angle $\chi = 8\pi \mu$, obtained from Eq.(III.9).
For general curves in the $xy$ plane we find, using Eqs. (II.14) and (II.4) with $e^+ = e^- = 1$ and $e^z = (1 - 4\mu)\gamma$, that in the cosmic string case, $U(C)$ is given by

$$U(C) = \exp(-8\pi i\mu J_{12})$$

$$= \begin{pmatrix}
\cos(8\pi\mu) & \sin(8\pi\mu) & 0 & 0 \\
-\sin(8\pi\mu) & \cos(8\pi\mu) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

(III.11)

where $8\pi\mu$ is the deficit angle associated with the spacetime of a cosmic string. We can obtain a similar result in the case of transport of spinors but in order to incorporate fermions we have to use the spinorial representation of the Lorentz group. So, we change $J_{12}$ by $\Sigma_{12} = \frac{1}{2}[\gamma_1, \gamma_2]$, where $\gamma_1$ and $\gamma_2$ are Dirac matrices in the standard representation. Then, for general closed curves in the $xy$-plane we get

$$U(C) = \exp(-4\pi i\mu \Sigma_{12}).$$

(III.12)

After the parallel transport, the spinor $\psi(\varphi = 2\pi)$ will be given in terms of the original one $\psi(\varphi = 0)$ by the relation

$$\psi(\varphi = 2\pi) = e^{-4\pi i\mu \Sigma_{12}} \psi(\varphi = 0).$$

(III.13)

From Eq. (I.1.13) we conclude that there will be no Aharonov-Bohm effect if and only if $4\pi\mu$ is an even integer. However, this condition is not always satisfied. Then, we have shown that if we parallel transport a spinor around a closed path in the $xy$-plane lying in the flat region, the transported one does not necessarily coincide with the original. Therefore, when we parallel transport a spinor in a region in which the curvature vanishes, it exhibits a physical effect arising from the enclosed non-zero curvature associated with the presence of the cosmic string. This is an example of the gravitational analogue of the Aharonov-Bohm effect. This effect should be regarded as basically classical, and it is associated with the non-triviality of the holonomy transformation for general curves in the $xy$-plane, due to the presence of the cosmic string. As in this case the geometry is locally flat, the phase shift acquired by the spinor when parallel transported around the source may be regarded as due to the coupling of its energy-momentum to the global geometrical properties of this space-time. The same analysis can be applied to a vector. In this case we use the expression for the holonomy transformation given by Eq. (III.11) concluding that the same effect occurs.

A similar result can be obtained in the case of a spinless particle solution (three-dimensional case). In the spinor case, the spinor acquires a phase given by $\exp(-4\pi i m \Sigma_{12})$, where $\Sigma_{12} = \frac{1}{2}[\sigma_1, \sigma_2]$ with $\sigma_1$ and $\sigma_2$ being Pauli's matrices, and $m$ is the mass of the particle that generates the gravitational field. Following the arguments of the string case, we conclude that we have an Aharonov-Bohm effect in this case. The same analysis can be extended to the transport of vectors, with a similar conclusion.

Similarly, we can consider the metric

$$ds^2 = dt^2 - d\rho^2 - G_0^2\rho^2 d\varphi^2 - (B_0 t + B_1)\rho^2 ds^2,$$  

(III.14)

where $G_0, B_0$ are $B_1$, and $\int B_0 dz$ are integration constants.

The metric given by Eq. (III.14) corresponds to a Minkowski space-time minus a wedge as was see by defining the coordinates $\Phi, Z$, and $T$, for $B_0 \neq 0$, by

$$\Phi \equiv G_0 \varphi,$$

$$Z \equiv \left( t + \frac{B_1}{B_0} \right) \sinh(B_0 z),$$

$$T \equiv \left( t + \frac{B_1}{B_0} \right) \cosh(B_0 z).$$

In the new coordinates, the metric given by Eq. (III.14) reads

$$ds^2 = d\Phi^2 - d\rho^2 - \rho^2 d\varphi^2 - dZ^2.$$  

(III.16)

Thus the above metric is locally flat but not globally. The deficit angle is $2\pi G_0$.

We can do the previous analysis showing that we have a gravitational analogue of the Aharonov-Bohm effect also, in the vector and spinor cases, with the holonomies in the $xy$-plane given by $U(C) = \exp(-2\pi i G_0 J_{12})$. Similarly, $U(C) = \exp(-\pi i G_0 \Sigma_{12})$, respectively.

As another application we shall study the space-time configuration of two moving cosmic strings. To do this we shall use Eq. (III.10) and a result showing that only strings enclosed by the circles contribute to the phase factor acquired by a vector when parallel transported in the background space-time of the multiple cosmic string solution.

Then, suppose that we transport a vector around a string 2 localized at $(a_2, 0, 0, 0)$. The phase factor acquired by this vector is $U_2 = \exp(-8\pi i \mu_2 J_{12})$. Now, carrying the resulting vector along a circle around string 1 localized at $(a_1, 0, 0, 0)$, it is easy to conclude that the resulting vector will have a phase given by the product $U_1 U_2$, where $U_1 = \exp(-8\pi i \mu_1 J_{12})$. Note that we can continue this process involving $N$ strings. After this, the vector will have acquired a phase given by $U_1 U_2 \ldots U_{K-1} U_K U_{K+1} \ldots U_{N-1} U_N$, where $U_K = \exp(-8\pi i \mu_K J_{12})$, $\mu_K$ being the linear mass density of the $K$th string.

Now consider a system of two moving strings. Consider string 1, initially at the origin with velocity $\vec{v}_1$, and string 2, initially at $(a_2, 0, 0, 0)$ with velocity $\vec{v}_2$. The phase factor acquired by the vector in string 1 is $U_1 = \exp(-8\pi i \mu_1 J_{12})$, and the phase factor acquired by the vector in string 2 is $U_2 = \exp(-8\pi i \mu_2 J_{12})$.
and string 2 located along the r-direction at \((a_2, 0, 0, 0)\), with velocity \(\vec{v}_2\), in the xy-plane. These strings can be viewed as strings at rest that were boosted. Then, if we take a vector and carry it along a circle around string 2, instead of the phase \(U_2\) the vector will acquire a phase \(L_2 U L_2^{-1}\), which corresponds to the transformation of the loop variable under the change of coordinate corresponding to the boost \(L_2\) which is given by

\[
L_2 = \begin{pmatrix}
\cosh \gamma_2 & 0 & 0 & \sinh \gamma_2 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\sinh \gamma_2 & 0 & 0 & \cosh \gamma_2
\end{pmatrix},
\]  
(III.17)

where \(\gamma_2\) is the boost parameter and such that \(|\vec{v}_2| = \tanh \gamma_2\).

If now, the resulting vector is parallel transported around string 1, along a circle, the final phase will be \(L_1 U_1 L_1^{-1} L_2 U_2 L_2^{-1}\), where \(L_1\) is given by the same expression for \(L_2\) with the interchange of \(\gamma_2\) by \(\gamma_1\) (\(|\vec{v}_1| = \tanh \gamma_1\)).

Let us now consider a third string that behaves globally like the first two. This string can be viewed as one boosted by

\[
L_3(\varphi_3, \gamma_3) = \begin{pmatrix}
1 - \cos^2 \varphi_3 (1 - \cosh \gamma_3) & -\cos \varphi_3 \sin \varphi_3 (1 - \cosh \gamma_3) & 0 & \cos \varphi_3 \sinh \gamma_3 \\
-\cos \varphi_3 \sin \varphi_3 (1 - \cosh \gamma_3) & 1 - \sin^2 \varphi_3 (1 - \cosh \gamma_3) & 0 & \sin \varphi_3 \sinh \gamma_3 \\
0 & 0 & 1 & 0 \\
(\cosh \gamma_1 \cosh \gamma_2 - \sin \gamma_1 \sin \gamma_2) & 0 & 0 & \cosh \gamma_3
\end{pmatrix},
\]  
(III.18)

and point particles, respectively.

IV. Concluding remarks

We have shown by explicit computation for metrics corresponding to spherically symmetric space-times, that the phase acquired by a particle (vector or spinor), when parallel transported along a given curve \(C\) in the background gravitational fields is given by the loop variables \(U(C) = P \exp(\int \Gamma_{\mu} dx^\mu)\) with \(\Gamma_{\mu} = \frac{1}{2} J_{\mu}^a J_{\mu}^a\), where \(J_{\mu}^a\) are the generators of the Lie algebra of the Lorentz group \(SO(3, 1)\) or of its covering group. Then, for a given curve in these space-times, the phase shift acquired by a particle is an element of the Lorentz group, or in general, the phase factor is an element of the covering group of the Lorentz group, in order to include fermions.

For the metrics corresponding to cylindrically symmetric space-times, the loop variables are elements of the Lorentz group \(SO(3, 1)\) or of its covering group\(^{13}\), and in the \((2 + 1)\)-dimensional case, they are elements of the \(SO(2, 1)\) group or of its covering group also in order to include fermions. These results permit us to study the gravitational analogue of the Aharonov-Bohm effect\(^{11}\).

As the loop variables for the static geometric structures under considerations are elements of the Lorentz group \(SO(3, 1)\), this means that these quantities are related to the holonomies of a flat \(SO(3, 1)\) connections.
and consequently that the space-time geometry is encoded in the holonomies of these flat $SO(3,1)$ connections.

The configuration of a space-time corresponding to two moving strings or particles ((2 + 1)-dimensional case) shows that there is a linking between the parameters that describe this space-time and the spacetimes generated by each of the two strings or particles, respectively.

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